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ON THE LEFT INVERSIVE SEMIRING CONGRUENCES ON ADDITIVE REGULAR SEMIRINGS

M. K. SEN AND A. K. BHUNIYA

ABSTRACT. An additive regular semiring S is left inversive if the set $E^+(S)$ of all additive idempotents is left regular. The set $\mathcal{LC}(S)$ of all left inversive semiring congruences on an additive regular semiring S is a lattice. The relations θ and k (*resp.*), induced by tr and ker (*resp.*), are congruences on $\mathcal{LC}(S)$ and each θ-class $\rho\theta$ (*resp.* each k-class ρk) is a complete modular sublattice with $\rho_{\min}$ and $\rho_{\max}$ (*resp.* with $\rho^{\min}$ and $\rho^{\max}$), as the least and greatest elements. $\rho_{\min}$, $\rho_{\max}$, $\rho^{\min}$ and $\rho^{\max}$, in particular $\epsilon_{\max}$, the maximum additive idempotent separating congruence has been characterized explicitly. A semiring is quasi-inversive if and only if it is a subdirect product of a left inversive and a right inversive semiring.

1. INTRODUCTION

A natural generalization of the class of inverse semigroups is the class of left inverse semigroups: A semigroup S is left inverse if it is regular and its set of idempotents E is a left regular band, that is, $efe = ef$ for all $e, f \in E$. In this paper we introduce the notion of left inversive semirings as a generalization of inversive semirings. Our objective is to study the lattice $\mathcal{LC}(S)$ of all left inversive semiring congruences on a regular semiring S. It was recognized by Feigenbaum [3] that every congruence ρ on a regular semigroup is uniquely determined by its kernel, ker ρ, equal to the set of elements which are ρ-equivalent to idempotents and its trace, tr ρ, equal to the restriction of ρ to the set E of idempotents on S. The importance of trace was realized earlier by Reilly & Scheiblich [14], where they defined a congruence θ, induced by tr on the lattice of all congruences on an inverse semigroup and gave expressions for the least element $\rho_{\min}$ and greatest element

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Further correspondences should be made with A. K. Bhuniya.

$\rho_{\max}$ in $\rho\theta$. Feigenbaum [2] first extended these results to an orthodox semigroup and later [3] to regular semigroups. Green [6] characterized the k-equivalent classes, where k is the relation on the lattice of all congruences on an inverse semigroup induced by kernel. Petrich & Reilly [13] determined the least element in a k-class and Pastijn & Petrich [12] generalizes these results to regular semigroups.

The unqualified success of these relations θ and k to study the lattice of all congruences on regular semigroups including its diverse ramifications gave a certain hope that this may also turn out to be the case for the lattice of all left inversive semiring congruences on additive regular semirings. Sen, Ghosh & Mukhopadhyay [15] studied the congruences on inversive semirings with the additive reduct commutative and Maity [10] improved this to the inversive semirings with the set of all additive idempotents a bisemilattice. The main aspect of this paper is the expressions for $\rho_{\min}$, $\rho_{\max}$, $\rho^{\min}$ and $\rho^{\max}$, which gives several new characterization of $\rho_{\max}$. This also helps us to characterize the maximum additive idempotent separating congruence. In the last section we determine the class of semirings which are subdirect product of a left inversive semiring and a right inversive semiring.

2. Preliminaries

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that the redacts $(S, +)$ and $(S, \cdot)$ are semigroups and in which the two distributive laws

$$x(y+z) = xy + xz, \quad (y+z)x = yz + zx$$

are satisfied. A semiring S is *additive regular* if its additive redact is regular, that is, if for all $a \in S$ there is $x \in S$ such that $a = a + x + a$. Let S be an additive regular semiring. Then $x \in S$ is called an inverse of $a \in S$ if $a = a + x + a$ and $x = x + a + x$. The set of all additive inverses of $a \in S$ is denoted by $V^+(a)$.

Let $\mathcal{R}^+$ be the *right Green's relation* on $(S, +)$ where S is an additive regular semiring. Then for $a, b \in S$ we have

$$a \; \mathcal{R}^+ \; b \text{ if and only if } a + a' = b + b' \text{ for some } a' \in V^+(a), \; b' \in V^+(b).$$

We denote the set of all additive idempotents of an additive regular semiring S by E^+ or sometimes by $E^+(S)$.

Recall that a regular semigroup $(S, +)$ is called a *left inverse semigroup* if its set of idempotents E is a left regular band, *i.e.*, if E satisfies the identity $e + f + e = e + f$.

Similarly we define inversive semiring as follows.

Definition 2.1. A *left inversive semiring* is an additive regular semiring $(S,+,\cdot)$ such that $(S,+)$ is a left inverse semigroup, that is, $e+f = e+f+e$ for all $e,f \in E^+(S)$.

Note that a left inversive semiring with the commutative additive reduct is an inversive semiring. In this paper the additive reduct of the semiring is not assumed to be commutative.

The following two lemmas are immediate.

Lemma 2.2 (Venkatesan [16, 17]). *Let S be an additive regular semiring. Then the following are equivalent:*

(i) *S is left inversive.*
(ii) *$a+a' = a+a''$ for all $a \in S$ and for all a', $a'' \in V^+(a)$.*
(iii) *$a+e+a' = a+e+a''$ for all $e \in E^+$, for all $a \in S$ and for all $a', a'' \in V^+(a)$.*
(iv) *$e\ \mathcal{R}^+\ f$ implies $e = f$ for all $e, f \in E^+$.*

Lemma 2.3 (Edwards [1, §1.3]). *Let S be a left inversive semiring. Then*

$b'+a' \in V^+(a+b)$ for all $a,b \in S$, for all $a' \in V^+(a)$ and for all $b' \in V^+(b)$.

In particular, the direct product of a right zero semiring, that is, a semiring with the additive reduct a right zero semigroup and an inversive semiring is a left inversive semiring. Since any subsemiring of a left inversive semiring is again left inversive so the subdirect products of a right zero semiring and an inversive semiring is also left inversive.

Example 2.4. Consider $(\mathbb{N},+,\cdot)$, where $\mathbb{N}$ is the set of all natural numbers and $\cdot$ is the usual multiplication in $\mathbb{N}$ and $m+n=m$ for all m, $n \in \mathbb{N}$. Suppose S be any inversive semiring. Let $R = \mathbb{N} \times S$. Then $E^+(R) = \mathbb{N} \times E^+(S)$. If $u, v \in E^+(R)$ then $u = (m,e), v = (n,f)$ for some $m,n \in \mathbb{N}$ and $e,f \in E^+(S)$. So

$$u+v+u = (m+n+m, e+f+e) = (m, e+f) = (m,e)+(n,f) = u+v.$$

Therefore R is a left inversive semiring. Note that R is neither a right zero semiring nor an inversive semiring.

