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SOME SUMMATION FORMULAS FOR THE SERIES ${}_3F_2(1)$

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ABSTRACT. We evaluate the sum of certain class of generalized hypergeometric series of unit argument. Summation formulas, contiguous to Watson's, Whipple's, Lavoie's and Choi's theorems in the theory of the generalized hypergeometric series, are obtained. Certain limiting cases of these results are given.

1. Two Results Contiguous to Watson's, Lavoie's and Choi's Theorems

The two following summation formulas for the series ${}_3F_2(1)$ are useful, interesting, and easily established.

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c+4 \end{matrix}; 1\right] \\
&= \frac{2^{a+b}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c+\frac{5}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)} \\
&\times \Bigg[\frac{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})}{\Gamma(c-\frac{a}{2}+2)\Gamma(c-\frac{b}{2}+2)}\left\{\frac{-ab(2c-a-b+1)}{2(c+2)} - \frac{ab(a+1)(b+1)}{2(c+2)(c+3)}\right\} \\
&+ \frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}+\frac{5}{2})\Gamma(c-\frac{b}{2}+\frac{5}{2})} \\
&\times \left\{\frac{(2c-a-b+1)(2c-a-b+3)}{4} + \frac{ab(2c-a-b+1)}{4(c+2)}\right. \\
&\left.+ \frac{ab(2c-a-b)}{2(c+2)} + \frac{ab(a+2)(b+2)}{2(c+2)(c+3)}\right\}\Bigg],
\end{aligned}$$

provided $\Re e(2c-a-b) > -9$. (1.1)

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$$
{}_3F_2\left[\begin{matrix} a,b,c \\ \frac{1}{2}(a+b+1),2c-4 \end{matrix};1\right]
= \frac{2^{a+b}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{3}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}-\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)}
$$
$$
\times\left[\frac{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})}{\Gamma(c-\frac{a}{2}-2)\Gamma(c-\frac{b}{2}-2)}\left\{\frac{ab(2c-a-b-7)}{4(c-2)}+\frac{ab(a+1)(b+1)}{2(c-1)(c-2)}\right\}\right.
$$
$$
+\frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{b}{2}-\frac{3}{2})}
\times\left\{\frac{(2c-a-b-7)(2c-a-b-5)}{4}+\frac{ab(2c-a-b-7)}{4(c-2)}\right.
$$
$$
\left.\left.+\frac{3(2c-a-b-7)}{2}+\frac{(a+2)(b+2)}{c-1}\right\}\right],
$$

provided $\Re e(2c-a-b)>7$. (1.2)

They are thus seen to be contiguous to Watson's theorem [1, p. 16, 3.3.1]

$$
{}_3F_2\left[\begin{matrix} a,b,c \\ \frac{1}{2}(a+b+1),2c \end{matrix};1\right]
= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{b}{2}+\frac{1}{2})},
$$

provided $\Re(2c-a-b)>-1$ (1.3)

and Lavoie's theorems [3, p. 269, Eqs. (1) and (2)], and Choi's theorem [2, pp. 99-102, Eqs. (1),(2),(3) and (4)].

Proofs. It is easy to show that the left-hand side in the following relation [2, p. 105] involving these ${}_3F_2(1)$ reduces to the right-hand side:

$$
{}_3F_2\left[\begin{matrix} a,b,c \\ \frac{1}{2}(a+b+1),2c+4 \end{matrix};1\right] - {}_3F_2\left[\begin{matrix} a,b,c+1 \\ \frac{1}{2}(a+b+1),2c+4 \end{matrix};1\right]
= -\frac{2ab}{(a+b+1)(2c+4)}\,{}_3F_2\left[\begin{matrix} a+1,b+1,c+1 \\ \frac{1}{2}(a+b+3),2c+5 \end{matrix};1\right].
$$

But two of these ${}_3F_2(1)$ can be evaluated by Choi's theorem [2, pp. 100-102, Eqs. (2) and (4)];

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a+1, b+1, c+1 \\ \frac{1}{2}(a+b+3), 2c+5 \end{matrix}; 1\right] \\
&= \frac{2^{a+b+2}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{3}{2})\Gamma(c+\frac{5}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a+2)\Gamma(b+2)} \\
&\times\left[\left(\frac{2c-a-b+1}{2}+\frac{(a+1)(b+1)}{c+3}\right)\frac{\Gamma(\frac{a}{2}+\frac{3}{2})\Gamma(\frac{b}{2}+\frac{3}{2})}{\Gamma(c-\frac{a}{2}+2)\Gamma(c-\frac{b}{2}+2)}\right. \\
&\left.+\frac{(a+1)(b+1)}{4}\left(a+b-2c-\frac{(a+2)(b+2)}{(c+3)}\right)\frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}+\frac{5}{2})\Gamma(c-\frac{b}{2}+\frac{5}{2})}\right],
\end{aligned}
$$

