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INJECTIVE HYPERBOLICITY OF PRODUCT DOMAIN

KI SEONG CHOI

ABSTRACT. Let $H_1(\triangle, M)$ be the family of all 1-1 holomorphic mappings of the unit disk $\triangle \subset \mathbf{C}$ into a complex manifold M. Following the method of Royden, Hahn introduces a new pseudo-differential metric S_M on M. The present paper is to study the product property of the metric S_M when M is given by the product of two domains D_1 and D_2 in the complex plane $\mathbf{C}$, thus investigating the hyperbolicity of the product domain $D_1 \times D_2$ with respect to S_M metric.

1. Introduction

Let M be a complex manifold of dimension n and $T(M)$ be the complex tangent bundle on M. We define a differential metric on M by an upper semicontinuous function

$$F_M : T(M) \rightarrow R^+ \cup \{0\}$$

such that for each $(z, \xi) \in T(M)$

$$F_M(z, \lambda\xi) = |\lambda| \; F_M(z, \xi), \;\; \lambda \in \mathbf{C}$$

and

$$F_M(z, \xi) > 0$$

for $\xi \neq 0$. We say that F_M is a pseudo-differential metric if it satisfies

$$F_M(z, \xi) \geqq 0$$

for $(z, \xi) \in T(M)$. M is said to be hyperbolic with respect to F_M if ,for each $z_0 \in M$, there exists a neighbourhood $U(z_0)$ and a constant $c > 0$ such that

$$F_M(z, \xi) \geqq c \; |\xi|$$

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for $z \in U(z_0)$ and $\xi \in T_z(M)$.

Let $H(\triangle, M)$ be the family of all holomorphic mappings of the unit disk $\triangle \subset \mathbf{C}$ into a complex manifold M. The Kobayashi-Royden metric (KR-metric) is defined by

$$K_M(z,\xi) = \inf\{|v| : \exists f \in H(\triangle, M) \text{ such that } f(0) = z, f'(0)v = \xi\}.$$

where the inf is taken over all holomorphic mappings of $\triangle$ into M such that $f(0) = z, f'(0)v = \xi$. In terms of a differential metric F_M, the KR-metric may also be written by

$$K_M(z,\xi) = \inf\left\{\frac{F_M(z,\xi)}{F_M(f(0), f'(0))} : \exists f \in H(\triangle, M) \text{ such that } f(0) = z\right\}$$

and the S-metric by

$$S_M(z,\xi) = \inf\left\{\frac{F_M(z,\xi)}{F_M(f(0), f'(0))} : \exists f \in H_1(\triangle, M) \text{ such that } f(0) = z\right\}$$

where $H_1(\triangle, M)$ denotes the class of all injective holomorphic maps of $\triangle$ into M and the inf is taken over all $f \in H_1(\triangle, M)$ such that$f(0) = z$.

The fact that the definition involves only injective holomorphic maps implies that S_M dominates the Kobayashi-Royden pseudo-differential metric K_M. However, it is shown in [1] that for $M = \mathbf{C} \setminus \{0\}$, S_M is a complete differential metric, while $K_M \equiv 0$.

From the work of Siu [5] and Minda [3], it is known that if M is a Riemann surface, then it is S-hyperbolic (i.e. hyperbolic with respect to S_M) unless it is covered by the complex plane or the extended complex plane.

The present paper is devoted to investigating the behaviour of $S_{D_1 \times D_2}$ for two domains D_1 and D_2 in the complex plane. We shall show in section 2 that if D_1 or D_2 is the unit disk in $\mathbf{C}$, then $S_{D_1 \times D_2} = K_{D_1 \times D_2}$. In section 3, it is shown that if D_2 is hyperbolic with respect to K_{D_2}, then $\triangle \times D_2$ is hyperbolic with respect to $S_{\triangle \times D_2}$.

