J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 11, Number 3 (August 2004), Pages 189–196

ISSN 1226-0657

EVALUATIONS OF THE IMPROPER INTEGRALS $\int_0^\infty [\sin^{2m}(\alpha x)]/(x^{2n})dx$ AND $\int_0^\infty [\sin^{2m+1}(\alpha x)]/(x^{2n+1})dx$

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ABSTRACT. In this article, using the L'Hospital rule, mathematical induction, the trigonometric power formulae and integration by parts, some integral formulae for the improper integrals $\int_0^\infty [\sin^{2m}(\alpha x)]/(x^{2n})dx$ and $\int_0^\infty [\sin^{2m+1}(\alpha x)]/(x^{2n+1})dx$ are established, where $m \geq n$ are all positive integers and $\alpha \neq 0$.

1. INTRODUCTION

The following improper integral is well-known and is synonymous with names of Laplace and Dirichlet

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{1}$$

In fact, in 1781, it was first obtained using the residue method by Euler. It can be found in standard textbooks for undergraduate students, for examples Rudin [12, pp. 226–227] and Staff Room of Higher Mathematics at Xi'an Jiaotong University [16, pp. 168–170].

Depending on the partial fraction decomposition

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{i=1}^{\infty} (-1)^i \left(\frac{1}{t - n\pi} + \frac{1}{t + n\pi} \right), \tag{2}$$

an elegant calculation of formula (1) is provided in Klambauer [6, pp. 436–437] and Klambauer [7, pp. 382–384], due to the noted geometrician N. I. Lobatscheuski. Another polished proof of identity (1) is given in Klambauer [7, pp. 381–382].

Received by the editors November 27, 2003 and, in revised form, June 30, 2004.

2000 *Mathematics Subject Classification.* 26A42, 26A06, 30E20.

Key words and phrases. evaluation, improper integral, integral formula, inequality, integration by parts, L'Hospital rule, mathematical induction.

The authors were supported in part by NSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), SF for Pure Research of Natural Science of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, China.

As exercises in She & Liu [13, p. 53, p. 147, and p. 335] and Spiegel [15, p. 495], using the Laplace transform, the Parseval identities of sine and cosine Fourier transforms and the residue theorem, the following formulae are requested to compute:

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^4 t}{t^2} dt = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^3 t}{t^3} dt = \frac{3\pi}{8}. \tag{3}$$

In Tan [17, pp. 74–75 and p. 84], using the Mellin transform and by approaches in theory of Fourier analysis or theory of residues, the following formulae are obtained:

$$\int_0^\infty \cos(tx) x^z \frac{dx}{x} = \Gamma(z) t^{-z} \cos\frac{\pi z}{2}, \quad \operatorname{Re}(z) > 0, \quad t > 0; \tag{4}$$

$$\int_0^\infty \sin(tx) x^z \frac{dx}{x} = \Gamma(z) t^{-z} \sin\frac{\pi z}{2}, \quad \operatorname{Re}(z) > -1, \quad t > 0; \tag{5}$$

$$\int_0^\infty \frac{\sin x}{x^z} dx = \Gamma(1-z) \cos\frac{\pi z}{2} = \frac{\pi}{2\Gamma(z)\sin\frac{\pi z}{2}}. \tag{6}$$

Especially, taking $t = 1$ and $z \to 0$ in (5) or taking $z = 1$ in (6) produces (1).

The following generalisation of formula (1) can be found in Erdélyi, Magnus, Oberhettinger & Tricomi [2, 3] and Gradshteyn & Ryzhik [4, p. 458, No. 3.836.5]:

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx) dx = n(2^{n-1} n!) \sum_{k=0}^{[r]} (-1)^k \binom{n}{k} (n - b - 2k)^{n-1}, \tag{7}$$

where $0 \le b < n$, $n \ge 1$, $r = \frac{n-b}{2}$, and $[r]$ is the largest integer contained in r.

In Sofo [14], some general results related to formulae (1) and (7) were obtained.

In Berger [1, p. 663] and Kuang [8, p. 606], the following inequality is given:

$$\int_{-\infty}^\infty \left|\frac{\sin t}{t}\right|^p dt \le \pi\sqrt{\frac{2}{p}}, \quad p \ge 2; \tag{8}$$

Equality is valid only if $p = 2$.