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c+1 \\ \frac{1}{2}(a+b+1), 2c+4 \end{matrix}; 1\right] \\
&= \frac{2^{a+b}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c+\frac{5}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)} \\
&\times\left[\frac{-2ab}{(a+1)(b+1)(c+2)}\frac{\Gamma(\frac{a}{2}+\frac{3}{2})\Gamma(\frac{b}{2}+\frac{3}{2})}{\Gamma(c-\frac{a}{2}+2)\Gamma(c-\frac{b}{2}+2)}\right. \\
&\left.+\left(\frac{2c-a-b+3}{2}+\frac{ab}{2(c+2)}\right)\frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}+\frac{5}{2})\Gamma(c-\frac{b}{2}+\frac{5}{2})}\right],
\end{aligned}
$$

and (1.1) is obtained when we make use of various familiar identities relating to the Γ-functions.

In the same way, (1.2) is obtained from the relation [2, p. 105]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c-4 \end{matrix}; 1\right] - {}_3F_2\left[\begin{matrix} a, b, c-1 \\ \frac{1}{2}(a+b+1), 2c-4 \end{matrix}; 1\right] \\
&= \frac{ab}{(a+b+1)(c-2)}{}_3F_2\left[\begin{matrix} 1+a, 1+b, c \\ \frac{1}{2}(a+b+3), 2c-3 \end{matrix}; 1\right].
\end{aligned}
$$

In this case, two of these ${}_3F_2(1)$ can be evaluated by Choi's theorem [2, pp. 100-101, Eqs. (1) and (3)];

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a+1, b+1, c \\ \frac{1}{2}(a+b+3), 2c-3 \end{matrix}; 1\right] \\
&= \frac{2^{a+b+2}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}+1)\Gamma(c+\frac{3}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(a+2)\Gamma(b+2)} \\
&\quad \times \left[\left(\frac{2c-a-b-7}{2}+\frac{(a+1)(b+1)}{2}\right)\frac{\Gamma(\frac{a}{2}+\frac{1}{2}+1)\Gamma(\frac{b}{2}+\frac{1}{2}+1)}{\Gamma(c-\frac{a}{2}-2)\Gamma(c-\frac{b}{2}-2)}\right. \\
&\quad \left. + \frac{(a+1)(b+1)}{8}\left(3(a+b-2c)+\frac{2(a+2)(b+2)}{(c-1)}\right)\frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}+\frac{3}{2})\Gamma(c-\frac{b}{2}+\frac{3}{2})}\right],
\end{aligned}
$$

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c-1 \\ \frac{1}{2}(a+b+1), 2c-4 \end{matrix}; 1\right] \\
&= \frac{2^{a+b}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{3}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}-\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)} \\
&\quad \times \left[\frac{2ab}{(a+1)(b+1)(c-2)}\frac{\Gamma(\frac{a}{2}+\frac{1}{2}+1)\Gamma(\frac{b}{2}+\frac{1}{2}+1)}{\Gamma(c-\frac{a}{2}-2)\Gamma(c-\frac{b}{2}-2)}\right. \\
&\quad \left. + \left(\frac{2c-a-b-5}{2}+\frac{ab}{2(c-2)}\right)\frac{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)}{\Gamma(c-\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{b}{2}-\frac{3}{2})}\right],
\end{aligned}
$$

and (1.2) is obtained when we make use of the properties of Γ-functions. □

2. Two Results Closely Related to Whipple's and Lavoie's Theorems

Formulas (1.1) and (1.2) lead, respectively, to the two summation formulas:

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(e)\Gamma(f)\Gamma(c-4)\Gamma(c-2+\frac{1}{2})\Gamma(a-4)}{\Gamma(a)\Gamma(2c-4)}
$$

$$
\times \Bigg[\frac{-2}{(c-2)\Gamma[\frac{1}{2}(e-a)]\Gamma[c-\frac{1}{2}(e-a)-2]\Gamma[\frac{1}{2}(f-a)]\Gamma[c-\frac{1}{2}(f-a)-2]}
$$

$$
\times \left\{2(a-4) + \frac{(e-a+1)(f-a+1)}{c-1}\right\}
$$

$$
+ \frac{1}{\Gamma[\frac{1}{2}(e-a+1)]\Gamma[c-\frac{1}{2}(e-a+1)-1]\Gamma[\frac{1}{2}(f-a+1)]\Gamma[c-\frac{1}{2}(f-a+1)-1]}
$$

$$
\times \Bigg\{(a-4)\left(a-3+\frac{(e-a)(f-a)}{2(c-2)}\right)
$$

$$
+ \frac{(e-a)(f-a)}{2(c-2)}\left(2a-9+\frac{(e-a+2)(f-a+2)}{c-1}\right)\Bigg\}\Bigg], \tag{2.1}
$$

provided the parameters satisfy the conditions $a + b = 5$ and $e + f = 1 + 2c$ with $\Re e(c) > 4$, and

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(e)\Gamma(f)\Gamma(c+4)\Gamma(c+\frac{5}{2})}{\Gamma(2c+4)}
$$

$$
\times \Bigg[\frac{2}{\Gamma[\frac{1}{2}(e-a)]\Gamma[\frac{1}{2}(f-a)]\Gamma[c-\frac{1}{2}(e-a)+2]\Gamma[c-\frac{1}{2}(f-a)+2]}
$$

$$
\times \left\{\frac{2a(c+3)+(e-a+1)(f-a+1)}{(c+2)(c+3)}\right\}
$$

$$
+ \frac{1}{\Gamma[\frac{1}{2}(e-a+1)]\Gamma[\frac{1}{2}(f-a+1)]\Gamma[c-\frac{1}{2}(e-a+1)+3]\Gamma[c-\frac{1}{2}(f-a+1)+3]}
$$

$$
\times \Bigg\{\frac{2a(a+1)(2+c)+a(e-a)(f-a)}{2(c+2)}
$$

$$
+ \frac{3a(c+3)+(e-a+2)(f-a+2)}{c+3}\Bigg\}\Bigg], \tag{2.2}
$$

provided the parameters satisfy the conditions $a + b = -3$ and $e + f = 1 + 2c$ with $\Re e(c) > -4$.

These results are closely related to Whipple's theorem [1, p. 6, 3.4.1]:

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; 1\right] = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma[\frac{1}{2}(e+a)]\Gamma[\frac{1}{2}(f+a)]\Gamma[\frac{1}{2}(e+b)]\Gamma[\frac{1}{2}(f+b)]}, \tag{2.3} $$

where $a+b=1$, $e+f=1+2c$ with $\Re e(c)>0$ and Lavoie's theorem [4, p. 270, Eqs. (3) and (4)] , and Choi's theorem [2, pp. 105-107, Eqs. (5),(6),(7) and (8)].

Proofs. Consider the following familiar transformation [1, p. 14, 3.2.1]:

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; 1\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} {}_3F_2\left[\begin{matrix} e-a, f-a, s \\ s+b, s+c \end{matrix} ; 1\right], \tag{2.4} $$

where $s=e+f-(a+b+c)$. As shown in [1, p. 16], Watson's theorem can be used to sum the series on the right of (2.4), provided $a+b=1$ and $e+f=1+2c$, and Whipple's theorem is obtained.

Similarly, when $a+b=5$ and $e+f=1+2c$, (1.1) can be used to evaluate the ${}_3F_2(1)$ on the right of (2.4);

$$ {}_3F_2\left[\begin{matrix} e-a, f-a, s \\ \frac{1}{2}\{(e-a)+(f-a)+1\}, 2s+4 \end{matrix} ; 1\right]. $$

Thus we obtain (2.1). Also, when $a+b=-3$ and $e+f=1+2c$, (1.2) can be used to evaluate the ${}_3F_2(1)$ on the right of (2.4);,

$$ {}_3F_2\left[\begin{matrix} e-a, f-a, s \\ \frac{1}{2}\{(e-a)+(f-a)+1\}, 2s-4 \end{matrix} ; 1\right]. $$

So, we obtain (2.2). □

3. Some Limiting Cases

Consider the following elementary relation [4, p.216]

$$ {}_4F_3\left[\begin{matrix} a+1, b+1, c+1, 1 \\ e+1, f+1, 2 \end{matrix} ; 1\right] = \frac{ef}{abc}\left({}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; 1\right] - 1\right). \tag{3.1} $$

