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DISTANCE-PRESERVING MAPPINGS ON RESTRICTED DOMAINS

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ABSTRACT. Let X and Y be n-dimensional Euclidean spaces with $n \geq 3$. In this paper, we generalize a classical theorem of Beckman and Quarles by proving that if a mapping, from a half space of X into Y, preserves a distance ρ, then the restriction of f to a subset of the half space is an isometry.

1. INTRODUCTION

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an *isometry* (or a *congruence*) if f satisfies

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x - y\| = \rho$ always implies $\|f(x) - f(y)\| \leq \rho$. Similarly, a distance ρ is said to be extensive (or non-shrinking) by f if the inequality $\|f(x) - f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x - y\| = \rho$. We say that ρ is conservative (or preserved) by f if ρ is contractive and extensive by f simultaneously.

If f is an isometry, then every distance $\rho > 0$ is conservative by f, and conversely. At this point, we can raise a question:

Is a mapping that preserves certain distances an isometry?

In 1970, Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $\rho > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $\rho = 1$ when X is a normed space (see Rassias [16]).

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Indeed, earlier than Aleksandrov [1], Beckman & Quarles [2] solved, in 1953, the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = E^n$:

Theorem of Beckman and Quarles. *If a mapping* $f : E^n \to E^n$ $(2 \le n < \infty)$ *preserves distance* 1, *then* f *is a linear isometry up to translation.*

For $n = 1$, they suggested the mapping $f : E^1 \to E^1$ defined by

$$f(x) = \begin{cases} x+1 & \text{for } x \in \mathbb{Z}, \\ x & \text{otherwise} \end{cases}$$

as an example for a non-isometric mapping that preserves distance 1. For $X = E^\infty$, Beckman and Quarles also presented an example for a unit distance preserving mapping that is not an isometry (*cf.* Rassias [13]).

We may find a number of papers on a variety of subjects in the Aleksandrov problem (see [3, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and also the references cited therein).

Let X and Y be n-dimensional Euclidean spaces with $n \ge 3$. In this paper, we generalize a classical theorem of Beckman and Quarles by proving that if a mapping, from a half space of X into Y, preserves a distance ρ, then the restriction of f to a subset of the half space is an isometry.

2. Preliminary lemmas and main theorem

Throughout this section, let X and Y denote n-dimensional Euclidean spaces, where $n \ge 3$ is a fixed integer, for which there exists a unit vector $w \in X$ and a subspace X_s of X such that $X = X_s \oplus \mathrm{Sp}(w)$ and X_s is orthogonal to $\mathrm{Sp}(w)$, where $\mathrm{Sp}(w)$ is the subspace of X which is spanned by w.

Let us define

$$r_0 = \theta, \quad r_1 = \theta + \rho, \quad r_2 = \theta + \rho + \rho_1, \quad r_3 = \theta + (1 + 1/n)\rho + \rho_1,$$

where θ is a real number, ρ is a positive real number and $\rho_1 = \sqrt{2(n+1)/n}\,\rho$. Using these r_k's we define

$$E_k = \{x + \lambda w \ : \ x \in X_s \, ; \ \lambda > r_k\}$$

for $k = 0, 1, 2, 3$. We remark that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset X$.

The author, jointly with Rassias, proved a theorem which ensures the validity of the following theorem (see Jung & Rassias [10]). Let us denote by x_s and y_s the X_s-components of x and y of X.

Theorem of Jung and Rassias. *Given an integer $N \geq 2$, if ρ is contractive and $N\rho$ is extensive by a mapping $f : E_2 \to Y$, then $f|_{E_3}$ is an isometry. In particular, if any points x and y of E_2 satisfy $x_s \neq y_s$, then $\|f(x) - f(y)\| = \|x - y\|$.*

Let E be a subset of an n-dimensional Euclidean space X. Following W. Benz, we will call a set of n distinct points of E a β-set in E if the points are pairwise of distance $\beta > 0$. Suppose that α and β are positive real numbers with

$$\gamma(\alpha, \beta) = 4\alpha^2 - 2\beta^2(1 - 1/n) > 0$$

and suppose that P is a β-set in E. The α-associated points of P are the uniquely determined two distinct points of X, which have distance α from each point of P, and the distance between α-associated points is $\sqrt{\gamma(\alpha, \beta)}$ (*cf.* Benz [4]).

Lemma 1. If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_1 = \sqrt{\gamma(\rho, \rho)}$ is preserved by $f|_{E_1}$.

Proof. Assume that x and y are points of E_1 satisfying $\|x - y\| = \rho_1$. According to 3) in Benz [4, §2] and the definition of E_k, there exists a ρ-set P in E_0 such that x and y are the ρ-associated points of P. Since f preserves ρ, $P' = f(P)$ is also a ρ-set in Y.

Due to 2) in Benz [4, §2], there are exactly two distinct ρ-associated points x' and y' of P' and they satisfy $\|x' - y'\| = \sqrt{\gamma(\rho, \rho)} = \rho_1$. Since there exist only two ρ-associated points of P', we have $\{f(x), f(y)\} \subset \{x', y'\}$, *i. e.*, $\|f(x) - f(y)\| = 0$ or ρ_1.