The integral (1) and other integral formulae stated above are useful and arising in research of damping vibration and other science or engineering. This was mentioned in Staff Room of Higher Mathematics at Xi'an Jiaotong University [16, p. 170].

Recently, Luo & Guo [9] and Luo & Qi [10, 11] obtained the following

Theorem A (Luo & Guo [9]). *For $k \in \mathbb{N}$ and $\alpha \neq 0$, we have*

$$\int_0^\infty \left(\frac{\sin(\alpha x)}{x}\right)^{2k+1} dx = \frac{\operatorname{sgn}\alpha \sum_{i=0}^k (-1)^i (2k-2i+1)^{2k} C_{2k+1}^i}{4^k (2k)!} \cdot \alpha^{2k} \cdot \frac{\pi}{2}, \tag{9}$$

$$\int_0^\infty \left(\frac{\sin(\alpha x)}{x}\right)^{2k} dx = \frac{\operatorname{sgn}\alpha \sum_{i=0}^{k-1}(-1)^i(k-i)^{2k-1}C_{2k}^i}{(2k-1)!}\cdot\alpha^{2k-1}\cdot\frac{\pi}{2}. \tag{10}$$

If taking $k=1$ in (9), the formula (1) follows.

In this article, using the L'Hospital rule, mathematical induction, trigonometric power formulae and integration by parts, we will establish integral formulae of the improper integrals

$$\int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}}dx \quad \text{and} \quad \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}}dx,$$

where $m \geq n$ are all positive integers and $\alpha \neq 0$. The following theorem holds.

Theorem 1. *Let $m, n \in \mathbb{N}$, $m \geq n$, and $\alpha \neq 0$. Then*

$$\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}}dx = \frac{(-1)^{m+n}\operatorname{sgn}\alpha \sum_{i=0}^{m}(-1)^i(2m-2i+1)^{2n}C_{2m+1}^i}{4^m(2n)!}\cdot\alpha^{2n}\cdot\frac{\pi}{2}, \tag{11}$$

$$\int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}}dx = \frac{(-1)^{m+n}\operatorname{sgn}\alpha \sum_{i=0}^{m-1}(-1)^i(m-i)^{2n-1}C_{2m}^i}{4^{m-n}(2n-1)!}\cdot\alpha^{2n-1}\cdot\frac{\pi}{2}. \tag{12}$$

As a direct consequence of Theorem 1, the following integral formula holds.

Corollary 1. *Let m be a nonnegative integer, then we have*

$$\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x}dx = \operatorname{sgn}\alpha\cdot\frac{(2m)!}{4^m(m!)^2}\cdot\frac{\pi}{2}. \tag{13}$$

It is obvious that Theorem 1 generalizes the formula (1), Theorem A and other results above.

2. Lemmas

The following trigonometric power formulae are the basis and key of our proof for Theorem 1.

Lemma 1. (Group of compilation [5, p. 41 and p. 280] and Weisstein [18]) For $\alpha > 0$ and $k \in \mathbb{N}$, we have

$$\int_0^\infty \frac{\sin(\alpha x)}{x}dx = \frac{\pi}{2}, \tag{14}$$

$$\sin^{2k+1} x = \frac{1}{2^{2k}} \sum_{i=0}^{k} (-1)^{k+i} C_{2k+1}^{i} \sin[(2k-2i+1)x], \tag{15}$$

$$\sin^{2k} x = \frac{1}{2^{2k-1}} \left[\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^{i} \cos[2(k-i)x] + \frac{1}{2} C_{2k}^{k} \right], \tag{16}$$

where $C_n^k = \frac{n!}{(n-k)!k!}$.

The following three combinatorial identities can be regarded as by-products, enabling us to employ the L'Hospital rule in the proof of Theorem 1. They can also be found in Luo & Guo [9].

