J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 10, Number 4 (November 2003), Pages 217–223
ISSN 1226-0657

ESSENTIAL SPECTRA OF w-HYPONORMAL OPERATORS

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ABSTRACT. Let $\mathcal{K}$ be the extension Hilbert space of a Hilbert space $\mathcal{H}$ and let ϕ be the faithful $*$-representation of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$. In this paper, we show that if T is an irreducible w-hyponormal operators such that $\ker(T) \subset \ker(T^*)$ and $T^*T - TT^*$is compact, then $\sigma_e(T) = \sigma_e(\phi(T))$.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space. The $*$-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For an operator T in $\mathcal{B}(\mathcal{H})$, we denote the spectrum, the point spectrum, the approximate point spectrum and the essential spectrum by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_e(T)$, respectively. A complex number z is a normal approximate propervalue of T if there exists a sequence $\{x_n\}$ of unit vectors such that $(T-z)x_n \to 0$ and $(T-z)^*x_n \to 0$. The set of all normal approximate propervalue is called the normal approximate spectrum of T and it denote by $\sigma_{na}(T)$.

Aluthge [1] first introduced p-hyponormality for operators; An operator T is said to be *p-hyponormal* for $p \in (0,1]$ if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called *hyponormal* and if $p = \frac{1}{2}$, T is called *semi-hyponormal.* It is well known that a p-hyponormal operator is a q-hyponormal operator for $0 < q \leq p$ by the Löwner-Heinz theorem.

Let $T = U|T|$ be the polar decomposition of T, where U is a partial isometry, $|T|$ is a positive square root of T^*T and $\ker T = \ker |T| = \ker U$. Aluthge [1] introduced the operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, which is called the Aluthge transformation of T.

Received by the editors June 27, 2003 and, in revised form, November 12, 2003.
2000 *Mathematics Subject Classification.* 47B20, 47A10.
Key words and phrases. w-hyponormal, approximate point spectrum, essential spectrum, irreducible operator.
This paper was supported by research fund of Hanyang University, Seoul, Korea, 2002.

Aluthge & Wang [2] first introduced a new operator that an operator T is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$. Evidently, if T is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal.

They proved that if an operator T is p-hyponormal, then it is w-hyponormal and also show the following results:

Theorem 1.1 (Aluthge & Wang [4]).

(1) *An operator T is w-hyponormal if and only if*

$$|T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}} \quad \textit{and} \quad |T^*| \leq (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}.$$

(2) *If T is a w-hyponormal operator, then $\sigma_{ap}(T) - \{0\} = \sigma_{na}(T) - \{0\}$.*

Fujii, Jung, S. H. Lee, M. Y. Lee & Nakamoto [11] introduced a new class $A(p,q)$ of operators that for $p, q > 0$, an operator T belongs to $A(p,q)$ if it satisfies an operator inequality

$$\left(|T^*|^q|T|^{2p}|T^*|^q\right)^{\frac{q}{p+q}} \geq |T^*|^{2q}.$$

Recently, Ito & Yamazaki [12] introdued a new class $wA(p,q)$ of operators that for $p, q > 0$, an operator T belongs to $wA(p,q)$ if it satisfies an operator inequalities

$$\left(|T^*|^q|T|^{2p}|T^*|^q\right)^{\frac{q}{p+q}} \geq |T^*|^{2q} \quad \textit{and} \quad |T|^{2p} \geq \left(|T|^p|T^*|^{2q}|T|^p\right)^{\frac{p}{p+q}}.$$

In Ito & Yamazaki [12], they obtained the following results:

Theorem 1.2 (Ito & Yamazaki [12]). *For each $p > 0$ and $q > 0$, the following assertions hold:*

(1) *Class $A(p,q)$ coincides with class $wA(p,q)$.*
(2) *Class $A(\frac{1}{2},\frac{1}{2})$ coincides with the class of w-hyponormal operators (i.e., class $wA(\frac{1}{2},\frac{1}{2})$).*

An operator T is said to be reducible if it has a nontrivial reducing subspace. If an operator is not reducible, then it is called irreducible.

Cha [6] constructed an extension $\mathcal{K}$ of $\mathcal{H}$ by means of all weakly convergent sequences in $\mathcal{H}$ and the Banach Limit, and obtained the faithful $*$-representation ϕ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$.

In this paper, using the faithful $*$-representation ϕ, for an irreducible w-hyponormal operator T with $\ker(T) \subset \ker(T^*)$, we investigate the relation between the essential spectrum of T and the essential spectrum of $\phi(T)$.