Lemma 2.5. *Let S be a left inversive semiring and r, a, $s \in S$. Then*

$$ra's \in V^+(ras)$$

for all $a' \in V^+(a)$.

This implies the following results trivially.

Lemma 2.6. *Let S be a left inversive semiring and $a \in S$. Then*

$$ras + e + ra's \in E^+$$

for all $a' \in V^+(a)$, for all $r, s \in S^1$ and for all $e \in E^+$.

Lemma 2.7. *Let S be a left inversive semiring and $a \in S$. Then*

$$ras + e + ra's = ras + e + ra''s$$

for all $a', a'' \in V^+(a)$, for all $r, s \in S^1$ and for all $e \in E^+$.

A congruence ρ on an additive regular semiring S is said to be a *left inversive semiring congruence* if S/ρ is a left inversive semiring.

Let ρ be a congruence on an additive regular semiring S. Then we define:

$$\operatorname{tr}\rho = \rho \cap (E^+ \times E^+) \quad \text{and} \quad \ker\rho = \{a \in S \mid a\rho \in E^+(S/\rho)\}.$$

Since $(S, +)$ is regular, by Lallement's Lemma [8],

$$\ker\rho = \{a \in S \mid a \;\rho\; e \text{ for some } e \in E^+\}$$

Also recall that on a regular semigroup $(S, +)$, Nambooripad [11] defined a *sandwich set* $S(e, f)$ for $e, f \in E^+$ by:

$$S(e, f) = \{g \in E^+ \mid g + e = g = f + g, e + g + f = e + f\}$$

Nambooripad [11] has shown that if $a, b \in S$; $a' \in V^+(a)$; $b' \in V^+(b)$ then

$$S(a' + a, b + b') \neq \varnothing,$$

and if $g \in S(a' + a, b + b')$ then $b' + g + a' \in V^+(a + b)$.

We use, whenever possible, the notations of Golan [4] and Howie [7]. Throughout the rest of this article a semiring means an additive regular semiring.

3. Lattice of Left Inversive Semiring Congruences

Let S be a semiring. Then the set of all left inversive semiring congruences on S is a lattice under inclusion, which let us denote by $\mathcal{LC}(S)$.

Reilly & Scheiblich [14] defined a relation θ on the lattice of congruences on inverse semigroups by:

$\rho\;\theta\;\xi$ if and only if ρ, ξ induce the same partition of the idempotents of S.

There they proved that θ is a complete congruence and each θ-class is a complete modular lattice. Let us now define a relation θ on $\mathcal{LC}(S)$ by:

$$\rho\,\theta\,\xi \text{ if and only if } \operatorname{tr}\rho = \operatorname{tr}\xi.$$

We will show that each θ-class $\rho\theta$ contains a greatest and a least element in $\mathcal{LC}(S)$. For $\rho \in \mathcal{LC}(S)$, we define a relation $\rho_{\max}$ on S by:

$$\rho_{\max} = \{(a,b) \in S \times S \mid \text{ there exist } a' \in V^+(a),\, b' \in V^+(b) \text{ such that } (ras + e + ra's)\ \rho\ (rbs + e + rb's) \text{ for all } e \in E^+ \text{ and for all } r,s \in S^1\}.$$

Proposition 3.1. *Let S be a semiring and ρ be a left inversive semiring congruence on S. Then $\rho_{\max}$ is the greatest element in its θ-class $\rho\theta$.*

Proof. It is obvious that $\rho_{\max}$ is reflexive and symmetric. Now suppose $a,b,c \in S$ are such that $a\,\rho_{\max}\,b$ and $b\,\rho_{\max}\,c$. Let $r,s \in S^1$ and $e \in E^+$. Then there are $a' \in V^+(a)$; $b', b'' \in V^+(b)$ and $c' \in V^+(c)$ such that

$$(ras + e + ra's)\ \rho\ (rbs + e + rb's) \text{ and } (rbs + e + rb''s)\ \rho\ (rcs + e + rc's).$$

Since ρ is left inversive semiring congruence, by Lemma 2.7,

$$(rbs + e + rb's)\ \rho\ (rbs + e + rb''s).$$

Hence the transitivity of ρ implies that $(ras + e + ra's)\ \rho\ (rcs + e + rc's)$. Thus $a\,\rho_{\max}\,c$, that is, $\rho_{\max}$ is transitive.

To prove that $\rho_{\max}$ is a congruence suppose $a,b \in S$ are such that $(a,b) \in \rho_{\max}$ and $c \in S$. There are $a' \in V^+(a)$ and $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \rho\ (rbs + e + rb's) \text{ for all } e \in E^+ \text{ and for all } r,s \in S^1.$$

Let $c' \in V^+(c)$. Then $(rcs+e+rc's)\rho \in E^+(S/\rho)$ for all $e \in E^+$ and for all $r,s \in S^1$ and so by Lallemant's Lemma (*cf.* [8]), for $e \in E^+$ and $r,s \in S^1$,

$$(rcs + e + rc's)\ \rho\ f \text{ for some } f \in E^+.$$

Therefore, for $e \in E^+$ and $r,s \in S^1$,

$$(ras + rcs + e + rc's + ra's)\ \rho\ (ras + f + ra's)\ \rho\ (rbs + f + rb's)\ \rho\ (rbs + rcs + e + rc's + rb's).$$

Now let $g \in S(a'+a, c+c')$ and $h \in S(b'+b, c+c')$. Then $c'+g+a' \in V^+(a+c)$ and $c'+h+b' \in V^+(b+c)$. In S/ρ, $(c'+a')\rho$ and $(c'+g+a')\rho$ are both inverses of $(a+c)\,\rho$. Hence

$$(rc's+ra's)\rho \quad \text{and} \quad (rc's+rgs+ra's)\rho$$

are both inverses of $(ras+rcs)\rho$ in S/ρ. Since S/ρ is left inversive, it follows that

$$(ras+rcs+e+rc's+ra's)\ \rho\ (ras+rcs+e+rc's+rgs+ra's)$$

for all $e \in E^+$ and for all $r, s \in S^1$. Similarly,

$$(rbs+rcs+e+rc's+rb's)\ \rho\ (rbs+rcs+e+rc's+rhs+rb's)$$

for all $e \in E^+$ and for all $r, s \in S^1$. Hence

$$(ras+rcs+e+rc's+rgs+ra's)\ \rho\ (rbs+rcs+e+rc's+rhs+rb's)$$

and so $(a+c)\,\rho_{\max}\,(b+c)$. Similarly $(c+a)\ \rho\ (c+b)$.