Legendre's duplication formula [3] yields

$$\Gamma(\frac{1}{2})\Gamma(1+a) = 2^a\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+1),$$

since

$$2\ln 2 = \frac{\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}.$$

If the parameters are such that the ${}_3F_2(1)$ in (3.1) can be summed by (1.2), then letting $b \to 0$;

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}a+1, c+1, 1, 1\\ \frac{1}{2}(a+3), 2c-3, 2\end{matrix}; 1\right]\\ &= \lim_{b\to 0}\frac{(a+b+1)(c-2)}{abc}\left({}_3F_2\left[\begin{matrix}a, b, c\\ \frac{1}{2}(a+b+1), 2c-4\end{matrix}; 1\right] - 1\right),\end{aligned}$$

and using L'Hospital's rule yields the following evaluation

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}a+1, c+1, 1, 1\\ \frac{1}{2}(a+3), 2c-3, 2\end{matrix}; 1\right]\\ &= \frac{a(a+1)\Gamma(\frac{1}{2})\Gamma(c-\frac{3}{2})\Gamma(\frac{a}{2}+\frac{1}{2})}{2ac(c-1)\Gamma(\frac{a}{2}+1)\Gamma(c-\frac{a}{2}-2)\Gamma(c-2)}\\ &\times\left\{(c-1)\Gamma(c-\frac{a}{2}-\frac{5}{2}) + (a+1)\Gamma(c-\frac{a}{2}-\frac{7}{2})\right\}\\ &+\frac{a(a+1)}{ac(2c-a-5)} - \frac{c-2}{ac}\\ &+\frac{1}{ac}\left[\frac{c-2}{2}\left\{(a+1)\Big(\Psi(a,c) - \frac{2}{2c-a-7} - \frac{2}{2c-a-5}\Big) + 2\right\}\right.\\ &+\frac{3(c-2)}{2c-a-5}\left\{(a+1)\Big(\Psi(a,c) - \frac{2}{2c-a-7}\Big) + 2\right\}\\ &\left.+\frac{4(c-2)(a+2)}{(2c-a-5)(2c-a-7)(c-1)}\left\{(a+1)(\Psi(a,c)+1) + 2\right\}\right],\end{aligned}$$

provided $\Re e(2c-a) > 7$, (3.2)

where $\Psi(a,c) = 2\psi(a) - \psi(\frac{a}{2}) + \psi(c-\frac{3}{2}) - \psi(c-\frac{a}{2}-\frac{7}{2}) + \gamma$, γ is the Euler's constant and $\psi(z) = \frac{d}{dz}[\ln\Gamma(z)]$.

If the parameters are such that the ${}_3F_2(1)$ in (3.1) can be summed by (1.1), then letting $b \to 0$;

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a+1, c+1, 1, 1 \\ \frac{1}{2}(a+3), 2c+5, 2 \end{matrix}; 1\right] \\
&= \lim_{b\to 0} \frac{(a+b+1)(c+2)}{abc}\left({}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c+4 \end{matrix}; 1\right] - 1\right),
\end{aligned}
$$

and using L'Hospital's rule yields the following evaluation

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a+1, c+1, 1, 1 \\ \frac{1}{2}(a+3), 2c+5, 2 \end{matrix}; 1\right] \\
&= \frac{1}{ac}\left[\frac{-\Gamma(\frac{1}{2})a(a+1)\Gamma(c+\frac{5}{2})\Gamma(\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+1)\Gamma(c-\frac{a}{2}+2)\Gamma(c+2)} \times \left\{\Gamma(c-\frac{a}{2}+\frac{3}{2}) + \frac{(a+1)}{2(c+3)}\right\}\right. \\
&\quad + \frac{(c+2)\Gamma(\frac{a}{2}+1)(a+1)}{2\Gamma(\frac{a}{2}+\frac{1}{2})} \times \left\{\frac{2}{a+1} - \frac{4(2c-a+2)}{(2c-a+1)(2c-a+3)}\right. \\
&\quad \left. + 2\psi(a+2) + \psi(c+\frac{5}{2}) - \psi(\frac{a}{2}+\frac{3}{2}) - \psi(c-\frac{a}{2}+1) + \gamma\right\} \\
&\quad \left. + \frac{a(a+1)}{2c-a+3} \times \left\{1 + \frac{2(2c-a)}{2c-a+1} + \frac{4(a+2)}{(2c-a+1)(2c-a+3)}\right\} - (c+2)\right],
\end{aligned}
$$

provided $\Re e(2c-a) > -9$. (3.3)

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