2. S-metric in the unit disk

Let M and N be two complex manifolds. Let F_M and F_N be differential metrics of M and N, respectively. Then $\max(F_M, F_N)$ defines a differential metric on

$M \times N$. Therefore, for each $(z, w) \in M \times N$,

$$K_{M\times N}((z,\omega),(\xi,\eta)) = \inf\{\frac{\max\{F_M(z,\xi), F_N(\omega,\eta)\}}{\max\{F_M(f(0),f'(0)), F_N(g(0),g'(0))\}} : (f,g) \in H(\triangle, M\times N), (f,g)(0) = (z,\omega)\}$$

for $(\xi, \eta) \in T_z(M) \times T_w(N)$. By definition (See [1]),

$$S_{\triangle\times D}((z,\omega),(\xi,\eta)) = \inf\{\frac{\max\{F_\triangle(z,\xi), F_D(\omega,\eta)\}}{\max\{F_\triangle(f(0),f'(0)), F_D(g(0),g'(0))\}} : (f,g) \in H_1(\triangle, \triangle\times D), (f,g)(0) = (z,\omega)\}$$

It is easily shown by definition that $S_{\triangle\times D}$ dominates $K_{\triangle\times D}$. Applying the distance decreasing property of KR-metric for the projections

$$P : M \times N \to M,\ Q : M \times N \to N,$$

we have

$$K_{M\times N} \geqq \max\{K_M, K_N\}.$$

The opposite inequality also holds for the KR-metric. In fact, we have:

Lemma 1. *Let M and N be two complex manifolds. Then*

$$K_{M\times N} = \max\{K_M, K_N\}.$$

Proof. See [1].

The same result is, however, not true for the S-metric (See [1]). But if we consider the unit disk $\triangle$ in **C** and domain D in **C**, we obtain the following result.

Theorem 1. *If $\triangle$ is the unit disk in* **C** *and D is the domain in* **C***, then*

$$S_{\triangle\times D} = \max\{K_\triangle, K_D\} = K_{\triangle\times D}.$$

Proof. Let $F_\triangle$ and F_D be any differential metric of $\triangle$ and D, respectively. By the definition of $K_D(\omega, \eta)$, $\exists \psi \in H(\triangle, D)$ with $\psi(0) = \omega$ and, for any $\varepsilon > 0$,

$$\frac{F_D(\omega,\eta)}{F_D(\psi(0),\psi'(0))} \leqq K_D(\omega,\eta) + \varepsilon.$$

By the definition of $S_\triangle(z,\xi)$, $\exists\varphi \in S(\triangle,\triangle)$ with $\varphi(0)=z$ and the fact that both KR-metric and S-metric coincide on the unit disc $\triangle$ (see [1]), we have

$$\frac{F_\triangle(z,\xi)}{F_\triangle(\varphi(0),\varphi'(0))} \leqq S_\triangle(z,\xi)+\varepsilon = K_\triangle(z,\xi)+\varepsilon$$

Therefore, there exists an $h=(\varphi,\psi)\in S(\triangle,\triangle\times D)$ with $h(0)=(z,\omega)$ and

$$\begin{aligned} &S_{\triangle\times D}((z,\omega),(\xi,\eta)) \\ &= \inf\{\frac{\max\{F_\triangle(z,\xi),F_D(\omega,\eta)\}}{\max\{F_\triangle(f(0),f'(0)),F_D(g(0),g'(0))\}} : \\ &\qquad (f,g)\in S(\triangle,\triangle\times D),(f,g)(0)=(z,\omega)\} \\ &\leqq \frac{\max\{F_\triangle(z,\xi),F_D(\omega,\eta)\}}{\max\{F_\triangle(\varphi(0),\varphi'(0)),F_D(\psi(0),\psi'(0))\}} \\ &\leqq \max\left\{\frac{F_\triangle(z,\xi)}{F_\triangle(\varphi(0),\varphi'(0))},\frac{F_D(\omega,\eta)}{F_D(\psi(0),\psi'(0))}\right\} \\ &\leqq \max\{K_\triangle(z,\xi)+\varepsilon,K_D(\omega,\eta)+\varepsilon\}. \end{aligned}$$

This proves that

$$S_{\triangle\times D}((z,\omega),(\xi,\eta)) \leqq \max\{K_\triangle(z,\xi),K_D(\omega,\eta)\}$$

Since $S_{\triangle\times D}$ dominates $K_{\triangle\times D}$ by definition, we have

$$S_{\triangle\times D}((z,\omega),(\xi,\eta)) = \max\{K_\triangle(z,\xi),K_D(\omega,\eta)\} = K_{\triangle\times D}((z,\omega),(\xi,\eta)).$$

3. S-hyperbolicity in product domain

Definition 1. *Let M be a complex manifold furnished with a pseudo-differential metric F_M. M is said to be hyperbolic with respect to F_M if, for each $z_0\in M$, there exists a neighbourhood $U(z_0)$ and a constant $c>0$ such that*

$$F_M(z,\xi)\geqq c\ |\xi|$$

for $z\in U(z_0)$ and $\xi\in T_z(M)$.