Again,

$$(ras+e+ra's)\ \rho\ (rbs+e+rb's) \ \text{ for all } \ e \in E^+ \ \text{ and for all } \ r,s \in S^1$$

implies that

$$(rcas+e+rca's)\ \rho\ (rcbs+e+rcb's) \ \text{ for all } \ e \in E^+ \ \text{ and for all } \ r,s \in S^1$$

Since in a regular semiring $ca' \in V^+(ca)$ and $cb' \in V^+(cb)$, it follows that $ca\,\rho_{\max}\,cb$. Similarly $ac\,\rho_{\max}\,bc$. Thus $\rho_{\max}$ is a congruence on S.

Let $\xi \in \rho\theta$ and $a, b \in S$ be such that $a\ \xi\ b$. Then $\xi \in \mathcal{LC}(S)$ implies that

$$(ras+e+ra's)\ \xi\ (rbs+e+rb's)$$

for all $a' \in V^+(a)$, $b' \in V^+(b)$, $e \in E^+(S)$, and $r, s \in S^1$. Since by Lemma 2.6, $ras+e+ra's \in E^+(S)$ for all $a' \in V^+(a)$, $e \in E^+(S)$, $r, s \in S^1$ and $\xi\ \theta\ \rho$, it follows that

$$(ras+e+ra's)\ \rho\ (rbs+e+rb's)$$

for all $a' \in V^+(a)$, $b' \in V^+(b)$ and for all $e \in E^+(S)$, $r, s \in S^1$. Hence $a\ \rho_{\max}\ b$, that is, $\xi \subseteq \rho_{\max}$. In particular $\rho \subseteq \rho_{\max}$. Since $\rho \in \mathcal{LC}(S)$ this implies that $\rho_{\max} \in \mathcal{LC}(S)$.

Now to show that $\rho_{\max} \in \rho\theta$. Suppose that $f, g \in E^+(S)$ be such that $(f, g) \in \rho_{\max}$. Then there exist $f' \in V^+(f)$ and $g' \in V^+(g)$ such that $(f+e+f')\ \rho\ (g+e+g')$ for all $e \in E^+(S)$. Since $\rho \in \mathcal{LC}(S)$ we get, by Lemma 2.7, that

$$(f+e+f')\ \rho\ (f+e+f)\ \rho\ (f+e).$$

Similarly $(g+e+g')\ \rho\ (g+e)$. Hence $(f+e)\ \rho\ (g+e)$ for all $e \in E^+S$. Taking $e = f$, we have $f = (f+f)\ \rho\ (g+f)$ and taking $g = e$, we have $(f+g)\ \rho\ (g+g) = g$. Hence by R-unipotent property of ρ, $f\ \rho\ (g+f)\ \rho\ (g+f+g)\ \rho\ (g+g) = g$. This implies that $f\ \rho\ g$. Hence $\operatorname{tr}\rho_{\max} \subseteq \operatorname{tr}\rho$. Since $\rho \subseteq \rho_{\max}$, it follows that $\operatorname{tr}\rho = \operatorname{tr}\rho_{\max}$. □

Lemma 3.2 (LaTorre [9]). *If ρ is a congruence on a regular semigroup G then the least congruence $\rho_{\min}$ in the lattice $\mathcal{C}(G)$ of all congruences on G, such that $\operatorname{tr}\rho = \operatorname{tr}\rho_{\min}$ is given by:*

$$\rho_{\min} = \{(a,b) \in G \times G \mid x + a = b + y,\ x\ \rho\ (a+a')\ \text{and}\ y\ \rho\ (b'+b)\ \text{for some}\ x,y \in U(G);\ a' \in V^+(a)\ \text{and}\ b' \in V^+(b)\}$$

where $U(G)$ is the least full, self-conjugate subsemigroup of G.

Gomes [5] showed that $\rho_{\min}$ is a left inverse semigroup congruence when ρ is so on regular semigroups.

An ideal I of S is called *full* if $E \subseteq I$ and is called *self-conjugate* if $x + I + x' \subseteq I$ for all $x \in S$ and for all $x' \in V^+(x)$. Let $\mathcal{C}$ be the collection of all full, self-conjugate ideals of S and $U = \bigcap \mathcal{C}$. Then U is the least full, self-conjugate ideal of S. Note that if S is left inversive semiring then $U = E^+$.

Lemma 3.3. *Let S be a semiring and $\rho \in \mathcal{LC}(S)$. Then $x \in U$ implies that $x\rho \in E^+(S/\rho)$.*

Thus we get the following lemma.

Lemma 3.4. *If ρ is a left inversive semiring congruence on a semiring S then the least element $\rho_{\min}$ of $\rho\theta$ in the lattice $\mathcal{LC}(S)$ of all left inversive semiring congruences is given by:*

$$\rho_{\min} = \{(a,b) \in S \times S \mid x + a = b + y,\ x\ \rho\ (a+a')\ \text{and}\ y\ \rho\ (b'+b)\ \text{for some}\ x,y \in U,\ a' \in V^+(a)\ \text{and}\ b' \in V^+(b)\}.$$

Theorem 3.5. *Let S be a semiring. Then*

(i) *θ is a complete congruence on $\mathcal{LC}(S)$.*
(ii) *for any $\rho \in \mathcal{LC}(S)$, $\rho\theta = [\rho_{\min}, \rho_{\max}]$.*
(iii) *$\rho\theta$ is a complete modular sublattice of $\mathcal{LC}(S)$.*

Proof. (i) Follows easily. (ii) Follows from Lemma 3.1 and Lemma 3.4.

(iii) $\mathcal{LC}(S)$ is a sublattice of $\mathcal{C}_\ell(S)$, the lattice of all left inverse semigroup congruences on the regular semigroup $(S, +)$. Now Gomes [5, Theorem 7(b)] implies that $[\rho] = \{\xi \in \mathcal{C}_\ell(S) \mid \operatorname{tr}\xi = \operatorname{tr}\rho\}$ is a complete modular lattice. Since $\rho\theta = \{\xi \in \mathcal{LC}(S) \mid \xi\ \theta\ \rho\}$ is a sublattice of the modular lattice $[\rho]$, it is modular. Also (ii) implies that $\rho\theta$ has least and greatest elements. Therefore each θ-class is a complete modular sublattice of $\mathcal{LC}(S)$. □

In the following theorem we characterize $\rho_{\max}$ alternatively.