Assume that $f(x) = f(y)$. Choose a $z \in E_0$ with $\|x - z\| = \rho_1$ and $\|y - z\| = \rho$. In view of 3) in Benz [4, §2], there exists a ρ-set Q in E_0 such that x and z are the ρ-associated points of Q (Because $x \in E_1$ and $\|x - q\| = \rho$ for each $q \in Q$, Q is a subset of E_0). Similarly, $Q' = f(Q)$ is a ρ-set in Y.

Due to 2) in Benz [4, §2], there exist exactly two distinct ρ-associated points x'' and z'' of Q' which satisfy

$$\|x'' - z''\| = \sqrt{\gamma(\rho, \rho)} = \rho_1.$$

Hence, $\{f(x), f(z)\} \subset \{x'', z''\}$, *i. e.*, $\|f(x) - f(z)\| = 0$ or ρ_1, *i. e.*, $\|f(y) - f(z)\| = 0$ or ρ_1 because we assumed $f(x) = f(y)$.

On the other hand, we get $\rho = \|y - z\| = \|f(y) - f(z)\| = 0$ or ρ_1, which is a contradiction. Altogether, we conclude that $\|f(x) - f(y)\| = \rho_1$. □

Lemma 2. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_2 = \sqrt{\gamma(\rho_1, \rho_1)} = (n+1)(2\rho/n)$ is preserved by $f|_{E_2}$.*

Proof. Assume that x and y are points of E_2 with $\|x - y\| = \rho_2$. According to 3) in Benz [4, §2], there exists a ρ_1-set P in E_1 such that x and y are the ρ_1-associated points of P (see also the definition of E_k). Since $f|_{E_1}$ preserves ρ_1 (see Lemma 1), $P' = f(P)$ is also a ρ_1-set in Y.

By 2) in Benz [4, §2], there exist only two distinct ρ_1-associated points x' and y' of P' whose distance is $\|x' - y'\| = \rho_2$. Thus, we get $\{f(x), f(y)\} \subset \{x', y'\}$, *i. e.*, $\|f(x) - f(y)\| = 0$ or ρ_2.

Assume $f(x) = f(y)$. Choose a $z \in E_1$ with $\|x - z\| = \rho_2$ and $\|y - z\| = \rho_1$ (Because of $y \in E_2$ and $\|y - z\| = \rho_1$, we conclude that $z \in E_1$). In view of 3) in Benz [4, §2], there exists a ρ_1-set Q in E_1 such that x and z are the ρ_1-associated points of Q (Because $x \in E_2$ and $\|x - q\| = \rho_1$ for all $q \in Q$, Q is a subset of E_1). Hence, $Q' = f(Q)$ is a ρ_1-set in Y (see Lemma 1).

By 2) in Benz [4, §2], there exist exactly two distinct ρ_1-associated points x'' and z'' of Q' and $\|x'' - z''\| = \rho_2$. Therefore, we have $\|f(x) - f(z)\| = 0$ or ρ_2, *i. e.*, $\|f(y) - f(z)\| = 0$ or ρ_2 because we assumed $f(x) = f(y)$.

Since $y, z \in E_1$, by Lemma 1, we get $\rho_1 = \|y - z\| = \|f(y) - f(z)\| = 0$ or ρ_2, a contradiction. Altogether, we conclude that $\|f(x) - f(y)\| = \rho_2$. □

Lemma 3. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_3 = \sqrt{\gamma(\rho, \rho_1)} = 2\rho/n$ is contractive by $f|_{E_2}$.*

Proof. Assume that x and y are points of E_2 with $\|x - y\| = \rho_3$. By 3) in Benz [4, §2], there exists a ρ_1-set P in E_1 such that x and y are the ρ-associated points of P ($x \in E_2$ and $\|x - p\| = \rho$ for all $p \in P$. Hence, P is a subset of E_1). By Lemma 1, $P' = f(P)$ is also a ρ_1-set in Y.

According to 2) in Benz [4, §2], there exist only two distinct ρ-associated points x' and y' of P' with $\|x' - y'\| = \rho_3$. Hence, we obtain $\|f(x) - f(y)\| = 0$ or ρ_3, *i. e.*, $\|f(x) - f(y)\| \le \rho_3$. □

We are now ready to prove the main theorem of this paper.

Theorem 4. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the restriction $f|_{E_3}$ is an isometry. In particular, if any x, y of E_2 satisfy $x_s \neq y_s$, where x_s and y_s are the X_s-components of x and y, then it holds that $\|f(x) - f(y)\| = \|x - y\|$.*

Proof. According to Lemmas 2 and 3, the distance $2\rho/n$ is contractive and the distance $(n+1)(2\rho/n)$ is extensive (preserved) by $f|_{E_2}$. Hence, by the theorem of Jung and Rassias and by the remark belonging to that theorem, the restriction $f|_{E_3}$ is an isometry.

In view of the theorem of Jung and Rassias again, the second part of this theorem is obviously true. □

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