2. The main Theorem

Let $C^*(T)$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by a single operator T and identity. By a character on a C^*-algebra we mean a multiplicative linear functional. If $\mathcal{A}$ is a C^*-algebra with identity, then its commutator ideal is the closed ideal generated by the commutator $ab - ba$ for $a, b \in \mathcal{A}$.

Bunce [5] established a kind of the reciprocity among the character of single generated C^*-algebra and the approximate spectra of the generators and he proved the following theorem:

Theorem 2.1 (Bunce [5]). *If T is a hyponormal operator, then for all $\lambda \in \sigma_{ap}(T)$ there is a character ψ on the C^*-algebra $C^*(T)$ such that $\psi(T) = \lambda$.*

Enomoto, Fujii & Tamaki [10] was generalized Bunce's result as following:

Theorem 2.2 (Enomoto, Fujii & Tamaki [10]). *A complex number $\lambda \in \sigma_{na}(T)$ if and only if there is a character ψ of $C^*(T)$ such that $\psi(T) = \lambda$.*

Let $C^*(T_i : i \in \Gamma)$ be the C^*-algebra generated by $\{T_i : i \in \Gamma\}$ and the identity operator, and let $\mathcal{I}$ be the commutator ideal of $C^*(T_i : i \in \Gamma)$.

S. G. Lee [13] obtained that the quotient algebra $C^*(T_i : i \in \Gamma)/\mathcal{I}$ is isometrically $*$-isomorphic to $C(\sigma_n(T_i : i \in \Gamma))$, where $\sigma_n(T_i : i \in \Gamma)$ is the joint normal spectrum of $\{T_i : i \in \Gamma\}$.

By Theorem 1.1 and Theorem 2.2, we have the following result:

Corollary 2.3. *Let T be a w-hyponormal operator with* $\ker(T) \subset \ker(T^*)$. *Then $\lambda \in \sigma_{ap}(T)$ if and only if there is a character ψ of $C^*(T)$ such that $\psi(T) = \lambda$.*

If $\Phi_{\mathcal{A}}$ is the set of all character on $\mathcal{A}$ and M is the commutator ideal of $\mathcal{A}$, then $M = \bigcap\{h^{-1}(0) : h \in \Phi_{\mathcal{A}}\}$ and $\Phi_{\mathcal{A}}$ is the maximal ideal space of $\mathcal{A}/M$. With this statement, we have $\mathcal{A}/M \cong C(\Phi_{\mathcal{A}})$ under the Gel'fand transform, $a + M \to \hat{a}$, where $\hat{a}(h) = h(a)$ for a in $\mathcal{A}$ and h in $\Phi_{\mathcal{A}}$ (Conway [8, 9]).

Lemma 2.4. *If an operator T is w-hyponormal with* $\ker(T) \subset \ker(T^*)$, *there is an isometric $*$-isomorphism of $C^*(T)/M$ onto $C(\sigma_{ap}(T))$, where $A + M$ is mapped to the function z.*

Proof. Let $\tau : \Phi_{C^*(T)} \to \sigma_{ap}(T)$ be defined by $\tau(\psi) = \psi(T)$. By Corollary 2.3 this map is surjective. If $\psi(T) = \psi'(T)$ for $\psi, \psi' \in \Phi_{C^*(T)}$, then $\psi = \psi'$, since T is generator of $C^*(T)$, and ψ, ψ' are continuous on $C^*(T)$. So τ is injective. It is also easy to see that τ is continuous. Since $\Phi_{C^*(T)}$ is compact, τ is a homeomorphism. Thus the map $\tau^{\#} : C(\sigma_{ap}(T)) \to C(\Phi_{C^*(T)})$ defined by $\tau^{\#}(f) = f \circ \tau$ is an isometric $*$-isomorphism. We define a map $\rho : C(\sigma_{ap}(T)) \to C^*(T)/M$ so that the following diagram commutes:

$$\begin{array}{ccc} C^*(T)/M & \xrightarrow{\quad\gamma\quad} & C(\Phi_{C^*(T)}) \\ \nwarrow{\scriptstyle\rho} & & \nearrow{\scriptstyle\tau^{\#}} \\ & C(\sigma_{ap}(T)) & \end{array}$$

where the Gel'fand transform $\gamma : C^*(T)/M \to C(\Phi_{C^*(T)})$ is an isometric $*$-isomorphism. Then ρ is clearly an isometric $*$-isomorphism. □

Cha [6] introduced an extension $\mathcal{K}$ of $\mathcal{H}$ by means of all weakly convergent sequences in $\mathcal{H}$ and the Banach Limit, and obtained the faithful $*$-representation ϕ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$.