Theorem 3.6. *Let S be a semiring. Then $\rho_{\max} = \rho \vee \mathcal{R}^{+*}$ for all $\rho \in \mathcal{LC}(S)$.*

Proof. Suppose that $\rho \in \mathcal{LC}(S)$ and $a\ \rho_{\max}\ b$ for $a, b \in S$. Then there are $a' \in V^+(a)$, $b' \in V^+(b)$ such that $(a+e+a')\ \rho\ (b+e+b')$ for all $e \in E^+$. Since ρ is left inversive semiring congruence this by Lemma 2.2, implies that $(a+a')\ \rho\ (b+b')$. Then

$$a\ \mathcal{R}^{+*}\ (a+a')\ \rho\ (b+b')\ \mathcal{R}^{+*}\ b.$$

Hence $\rho_{\max} \subseteq \rho \vee \mathcal{R}^{+*}$.

Again $a\mathcal{R}^+b$ implies that $a+a' = b+b'$. So $ras+ra's = rbs+rb's$ for all $r, s \in S^1$, which gives that $(ras+ra's)\ \rho\ (rbs+rb's)$ for all $r, s \in S^1$. Since ρ is left inversive semiring congruence so Lemma 2.2 implies that

$$(ras+e+ra's)\ \rho\ (rbs+e+rb's) \text{ for all } r, s \in S^1,\ e \in E^+.$$

Hence $a\ \rho_{\max}\ b$. Thus $\mathcal{R}^{+*} \subseteq \rho_{\max}$. Therefore $\rho \vee \mathcal{R}^{+*} \subseteq \rho_{\max}$. Hence $\rho_{\max} = \rho \vee \mathcal{R}^{+*}$. □

Theorem 3.7. *Let $\mathcal{F}$ be a nonempty family of left inversive semiring congruences on a semiring S. Then*

$$\bigvee_{\rho\in\mathcal{F}} \rho_{\min} = \Big(\bigvee_{\rho\in\mathcal{F}} \rho\Big)_{\min} \quad \textit{and} \quad \bigcap_{\rho\in\mathcal{F}} \rho_{\max} = \Big(\bigcap_{\rho\in\mathcal{F}} \rho\Big)_{\max}$$

Proof. Let $a, b \in S$ and

$$a\ \Big(\bigcap_{\rho\in\mathcal{F}} \rho_{\max}\Big)\ b.$$

Then $a\ \rho_{\max}\ b$ for all $\rho \in \mathcal{F}$. Suppose $\rho, \sigma \in \mathcal{F}$. Then there exist $a', a'' \in V^+(a)$, $b', b'' \in V^+(b)$ such that

$$(ras+e+ra's)\ \rho\ (rbs+e+rb's) \text{ and } (ras+e+ra''s)\ \sigma\ (rbs+e+rb''s)$$

for all $e \in E^+, r, s \in S^1$. Since σ is a left inversive semiring congruence, Lemma 2.7 implies that $(ras+e+ra's)\ \sigma\ (ras+e+ra''s)$ and $(rbs+e+rb''s)\ \sigma\ (rbs+e+rb's)$.

This implies that $(ras + e + ra's)\ \sigma\ (rbs + e + rb's)$. This shows that there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \rho\ (rbs + e + rb's)$$

for all $e \in E^+$; $r, s \in S^1$ and for all $\rho \in \mathcal{F}$, that is, there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\,(\bigcap_{\rho\in\mathcal{F}} \rho)\ (rbs + e + rb's) \text{ for all } e \in E^+,\ r, s \in S^1.$$

Thus $a\ (\bigcap_{\rho\in\mathcal{F}} \rho)_{\max}\ b$.

Conversely let $a,\ b \in S$. Then

$$a\,(\bigcap_{\rho\in\mathcal{F}} \rho)_{\max}\ b$$

implies that there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ (\bigcap_{\rho\in\mathcal{F}} \rho)\ (rbs + e + rb's)$$

for all $e \in E^+, r, s \in S^1$. This implies that there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \rho\ (rbs + e + rb's)$$

for all $e \in E^+$, $r, s \in S^1$ and for all $\rho \in \mathcal{F}$. This implies that $a\ \rho_{\max}\ b$ for all $\rho \in \mathcal{F}$. Therefore $a\ (\bigcap_{\rho\in\mathcal{F}} \rho_{\max})\ b$. □

As in the lattice of congruences on inverse semigroups, we define another relation κ on $\mathcal{LC}(S)$ by

$$\rho\ \kappa\ \xi \text{ if and only if } \ker\rho = \ker\xi$$

A congruence ρ on a semiring S is said to *saturate* a non-empty subset H of S if H is a union of some ρ-classes.

Maity [10] determined the greatest congruence τ^H on an inversive semiring S, which saturates a given nonempty subset H as follows:

$$\begin{aligned} a\ \tau^H\ b & \\ &\text{if and only if } x + ras + y \in H \text{ for all } x, y \in S^o;\ r, s \in S^1. \\ &\text{if and only if } x + rbs + y \in H \text{ for all } x, y \in S^o;\ r, s \in S^1. \end{aligned}$$

For a congruence ρ on an inversive semiring S Maity [10] proved that, $\rho^{\max} = \tau^{\ker\rho}$ is the largest element in the κ-class of ρ on an inversive semiring S.

We state the following lemma which can be proved easily.

Lemma 3.8. *Let S be a semiring and H be a nonempty subset of S. Then τ^H, as defined above, is the largest congruence on S, which saturates H.*

For a left inversive semiring congruence ρ on a semiring S, we define a relation $\rho^{\max}$ on S by:

$$\rho^{\max} = \tau^{\ker \rho}.$$

Theorem 3.9. *Let S be a semiring and ρ is a left inversive semiring congruence on S. $\rho^{\max}$ is the greatest element of $\rho\kappa$, the κ-class of ρ in $\mathcal{LC}(S)$.*

Proof. First note that, since S is an additive regular semiring $x\xi \in E^+(S/\xi)$ implies that there is $e \in E^+$ such that $x \; \xi \; e$ for all congruence ξ on S. Let $e, f \in E^+$. Since ρ is left inversive semiring congruence, so $(e+f) \; \rho \; (e+f+e)$ which implies that

$$(x + r(e+f)s + y) \; \rho \; (x + r(e+f+e)s + y) \text{ for all } x, y \in S^o; \, r, s \in S^1.$$

Thus

$$x + r(e+f)s + y \in \ker \rho \text{ if and only if } x + r(e+f+e)s + y \in \ker \rho$$

for all $x, y \in S^o$; $r, s \in S^1$. Hence $(e+f) \; \tau^{\ker\rho} \; (e+f+e)$, that is, $\tau^{\ker\rho} = \rho^{\max}$ is a left inversive semiring congruence on S.