Lemma 2. *For $1 \le m \le k$ and $k \in \mathbb{N}$, we have*

$$\sum_{i=0}^{k} (-1)^{i} (2k-2i+1)^{2m-1} C_{2k+1}^{i} = 0, \tag{17}$$

$$\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^{i} + \frac{1}{2} C_{2k}^{k} = 0. \tag{18}$$

For $1 \le \ell \le k-1$ and $2 \le k \in \mathbb{N}$, we have

$$\sum_{i=0}^{k-1} (-1)^{i} (k-i)^{2\ell} C_{2k}^{i} = 0. \tag{19}$$

Proof. By the trigonometric power formula (15), it is easy to see that

$$\lim_{x \to 0} \frac{\sum_{i=0}^{k} (-1)^{k+i} C_{2k+1}^{i} \sin[(2k-2i+1)x]}{x^{2k}} = 2^{2k} \lim_{x \to 0} \frac{\sin^{2k+1} x}{x^{2k}} = 0, \tag{20}$$

thus

$$\sum_{i=0}^{k} (-1)^{i} C_{2k+1}^{i} \sin[(2k-2i+1)x] = o(x^{2k}) \quad \text{as } x \to 0, \tag{21}$$

then, for $0 \le j \le 2k$,

$$\left(\sum_{i=0}^{k} (-1)^{i} C_{2k+1}^{i} \sin[(2k-2i+1)x] \right)^{(j)} = o(x^{2k-j}) \quad \text{as } x \to 0, \tag{22}$$

therefore, for any natural number $1 \le m \le k$, we have

$$\begin{aligned} 0 &= \lim_{x \to 0} \left(\sum_{i=0}^{k} (-1)^{i} C_{2k+1}^{i} \sin[(2k-2i+1)x] \right)^{(2m-1)} \\ &= \lim_{x \to 0} \left((-1)^{m-1} \sum_{i=0}^{k} (-1)^{i} (2k-2i+1)^{2m-1} C_{2k+1}^{i} \cos[(2k-2i+1)x] \right) \end{aligned} \tag{23}$$

$$= (-1)^{m-1} \sum_{i=0}^{k} (-1)^i (2k-2i+1)^{2m-1} C_{2k+1}^i.$$

Identity (17) follows.

By the trigonometric power formula (16), it is not difficult to obtain

$$\lim_{x \to 0} \frac{\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k-i)x] + \frac{1}{2} C_{2k}^k}{x^{2k-1}} = 2^{2k-1} \lim_{x \to 0} \frac{\sin^{2k} x}{x^{2k-1}} = 0, \tag{24}$$

hence

$$\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k-i)x] + \frac{1}{2} C_{2k}^k = o(x^{2k-1}) \quad \text{as } x \to 0, \tag{25}$$

consequently

$$\begin{aligned} 0 &= \lim_{x \to 0} \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k-i)x] + \frac{1}{2} C_{2k}^k \\ &= \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i + \frac{1}{2} C_{2k}^k \end{aligned} \tag{26}$$

and, for $1 \le j \le 2k-1$,

$$\left(\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k-i)x] + \frac{1}{2} C_{2k}^k \right)^{(j)} = o(x^{2k-j-1}) \quad \text{as } x \to 0, \tag{27}$$

then, for any positive integer ℓ such that $1 \le \ell \le k-1$, we have

$$\begin{aligned} 0 &= \lim_{x \to 0} \left(\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k-i)x] + \frac{1}{2} C_{2k}^k \right)^{(2\ell)} \\ &= \lim_{x \to 0} \left((-1)^\ell \sum_{i=0}^{k} (-1)^i (2k-2i+1)^{2\ell} C_{2k+1}^i \cos[(2k-2i+1)x] \right) \\ &= (-1)^\ell \sum_{i=0}^{k} (-1)^i (2k-2i+1)^{2m-1} C_{2k+1}^i. \end{aligned} \tag{28}$$

Identities (18) and (19) follow. The proof is complete. □

3. Proof of Theorem 1

Let $t = \alpha x$, then, by straightforward computation, we have

$$\int_0^\infty \frac{\sin^r(\alpha x)}{x^s} dx = \alpha^{s-1} \operatorname{sgn} \alpha \int_0^\infty \frac{\sin^r t}{t^s} dt. \tag{29}$$

From Lemma 1 and Lemma 2, using the L'Hospital rule and integration by parts yields