In order to show our results, we use the following propositions.

Proposition 2.5 (Cha [6, 7]). *There exists a faithful $*$-representation ϕ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$ with the following properties:*

(1) $\|\phi(T)\| = \|T\|$.
(2) $\sigma(T) = \sigma(\phi(T))$.
(3) $\sigma_{ap}(T) = \sigma_p(\phi(T))$.
(4) *If T is a compact operator on $\mathcal{H}$, then so is $\phi(T)$ on $\mathcal{K}$.*
(5) *If T is an irreducible operator on $\mathcal{H}$, then so is $\phi(T)$ on $\mathcal{K}$.*

Remark. The Proposition 2.5 (5) does not mean a representation of a C^*-algebra is irreducible. It implies the concept of a simple irreducible operators.

Proposition 2.6 (Cha [7]). *We have the following properties.*

(1) *The C^*-algebra $C^*(T)$ is isometrically $*$-isomorphic to the C^*-algebra $C^*(\phi(T))$.*
(2) *If M is the maximal ideal of $C^*(T)$, then $\phi(M)$ is the maximal ideal of $C^*(\phi(T))$.*
(3) *Let $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ be the maximal ideal space of $C^*(T)$ and $C^*(\phi(T))$, respectively. Then $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ are isometrically isomorphic.*

Proposition 2.7 (Cha [7]). *We have the following properties.*

(1) $M = \bigcap\{f^{-1}(0) : f \in \Phi_{C^*(T)}\} \cong N = \bigcap\{h^{-1}(0) : h \in \Phi_{C^*(\phi(T))}\}$.
(2) $C^*(T)/M \cong C^*(\phi(T))/N$.

Related to above propositions, we obtained the following results for w-hyponormal operators.

To show this property, we need the following proposition:

Proposition 2.8. *If an operator T is w-hyponormal, then so is $\phi(T)$.*

Proof. Since operators in $A(\frac{1}{2},\frac{1}{2})$ is w-hyponormal, we need only to show that

$$\left(|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2}\right)^{1/2} \geq |\phi(T)^*|.$$

It is easily check that $|\phi(T)| = \phi(|T|)$ and ϕ preserves the positive property. Thus we have

$$\begin{aligned} |\phi(T)^*| &= \phi(|T^*|) \\ &\leq \phi((|T^*|^{1/2}|T||T^*|^{1/2})^{1/2}) \\ &= (|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2})^{1/2}. \end{aligned}$$

Therefore,

$$\left(|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2}\right)^{1/2} \geq |\phi(T)^*|.$$

Thus, $\phi(T)$ is w-hyponormal. □

With the notation of Proposition 2.5, Proposition 2.7 and Lemma 2.4, we have the following:

Theorem 2.9. *If T is a w-hyponormal operator with* $\ker(T) \subset \ker(T^*)$, *then*

$$C^*(T)/M \cong C^*(\phi(T))/N \cong C(\sigma_p(\phi(T))).$$

We need the following proposition in order to give proofs of Theorem 2.11 and Corollary 2.12.

Proposition 2.10 (Conway [9]). *If T is an irreducible operator such that $T^*T - TT^*$ is compact, then the commutator ideal M of $C^*(T)$ is $K(\mathcal{H})$, where $K(\mathcal{H})$ is the ideal of all compact operators on $\mathcal{H}$.*

We have the results for irreducible w-hyponormal operators.

Theorem 2.11. *If T is an irreducible w-hyponormal operators such that* $\ker(T) \subset \ker(T^*)$ *and $T^*T - TT^*$ is compact, then*

$$\sigma_{ap}(T) = \sigma_e(T) \text{ and } \sigma_p(\phi(T)) = \sigma_e(\phi(T)).$$

Proof. The fact that $\sigma_{ap}(T) = \sigma_e(T)$ follows immediately from Proposition 2.10 and Lemma 2.4. The second assertion is clear from Proposition 2.10 and Proposition 2.5. □

It is easy to see that if T is a Fredholm operator on $\mathcal{H}$, then so is $\phi(T)$ on $\mathcal{K}$, and so $\sigma_e(\phi(T)) \subset \sigma_e(T)$ for any operator T.

Corollary 2.12. *If T is an irreducible w-hyponormal operators such that* $\ker(T) \subset \ker(T^*)$ *and $T^*T - TT^*$ is compact, then $\sigma_e(T) = \sigma_e(\phi(T))$.*

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