Let $\xi \in \mathcal{LC}(S)$ and $\xi \; \kappa \; \rho$. Then $\ker \xi = \ker \rho$. Let $a, b \in S$ be such that $a \; \xi \; b$. Then

$$(x + ras + y) \; \xi \; (x + rbs + y) \text{ for all } x, y \in S^o; \, r, s \in S^1.$$

This implies that,

$$x + ras + y \in \ker \xi \text{ if and only if } x + rbs + y \in \ker \xi$$

for all $x, y \in S^o$; $r, s \in S^1$. Hence

$$x + ras + y \in \ker \rho \text{ if and only if } x + rbs + y \in \ker \rho$$

for all $x, y \in S^o$;, $r, s \in S^1$. Thus $a \, \rho^{\max} \, b$ and so $\xi \subseteq \rho^{\max}$. That $ker\rho^{\max} = ker\rho$ follows similarly as in Maity [10]. □

For a left inversive semiring congruence ρ on a semiring S, we define another relation $\rho^{\min}$ on S by:

$$\rho^{\min} = \big(\{(x, x+x) \mid x \; \rho \; (x+x)\} \cup \varrho\big)^*$$

where $\varrho = \{(e+f, e+f+e) \in S \times S \mid e, f \in E^+\}$.

Theorem 3.10. *Let S be a semiring and ρ be a congruence on S. Then $\rho^{\min}$ is the smallest left inversive semiring congruence in the κ-class of ρ.*

Proof. $\varrho \subseteq \rho^{\min}$ implies that $\rho^{\min}$ is a left inversive semiring congruence on S. Let $\xi \in \kappa\rho$. Then $\ker \xi = \ker \rho$ and this implies that

$$x \; \rho \; (x+x) \text{ if and only if } x \; \xi \; (x+x)$$

for all $x \in S$. Hence $\{(x, x+x) \mid x \; \rho \; (x+x)\} = \{(x, x+x) \mid x \; \xi \; (x+x)\} \subseteq \xi$. Also ξ being left inversive $\varrho \subseteq \xi$. Therefore $\rho^{\min} \subseteq \xi$. In particular $\rho^{\min} \subseteq \rho$. This implies that $\ker \rho^{\min} \subseteq \ker \rho$. Again $x \in \ker \rho$ implies that $x \; \rho \; (x+x)$. Hence $x \, \rho^{\min} \, (x+x)$, that is, $x \in \ker \rho^{\min}$. So $\ker \rho \subseteq \ker \rho^{\min}$. Therefore $\ker \rho = \ker \rho^{\min}$. Thus $\rho^{\min}$ is the least element of $\rho\kappa$ in the lattice $\mathcal{LC}(S)$. □

The following theorem can be proved similarly as Theorem 3.5.

Theorem 3.11. *Let S be a semiring. Then*

(i) *κ is a $\cap$-complete congruence on $\mathcal{LC}(S)$.*
(ii) *$\rho\kappa = [\rho^{\min}, \rho^{\max}]$ for any $\rho \in \mathcal{LC}(S)$.*
(iii) *$\rho\kappa$ is a complete sublattice of $\mathcal{LC}(S)$ for all $\rho \in \mathcal{LC}(S)$.*

From Theorems 3.5 and 3.12 the following theorem follows immediately.

Theorem 3.12. *Let S be a semiring. Then for any left inversive semiring congruence ρ on S,*

$$\rho = \rho_{\min} \vee \rho^{\min} = \rho_{\max} \cap \rho^{\max}.$$

4. Idempotent Separating Congruences

A congruence ρ on a semiring S is *idempotent separating* if, for $e, \; f \in E^+$,

$$e \; \rho \; f \text{ implies that } e = f.$$

Let S be a left inversive semiring. It is clear that ε, the equality relation on S, is the minimum idempotent separating congruence on S. Theorem 3.12 implies that $\varepsilon_{\max}$ is the maximum idempotent separating congruence on S. We denote it by μ or sometimes by μ_S. Therefore

$$a \; \mu \; b \text{ if and only if } ras + e + ra's = rbs + e + rb's \text{ for all } e \in E^+;\; r, s \in S^1.$$

Theorem 4.1. *Let S be a regular semiring and $\rho, \xi \in \mathcal{LC}(S)$. Then the following are equivalent.*

(i) $\rho \; \theta \; \xi$

(ii) $\rho \subseteq \xi_{\max}$, $\xi_{\max}/\rho = \mu_{S/\rho}$

(iii) *For* $a, \in S$, $(a\rho)\mu_{S/\rho}(b\rho) \Leftrightarrow (a\xi)\mu_{S/\xi}(b\xi)$.

(iv) *For* $a, b \in S$, $(a\rho)\mathcal{R}^+_{S/\rho}(b\rho) \Leftrightarrow (a\xi)\mathcal{R}^+_{S/\xi}(b\xi)$.

(v) $\rho/(\rho \cap \xi)$ *and* $\xi/(\rho \cap \xi)$ *are idempotents separating congruences on* S.

Proof. (i) $\Rightarrow$ (iii): For a, $b \in S$, we have,

$(a\rho)\,\mu_{S/\rho}\,(b\rho)$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(r\rho)(a\rho)(s\rho) + (e\rho) + (r\rho)(a\rho)'(s\rho) = (r\rho)(b\rho)(s\rho) + (e\rho) + (r\rho)(b\rho)'(s\rho)$$

for all $r, s \in S$, $e \in E^+$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \rho\ (rbs + e + rb's); \text{ for all } r, s \in S, e \in E^+$$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \tau\ (rbs + e + rb's); \text{ for all } r, s \in S, e \in E^+$$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(r\tau)(a\tau)(s\tau) + (e\tau) + (r\tau)(a\tau)'(s\tau) = (r\tau)(b\tau)(s\tau) + (e\tau) + (r\tau)(b\tau)'(s\tau)$$

for all $r, s \in S$, $e \in E^+$

if and only if $(a\,\tau)\mu_{S/\tau}\,(b\tau)$.

(iii) $\Rightarrow$ (iv): First note that, since $(S, +)$ is regular for a, $b \in S$;

$a\,\mathcal{R}^+\,b$ if and only if there exist

$$a' \in V^+(a),\ b' \in V^+(b) \text{ such that } a + a' = b + b'.$$

Now let $a, b \in S$. Then

$(a\rho)\,\mathcal{R}^+_{S/\rho}\,(b\rho)$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(a\rho) + (a'\rho) = (b\rho) + (b'\rho)$$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(a + a')\rho = (b + b')\rho$$

if and only if there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$(a+a')\ \rho\ \mu_{S/\rho}(b+b')\rho$$

if and only if there exist $a' \in V^+(a),\ b' \in V^+(b)$ such that

$$(a+a')\xi\mu_{S/\xi}(b+b')\xi$$

if and only if there exist $a' \in V^+(a),\ b' \in V^+(b)$ such that

$$(a\xi)+(a'\xi)=(b\xi)+(b'\xi)$$

if and only if $(a\xi)\,\mathcal{R}^+_{S/\xi}\,(b\xi)$.