$$
\begin{aligned}
&\int_0^\infty \frac{\sin^{2m+1} x}{x^{2n+1}}dx \\
&= \frac{1}{2^{2m}}\int_0^\infty \frac{\sum_{i=0}^m (-1)^{m+i} C_{2m+1}^i \sin[(2m-2i+1)x]}{x^{2n+1}}dx \\
&= \frac{(-1)^{2j-1}(2n-j)!}{2^{2m}(2n)!} \\
&\quad \times \left\{ \frac{\sum_{i=0}^m (-1)^{m+i}(2m-2i+1)^{j-1} C_{2m+1}^i \sin\left[(2m-2i+1)x+\frac{(j-1)\pi}{2}\right]}{x^{2n-j+1}}\Bigg|_0^\infty \right. \\
&\quad \left. - \int_0^\infty \frac{\sum_{i=0}^m (-1)^{m+i}(2m-2i+1)^{j} C_{2m+1}^i \sin\left[(2m-2i+1)x+\frac{j\pi}{2}\right]}{x^{2n-j+1}}dx \right\} \\
&= \frac{(-1)^n}{2^{2m}(2n)!}\int_0^\infty \frac{\sum_{i=0}^m (-1)^{m+i}(2m-2i+1)^{2n} C_{2m+1}^i \sin[(2m-2i+1)x]}{x}dx \\
&= \frac{(-1)^{m+n}}{2^{2m}(2n)!}\sum_{i=0}^m (-1)^i (2m-2i+1)^{2n} C_{2m+1}^i \int_0^\infty \frac{\sin[(2m-2i+1)x]}{x}dx \\
&= \frac{(-1)^{m+n}\sum_{i=0}^m (-1)^i (2m-2i+1)^{2n} C_{2m+1}^i}{2^{2m}(2n)!}\cdot\frac{\pi}{2},
\end{aligned}
$$

where $1 \le j \le 2n$. Combining with (29), the formula (11) follows.

By formula (16), using the L'Hospital rule, from Lemma 2, integration by parts gives us

$$
\begin{aligned}
&\int_0^\infty \frac{\sin^{2m} x}{x}^{2n} dx \\
&= \frac{1}{2^{2m-1}}\int_0^\infty \frac{\sum_{i=0}^{m-1}(-1)^{k+i} C_{2m}^i \cos[2(m-i)x] + \frac{1}{2}C_{2m}^m}{x^{2n}}dx \\
&= -\frac{1}{2^{2m-1}}\cdot\frac{1}{2n-1}\int_0^\infty \left[\sum_{i=0}^{m-1}(-1)^{m+i} C_{2m}^i \cos[2(m-i)x] + \frac{1}{2}C_{2m}^m\right] d\left(\frac{1}{x^{2n-1}}\right) \\
&= -\frac{1}{(2m-1)\cdot 2^{2n-1}}\left[\frac{\sum_{i=0}^{m-1}(-1)^{m+i} C_{2m}^i \cos[2(m-i)x] + \frac{1}{2}C_{2m}^m}{x^{2n-1}}\Bigg|_0^\infty \right. \\
&\qquad \left. + \int_0^\infty \frac{\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)] C_{2m}^i \sin[2(m-i)x]}{x^{2n-1}}dx \right] \\
&= -\frac{1}{(2m-1)\cdot 2^{2n-1}}\int_0^\infty \frac{\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)] C_{2m}^i \sin[2(m-i)x]}{x^{2n-1}}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(2n-j-2)!}{2^{2m-1}(2n-1)!}\left\{\frac{\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)]^{j}C_{2m}^{i}\sin\left[2(m-i)x\right]+\frac{(j-1)\pi}{2}]}{x^{2n-j-1}}\Bigg|_{0}^{\infty}\right.\\
&\qquad\left.-\int_{0}^{\infty}\frac{\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)]^{j+1}C_{2m}^{i}\sin\left[2(m-i)x\right]+\frac{j\pi}{2}]}{x^{2n-j-1}}dx\right\}\\
&= \frac{(-1)^{n}}{2^{2m-1}(2n-1)!}\int_{0}^{\infty}\frac{\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)]^{2n-1}C_{2m}^{i}\sin[2(m-i)x]}{x}dx\\
&= \frac{(-1)^{n}}{2^{2m-1}(2n-1)!}\sum_{i=0}^{m-1}(-1)^{m+i}[2(m-i)]^{2n-1}C_{2m}^{i}\int_{0}^{\infty}\frac{\sin[2(m-i)x]}{x}dx\\
&= \frac{(-1)^{m+n}\sum_{i=0}^{m-1}(-1)^{i}(m-i)^{2n-1}C_{2m}^{i}}{2^{2(m-n)}(2n-1)!}\cdot\frac{\pi}{2},
\end{aligned}
$$

where $1 \le j \le 2(n-1)$. On combining with (29), the second formula (12) follows.

The proof of Theorem 1 is thus complete.

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