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ON REGULAR–QUASICONFORMAL MAPPINGS

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A C^∞ manifold is a pair (M, C) where

a) M is a Hausdorff topological space such that every point $x \in M$ has a neighborhood homeomorphic to an open subset of R^n.

b) C is a collection of these homeomorphisms whose domains cover M. If $\phi, \psi \in C$ then $\phi \circ \psi^{-1}$ is C^∞.

c) C is maximal with respect to (b).

A Riemannian manifold is a C^∞ manifold (M, C) for which is given at each $p \in M$ a positive definite symmetric form $<, >$ on M_p, the tangent space at p, and this assignment is C^∞ in the sense that for any coordinate system $(x_1, x_2, \cdots, x_n)$ the functions $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \rangle$ are C^∞. Such assignment is called a Riemann Metric on M. Let M and N be manifolds. A map $f : M \to N$ is measurable if $f^{-1}(G)$ of each open set $G \subset N$ is a Borel set. A measurable real-valued map $f : M \to N$ is called a Baire function.

Let M and N be a manifold and $f : M \to N$ a continuous map which is differentiable almost everywhere. Then the derivative map $Df : TM \to TN$ is measurable where TM and TN are tangent bundles of M and N respectively. If M and N are Riemannian manifolds, then $\|Df\|$ and $\det Df$ are Baire functions.

The following fact is well-known ([1]).

Theorem 1. *With each Riemannian manifold M, we can asociate a unique Borel measure τ_M so that the following conditions are satisfied:*

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(a) If N is an open Riemannian submanifold of a Riemannian manifold M, then $\tau_M(E) = \tau_N(E)$ for all Borel sets $E \subset N$.
(b) If $f : M \to N$ is a C^1-diffeomorphism of Riemannian manifolds M and N, then $\tau_M(f(E)) = \int_E J_f \, d\tau_M$ for all Borel sets $E \subset M$.
(c) If $M = R^n, n = 1, 2, \cdots,$ τ_M is the Lebesgue measure.

We define the extremal lenght of the family Γ of rectifiable curves on the Riemannian manifold M.

Definition 1. For a nonnegative Baire function ρ on M, we define the ρ-length of a curve γ by

$$L(\gamma, \rho) = \int_\gamma \rho \, ds$$

where S is the Riemannian arc length and $L(\gamma, \rho)$ can be ∞.

Definition 2. For each positive real number p, we define the p-vloume by

$$V_p(M, \rho) = \int_M \rho^p d\tau$$

where τ is the Lebesgue measure.

Definition 3. The minimal length of Γ a family of curves on M is defined by

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho)$$

and the extremal length of Γ a family of curves on M is defined to be

$$\lambda_M(\gamma) = \sup_\rho \frac{L(\Gamma, \rho)^p}{V_p(M, \rho)}$$

where ρ is a Baire function which is not identically equal to zero([2]).

Following Theorems are generalized verson of the corresponding theorems for the extremal length of the Riemann Surface ([2]).

Theorem 2. *If each $\gamma \in \Gamma$ contains a γ' in Γ', then $\lambda(\Gamma) \geq \lambda(\Gamma')$.*

Theorem 3. *Let Γ_1 and Γ_2 be families of curves in G_1,G_2 disjoint open sets in M.*
(1) If each $\gamma \in \Gamma^*$contains $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ such that $\gamma_1 \subset \gamma$ and $\gamma_2 \subset \gamma$, then $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$.
(2) If each $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ contains $\gamma \in \Gamma$, then

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Now we are ready to define quasiconformal mappings on a Riemannian manifold.

Definition 4. Let M and N be Riemannian manifolds of dimension n. A homeomorphism $f : M \to N$ is called a quasiconformal map if there exists a constant K, $1 \leq K < \infty$ such that

$$\frac{1}{K}\lambda(\Gamma) \leq \lambda(\Gamma^*) \leq K\lambda(\Gamma)$$

where Γ is a family of curves on M and $\Gamma^* = f(\Gamma)$, $\lambda(\Gamma)$, and $\lambda(\Gamma^*)$ are extremal lengths of Γ and Γ^* respectively.

Definition 5. Let M be a Riemannian manifold and U_i, U_j be coordinate neighborhoods with nonempty intersection. If $\phi_i : U_i \to R^n$, $\phi_j : U_j \to R^n$ are homeomorphisms such that $\phi_j \circ \phi_i^{-1}$ maps every family of curves with extremal length zero onto a family of curves with extremal length zero, then M is called a quasiconformal manifold.

Definition 6. Let M and N be quasiconformal manifolds. If a homeomorphism $f : M \to N$ maps every family of curves with extremal length zero on M onto a family of curves with extremal length zero on N, then f is called a regular homeomorphism.

The following Theorem is given in [2].

Theorem 4. *Let D and D^* be two domains in the Euclidean n-space R^n. A homeomorphism $g : D \to D^*$ is a quasiconformal mapping if and only if g maps every family of curves with extremal length zero onto a family of curves with extremal length zero.*

We extend this theorem to the case of a regular-homeomorphism.

Theorem 5. *Let M and N be quasiconformal manifolds. A homeomorphism $f : M \to N$ is a quasiconformal mapping if and only if f is a regular-homeomorphism.*

Proof. Let ϕ_i be a homeomorphism mapping a coordinate neighborhood U_i into R^n, and ϕ_j be a homeomorphism mapping a coordinate neighborhood U_j into R^n. Then $\phi_i(U_i)$ and $\phi_j(U_j)$ are domains in R^n. Set $\phi_i(U_i) = D, \phi_j(U_j) = D^*$ and apply the Theorem 4 to obtain the desired result.

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