(iv) $\Rightarrow$ (v): For $e, f \in E^+$, $e(\rho\cap\xi)\ \rho/(\rho\cap\xi)\ f(\rho\cap\xi)$ implies that $e\ \rho\ f$ and so $e\rho = f\rho$. Hence $e\rho\ \mathcal{R}^+_{S/\rho}\ f\rho$ which implies that $e\xi\ \mathcal{R}^+_{S/\xi}\ f\xi$. Since S/ξ is left inversive, by Lemma 2.2, this implies that $e\xi = f\xi$. Hence $f\ (\rho\cap\xi)\ e$. Thus $e(\rho\cap\xi) = f(\rho\cap\xi)$.

(v) $\Rightarrow$ (ii): Suppose $a, b \in S$ be such that $a\rho b$. Since ρ is a left inversive semiring congruence, from Lemma 2.6, it follows that, for all $a' \in V^+(a)$ for all $b' \in V^+(b)$,

$$(ras+e+ra's)\ \rho\ (rbs+e+rb's)\ \text{ for all }\ r,s\in S^1;\ e\in E^+.$$

This implies that

$$(ras+e+ra's)(\rho\cap\xi)\ \rho/\rho\cap\xi\ (ras+e+ra's)(\rho\cap\xi)\ \text{ for all }\ r,s\in S^1;\ e\in E^+.$$

Since $\rho\cap\xi$ is a left inversive semiring congruence, Lemma 2.6 implies that

$$(ras+e+ra's)(\rho\cap\xi)\ \text{ and }\ (ras+e+ra's)(\rho\cap\xi)\ \text{ are idempotents}$$

for all $a' \in V^+(a)$; $b' \in V^+(b)$; $r, s \in S^1$; $e \in E^+$. Hence by (v),

$$(ras+e+ra's)(\rho\cap\xi) = (ras+e+ra's)(\rho\cap\xi)$$

for all $a' \in V^+(a)$; $b' \in V^+(b)$; $r, s \in S^1$; $e \in E^+$. So

$$(ras+e+ra's)\ \xi\ (ras+e+ra's)$$

for all $a' \in V^+(a)$; $b' \in V^+(b)$; $r, s \in S^1$; $e \in E^+$ which gives $a\ \xi_{\max}\ b$. Hence $\rho \subseteq \xi_{\max}$.

For any $a, b \in S$,

$(a\rho)\ \xi_{\max}/\rho\ (b\rho)$

if and only if there exist $a\ \xi_{\max}\ b$

if and only if there exist $a' \in V^+(a),\ b' \in V^+(b)$ such that

$$(ras+e+ra's)\ \xi\ (ras+e+ra's)\ \text{ for all }\ r,s\in S^1, e\in E^+.$$

This, similarly to the first part,

if and only if there exist $a' \in V^+(a), b' \in V^+(b)$ such that

$$(ras + e + ra's)\ \rho\ (ras + e + ra's) \text{ for all } r, s \in S, e \in E^+$$

if and only if there exist $a' \in V^+(a), b' \in V^+(b)$ such that

$$(r\rho)(a\rho)'(s\rho) + (e\rho) + (r\rho)(a\rho)(s\rho) = (r\rho)(b\rho)'(s\rho) + (e\rho) + (r\rho)(b\rho)(s\rho)$$
$$\text{for all } r, s \in S^1, e \in E^+$$

if and only if $(a\rho)\mu_{S/\rho}(b\rho)$.

(ii) $\Rightarrow$ (i): $\rho \subseteq \xi_{\max}$ implies that $\operatorname{tr}\rho \subseteq \operatorname{tr}\xi$. Further we have, for any $e, f \in E^+$, $e\ \xi\ f$ implies that $e\ \xi_{\max}\ f$, and this implies that $e\rho\ (\xi_{\max}/\rho)\ f\rho$. Again this implies that $e\rho\ (\mu_{S/\rho})\ f\rho$, and this implies that $e\rho = f\rho$, implies that $e\ \rho\ f$. Hence it follows that $\operatorname{tr}\xi \subseteq \operatorname{tr}\rho$. Therefore $\operatorname{tr}\rho = \operatorname{tr}\xi$. □

The following corollary is immediate from the equivalence of (i) and (ii) in the above theorem.

Corollary 4.2. *Let ρ be a congruence on a semiring S. Then $\mu_{S/\rho} = \rho_{\max}/\rho$.*

Definition 4.3. A semiring S is called *fundamental* if $\varepsilon = \mu$.

Corollary 4.4. *Let ρ be a left inversive semiring congruence on S. Then*

$$\rho = \rho_{\max} \textit{ if and only if } S/\rho \textit{ is fundamental.}$$

Proof.

$$\rho = \rho_{\max} \text{ if and only if } \rho_{\max}/\rho = \varepsilon \text{ if and only if } \mu_{S/\rho} = \varepsilon$$
$$\text{if and only if } S/\rho \text{ is fundamental.} \quad \square$$

Lemma 4.5. *Let S be a semiring and $\rho \in \mathcal{LC}(S)$. Then*

(i) $\ker\rho_{\min} = \{a \in S \mid u + a \in U,\ u\ \rho\ (a + a')$ *for some* $u \in U,\ a' \in V^+(a)\}$.

(ii) $\ker\rho_{\max} = \{a \in S \mid (ras + ra's + e)\ \rho\ (ras + e + ra's)$ *for all* $a' \in V^+(a);\ e \in E^+;\ r, s \in S^1\}$.

Proof. (i) This is similar to Gomes [5, Proposition 15].

(ii) Since $\rho_{\max}$ is left inversive semiring congruence,

$$(a + a')\ \rho_{\max}\ (a + a'') \text{ for all } a \in S;\ a', a'' \in V^+(a).$$

Hence

$$
\begin{aligned}
& a \in \ker \rho_{\max} \\
& \quad \text{if and only if } \; a\, \rho_{\max}\, (a+a') \; \text{ for all } \; a' \in V^+(a) \\
& \quad \text{if and only if, for all } \; a' \in V^+(a), r, s \in S^1, e \in E^+, \\
& \qquad (ras + e + ra's) \; \rho \; (r(a+a')s + e + r(a+a')s) \\
& \qquad\qquad = (ras + ra's + r(a+a')s + e + r(a+a')s) \\
& \qquad\qquad\qquad \rho \; (ras + ra's + r(a+a')s + e) = ras + ra's + e,
\end{aligned}
$$

by Lemma 2.3. □

The following theorem shows that each idempotent separating congruence on a regular semiring is the least element of its kernel class.