Theorem 2. *Let M be any domain in $\mathbf{C}$ with $M \neq \mathbf{C}$. Then*

$$S_M(z,\xi) \geqq \frac{|\xi|}{4\delta(z)},$$

where $\delta(z)$ denotes the distance from $z \in M$ to $\mathbf{C} \setminus M$.

Proof. See [1].

From Theorem 2, $M = \mathbf{C} - \{0\}$ is S-hyperbolic (i.e. hyperbolic with respect to S_M). In the following example, we will show that $M \times N$ may not be S-hyperbolic (i.e. hyperbolic with respect to $S_{M\times N}$) even if M and N are S-hyperbolic.

Example 1. Let $(1,1) \in M \times M$ and $(1,\sqrt{2}) \in \mathbf{C}^2$. The function $h_n \in H(\triangle, M \times M)$ defined by

$$h_n(\lambda) = (f_n, g_n)(\lambda) = (e^{n\lambda},\ e^{\sqrt{2}n\lambda})$$

is injective. Indeed, if λ_1 and λ_2 are distinct points of $\triangle$ such that $h_n(\lambda_1) = h_n(\lambda_2)$, then

$$n(\lambda_2 - \lambda_1) = 2\pi k_1 i,\ \ k_1 \in \mathbf{Z}$$

$$n(\lambda_2 - \lambda_1)\sqrt{2} = 2\pi k_2 i,\ \ k_2 \in \mathbf{Z}.$$

Therefore, we get $\sqrt{2} = \frac{k_2}{k_1} \in \mathbf{Q}$, which is a contradiction. Since,

$$h_n(0) = (f_n(0), g_n(0)) = (1,1)$$

$$h_n'(0) = (f_n'(0), g_n'(0)) = (n, \sqrt{2}n),$$

$$S_{M\times M}\{(1,1),(1,\sqrt{2})\} \leqq \frac{\max\{F_M(1,1), F_M(1,\sqrt{2})\}}{\max\{F_M(1,n), F_M(1,\sqrt{2}n)\}} = \frac{1}{n}.$$

Since n is arbitrary, $S_{M\times M}\{(1,1),(1,\sqrt{2})\} = 0$. i.e. $M \times M$ is not S-hyperbolic.

Theorem 3. *Let D be a domain in $\mathbf{C}$. If D is hyperbolic with respect to K_D, then $\triangle \times D$ is S-hyperbolic.*

Proof. Since $\triangle$ is S-hyperbolic, there exists a neighbourhood $U(z_0)$ of z_0 and a constant $c_1 > 0$ such that if $z \in U(z_0)$, then $S_\triangle(z,\xi) \geqq c_1\ |\xi|$. We can choose $\varepsilon_1 > 0$ so that $\{z \in \mathbf{C} |\ |z - z_0| < \varepsilon_1\} \subset U(z_0)$.

Since D is hyperbolic with respect to K_D, there exists a neighbourhood $U(\omega_0)$ of ω_0 and a constant $c_2 > 0$ such that if $\omega \in U(\omega_0)$, then $K_D(\omega, \eta) \geqq c_2 \, |\eta|$. We can choose $\varepsilon_2 > 0$ so that $\{\omega \in \mathbf{C} | \; |\omega - \omega_0| < \varepsilon_2\} \subset U(\omega_0)$.

Put $c = \min\{c_1, c_2\}$. Then, by Theorem 1,

$$\begin{aligned} S_{\triangle \times D}((z,\omega),(\xi,\eta)) &= \max\{K_\triangle(z,\xi), K_D(\omega,\eta)\} \\ &\geqq \max\{c_1 \; |\xi|, \; c_2 \; |\eta|\} \\ &\geqq c \; \max\{|\xi|, |\eta|\} \end{aligned}$$

for all (z,ω) in $\{(z,\omega) : \; |z - z_0| < \varepsilon_1, \; |\omega - \omega_0| < \varepsilon_2\}$.

Hence $\triangle \times D$ is S-hyperbolic.

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Department of Mathematics, Konyang University, Nonsan 320-800, Korea.