Theorem 4.6. *Let S a semiring and ρ be an idempotent separating congruence on S. Then $\rho = \rho^{\min}$. In particular $\mu = \mu^{\min}$.*

5. Quasi-inversive Semirings

In Yamada [18], characterized the quasi-inverse semigroups as a subdirect product of a left inverse semigroup and a right inverse semigroup. Aim of this concluding section is to characterize the quasi-inversive semirings as the direct product of a left inversive semiring and a right inversive semiring.

If the set $E^+(S)$ of all additive idempotents of a semiring S is a subsemiring of S then we call S to be an *orthodox semiring.*

We state the following lemma without proof:

Lemma 5.1. *Let S be an additive regular semiring. S is orthodox if and only if for all $a, b \in S$,*

$$V^+(a) \cap V^+(b) \neq \varnothing \; \text{ implies that } \; V^+(a) = V^+(b)$$

Let S be a left inversive semiring. For $a, b \in S$, define

$$a \,\sigma\, b \; \text{ if and only if } \; V^+(a) = V^+(b)$$

Theorem 5.2. *Let S be an orthodox semiring. Then σ is the least inversive semiring congruence on S.*

Proof. Since S is left inversive so $(S,+)$ is orthodox. Hence form it follows that σ is the least inverse semigroup congruence on $(S,+)$. Now we shall show that σ is also multiplicative congruence on the semiring S. Let $a, b \in S$ be such that $a \; \sigma \; b$ and $c \in S$. Then $V^+(a) = V^+(b)$ and this implies that $V^+(a) \cap V^+(b) \neq \varnothing$. Let $x \in V^+(a) \cap V^+(b)$. Then $xc \in V^+(ac) \cap V^+(bc)$. Thus it follows that $ac \; \sigma \; bc$. Similarly $ca \; \sigma \; cb$. Hence σ is the least inversive semiring congruence on S. □

Now let us introduce a relation ρ_1 in S as follows:

$$a \; \rho_1 \; b \text{ if and only if there exist } a' \in V^+(a),\; b' \in V^+(b) \text{ such that}$$
$$ras + e + ra's + rbs + e + rb's = rbs + e + rb's, \text{ and}$$
$$rbs + e + rb's + ras + e + ra's = ras + e + ra's$$
$$\text{for all } r, s \in S^1;\; e \in E^+.$$

Lemma 5.3. *Let S be an orthodox semiring. Then ρ_1 is a congruence on S.*

Proof. Let $a \in S$ and $a' \in V^+(a)$. Then $ras+e+ra's \in E^+$ for all $r, s \in S^1$; $e \in E^+$. This implies that $a \; \rho_1 \; a$ for all $a \in S$. That ρ_1 is symmetric follows from its definition. Suppose $a, b, c \in S$ be such that $a \; \rho_1 \; b$ and $b \; \rho_1 \; c$. Then there is $a' \in V^+(a)$; $b', b'' \in V^+(b)$ and $c' \in V^+(c)$ such that

$$ras + e + ra's + rbs + e + rb's = rbs + e + rb's,$$
$$rbs + e + rb's + ras + e + ra's = ras + e + ra's,$$
$$rbs + e + rb''s + rcs + e + rc's = rcs + e + rc's, \text{ and}$$
$$rcs + e + rc's + rbs + e + rb''s = rbs + e + rb''s$$

for all $r, s \in S^1$; $e \in E^+$. Now

$$\begin{aligned} &rbs + e + rb's + rbs + e + rb''s \\ &= rbs + (rb's + rbs + e) + (rb's + rbs + e) + rb''s \\ &= rbs + rb's + rbs + e + rb''s \qquad \text{(since } rb's + rbs + e \in E^+) \\ &= rbs + e + rb''s \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$. Then, for all $r, s \in S^1$ and $e \in E^+$,

$$ras + e + ra's + rbs + e + rb's = rbs + e + rb's$$

implies that

$$ras + e + ra's + rbs + e + rb's + rbs + e + rb''s = rbs + e + rb's + rbs + e + rb''s,$$

that is,

$$ras + e + ra's + rbs + e + rb''s = rbs + e + rb''s$$

for all $r, s \in S^1$; $e \in E^+$. Hence

$$\begin{aligned} &ras + e + ra's + rcs + e + rc's \\ &\quad = ras + e + ra's + rbs + e + rb''s + rcs + e + rc's \\ &\quad = rbs + e + rb''s + rcs + e + rc's = rcs + e + rc's \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$. Similarly

$$rcs + e + rc's + ras + e + ra's = ras + e + ra's$$

for all $r, s \in S^1$; $e \in E^+$. Therefore $a \ \rho_1 \ c$. Thus ρ_1 is an equivalence relation.

Suppose $a, b, c \in S$ and $a \ \rho_1 \ b$. Then there is $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$\begin{aligned} ras + e + ra's + rbs + e + rb's &= rbs + e + rb's, \\ rbs + e + rb's + ras + e + ra's &= ras + e + ra's \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$. Then

$$\begin{aligned} &rcs + rbs + e + rb's + rc's \\ &\quad = rcs + (rc's + rcs + ras + e + ra's) + rbs + e + rb's + rc's \\ &\quad = rcs + (rc's + rcs + ras + e + ra's) + (rc's + rcs + ras + e + ra's) \\ &\qquad\qquad + rbs + e + rb's + rc's \qquad (\text{since } rc's + rcs + ras + e + ra's \in E^+) \\ &\quad = rcs + ras + e + ra's + rc's + rcs + rbs + e + rb's + rc's \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$. Similarly

$$\begin{aligned} &rcs + rbs + e + rb's + rc's + rcs + ras + e + ra's + rc's \\ &\qquad\qquad\qquad\qquad = rcs + ras + e + ra's + rc's \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$. Since $a' + c' \in V^+(c + a)$ and $b' + c' \in V^+(c + b)$ therefore $(c + a) \ \eta_1 \ (c + b)$. Again $rcs + e + rc's \in E^+$ for all $r, s \in S^1$, $e \in E^+$ and

$$\begin{aligned} ras + e + ra's + rbs + e + rb's &= rbs + e + rb's, \\ rbs + e + rb's + ras + e + ra's &= ras + e + ra's \end{aligned}$$

for all $r, s \in S^1$; $e \in E^+$ implies that

$$ras + rcs + e + rc's + ra's + rbs + rcs + e + rc's + rb's = rbs + rcs + e + rc's + rb's,$$

$$rbs + rcs + e + rc's + rb's + ras + rcs + e + rc's + ra's = ras + rcs + e + rc's + ra's$$

for all $r, s \in S^1$; $e \in E^+$. Therefore $(a+c)\ \rho_1\ (b+c)$.

Suppose $a, b, c \in S$ and $a\ \rho_1\ b$. Then there are $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$ras + e + ra's + rbs + e + rb's = rbs + e + rb's,$$

$$rbs + e + rb's + ras + e + ra's = ras + e + ra's$$

for all $r, s \in S^1$, $e \in E^+$. This implies that

$$(rc)as + e + (rc)a's + (rc)bs + e + (rc)b's = (rc)bs + e + (rc)b's,$$

$$(rc)bs + e + (rc)b's + (rc)as + e + (rc)a's = (rc)as + e + (rc)a's$$

for all $r, s \in S^1$, $e \in E^+$, which again implies that

$$r(ca)s + e + r(ca)'s + r(cb)s + e + r(cb)'s = r(cb)s + e + r(cb)'s,$$

$$r(cb)s + e + r(cb)'s + r(ca)s + e + r(ca)'s = r(ca)s + e + r(ca)'s$$

for all $r, s \in S^1$, $e \in E^+$. Therefore $ca\ \rho_1\ cb$. Similarly $ac\ \rho_1\ bc$.

Definition 5.4. A semiring S is called a *quasi-inversive semiring* if $E^+(S)$ is a subsemiring which satisfies the identity $x + y + z + x = x + y + x + z + x$.

Of course, both a left inversive semiring and a right inversive semiring (a semiring such that $e + f + e = f + e$ for all $e, f \in E^+$) are quasi-inversive semirings. In this section we describe the structure of quasi-inversive semirings as a subdirect product of a left inversive semiring and a right inversive semiring.

Now, define a relation η_1 as

$$\eta_1 = \sigma \cap \rho_1.$$

Lemma 5.5.

(i) *Let S be a quasi-inversive semiring and $x \in S$. Then $\tilde{x}$, the η_1-class of x, is an additive idempotent if and only if x is an additive idempotent.*

(ii) *$E^+(S/\eta_1)$ is a left regular additive idempotent semiring and hence S/η_1 is a left inversive semiring.*

Proof. (i) Follows from Yamada [18, Lemma 13].

(ii) Due to (i) it is sufficient to show that $(e+f+e)\eta_1(e+f)$ for all $e, f \in E^+$. Since S is an orthodox semiring $e+f+e, e+f \in E^+$ and so $e+f+e \in V^+(e+f+e)$ and $e+f \in V^+(e+f)$. Now

$$\begin{aligned} r(e+f+e)s+g+r(e+f+e)s+r(e+f)s+g+r(e+f)s \\ &= res+rfs+res+g+res+rfs \\ &= res+rfs+res+rfs+(g+res)+rfs \quad \text{(since } S \text{ is quasi-inversive)} \\ &= r(e+f)s+g+r(e+f)s \end{aligned}$$

and

$$\begin{aligned} r(e+f)s+g+r(e+f)s+r(e+f+e)s+g+r(e+f+e)s \\ &= (res+rfs+g+res)+rfs+res \\ &= (res+rfs+res+g+res)+rfs+res \quad \text{(since } S \text{ is quasi-inversive)} \\ &= r(e+f+e)s+g+r(e+f+e)s \end{aligned}$$

for all $r, s \in S^1$ and $g \in E^+$ implies that $(e+f+e)\eta_1(e+f)$.

Next, let us define a relation η_2 as

$$\eta_2 = \sigma \cap \rho_2.$$

Where ρ_2 is defined by:

$$\begin{aligned} a\ \rho_2\ b \text{ if and only if there exist } a' \in V^+(a),\ b' \in V^+(b) \text{ such that} \\ ras+e+ra's+rbs+e+rb's = ras+e+ra's, \quad \text{and} \\ rbs+e+rb's+ras+e+ra's = rbs+e+rb's \\ \text{for all } r, s \in S^1,\ e \in E^+. \end{aligned}$$

By a similar process to that for η_1, we can prove that S/η_2 is a right inversive semiring. □

Lemma 5.6. *Let S be a quasi-inversive semiring and $a, b \in S$.*

(i) *If $a\ \eta_1\ b$, then there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that*

$$\begin{aligned} ras+ra's+rbs+rb's = rbs+rb's, \quad \textit{and} \\ rbs+rb's+ras+ra's = ras+ra's \end{aligned}$$

for all $r, s \in S^1$.

(ii) *If $a\ \eta_2\ b$, then there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that*

$$ras + ra's + rbs + rb's = ras + ra's, \quad and$$
$$rbs + rb's + ras + ra's = rbs + rb's$$

for all $r, s \in S^1$.

Proof. Let $a\ \eta_1\ b$. Then there exist $a' \in V^+(a)$, $b' \in V^+(b)$ such that

$$ras + e + ra's + rbs + e + rb's = rbs + e + rb's, \quad \text{and}$$
$$rbs + e + rb's + ras + e + ra's = ras + e + ra's$$

for all $r, s \in S^1$, $e \in E^+$. For $r, s \in S^1$ take $e = ra's + ras$. Then we get

$$ras + ra's + rbs + ra's + ras + rb's = rbs + ra's + ras + rb's.$$

Hence

$$ras + ra's + rbs + (rb's + rbs + ra's + ras + rb's + rbs) + rb's$$
$$= rbs + (rb's + rbs + ra's + ras + rb's + rbs) + rb's.$$

Now $a\ \sigma\ b$ implies that $ras\ \sigma\ rbs$. Therefore

$$rb's + rbs + ra's + ras + rb's + rbs = rb's + rbs.$$

Therefore

$$ras + ra's + rbs + rb's + rbs + rb's = rbs + rb's + rbs + rb's$$

that is $ras + ra's + rbs + rb's = rbs + rb's$. Similarly

$$rbs + rb's + ras + ra's = ras + ra's. \qquad \square$$

The following lemma can be proved easily as in Yamada [18, Lemma 15].

Lemma 5.7. *Let S be a quasi-inversive semiring and $a, b \in S$. Then $a\ \eta_1\ b$ and $a\ \eta_2\ b$ implies that $a = b$.*

Thus we get the following theorem.

Theorem 5.8. *A regular semiring is a quasi-inversive semiring if and only if it is a subdirect product of a left inversive semiring and a right inversive semiring.*

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(M. K. SEN) DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA, 35, BALLYGUNGE CIRCULAR ROAD, KOLKATA-700019, INDIA
Email address: senmk6@eth.net

(A. K. BHUNIYA) DEPARTMENT OF MATHEMATICS, VISVA-BHARATI UNIVERSITY, SANTINIKETAN, WEST BENGAL, INDIA
Email address: anjankbhuniya@yahoo.co.in