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EXAMPLE AND COUNTEREXAMPLES IN DOUBLE INTEGRAL AND ITERATED INTEGRAL

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I. Multiple Riemann Integrals

[1] Show that $\int_0^1 \left[\int_0^1 f(x,y)dy\right] dx = \int_0^1 \left[\int_0^1 f(x,y)dx\right]$

Counterexample : If p_k denotes the k-th prime number, let $S(p_k) = \left\{\left(\frac{n}{p_k}, \frac{n}{p_k}\right); n = 1, 2, \cdots, p_k - 1\right\}$, let $S = \cup_{k=1}^{\infty} S(p_k)$, and let $Q = [0,1] \times [0,1]$. Define f on Q as follows ; $f(x,y) = 0$ $(x,y) \in S$, $f(x,y) = 1$ $(x,y) \in Q - S$.

Solution : Let $y \in [0,1]$.

case i) $y \neq \frac{n}{p_k}$ for all $n < p_k$ and for anll $k \in N$.

In this case $(x,y) \in Q - S$ for all $x \in [0,1]$. Hence $f(x,y) = 1$ so that $\int_0^1 f(x,y)dx = 1$.

case ii) $y = \frac{n}{p_k}$ for some $k \in N$ and $n < p_k$.

Since $S(p_1), S(p_2), \cdots$ are disjoint, there are finitely many $x \in [0,1]$ namely $x = \frac{1}{p_k}, \frac{2}{p_k}, \cdots, \frac{p_k - 1}{p_k}$ for which $(x,y) \in S$. Hence $f(x,y) = 1$ for all but finitely many $x \in [0,1]$, so that $\int_0^1 f(x,y)dx = 1$.

Consequently, we have $\int_0^1 f(x,y)dx = 1$ for all $y \in [0,1]$ and $\int_0^1 \int_0^1 f(x,y)dxdy = 1$. Symmetrically $\int_0^1 \int_0^1 f(x,y)dydx = 1$.

To show that S is dense in Q, let $(x,y) \in Q$ and let $\epsilon > 0$ be given.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

There are natural numbers ϵ, r, s such that $\epsilon < s, r < s$ and $\left|\left(\frac{q}{s}, \frac{r}{s}\right) - (x,y)\right| < \epsilon$. Choose a $k \in N$ such that $\frac{1}{p_k} < \epsilon, s < p_k$. Set $n = \left[\frac{qp_k}{s}\right], m = \left[\frac{rp_k}{s}\right]$ then $\left|\left(\frac{n}{p_k}, \frac{m}{p_k}\right) - \left(\frac{q}{s}, \frac{r}{s}\right)\right| < 2\epsilon$.

Thus $\left|\left(\frac{n}{p_k}, \frac{m}{p_k}\right) - (x,y)\right| < 3\epsilon$.

It is clear that $Q - S$ is dense in Q.

Since every rectangle $R \subset Q$ contains a point of S as well as a point of $Q - S$, $\bar{\int}\int_Q f = 1$, $\underline{\int\int}_Q f = 0$. Hence $\int_Q f(x,y)d(x,y)$ does not exist.

[2] Show that $\int_0^1 f(x,y)dx = \int_0^1 \left[\int_0^1 f(x,y)dx\right] dy = \int_Q f(x,y)d(x,y) = 0$ but that $\int_0^1 f(x,y)dy$ does not exist for ratimal x.

Counterexample : Define f on the square $Q = [0,1] \times [0,1]$ as follows :

$$f(x,y) = \begin{cases} 0 & \text{if at least one of } x, y \text{ is irrational,} \\ 1 & \text{if } y \text{ is rational and } x = \frac{m}{n}, \\ & \text{where } m \text{ and } n \text{ are relatively prime integers, } n > 0 \end{cases}$$

Solution : Let $y \in [0,1]$.

case i) y is irrational.

$$\int_0^1 f(x,y)dx = \int_0^1 0dx = 0$$

case ii) y is rational.

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

Let $\epsilon > 0$ be given. Choose $N \in N$ $s.t \frac{1}{N} < \epsilon$. Put $S = \left\{x \in [0,1] | x = \frac{m}{n}, gcd(m,n), n < N\right\}$. Then S is a finite set.

Take a $\delta > 0$ such that $\delta \sum_{\frac{m}{n} \in S} \frac{1}{n} < \epsilon$.

If $\Delta = \{\Delta_1, \Delta_2, \cdots, \Delta_s\}$ is a partition of $[0,1]$ into finite closed intervals, $\|\Delta\| < \delta$ and ξ is a choice function on Δ then

$$\begin{aligned}
0 &\le S_{\Delta\xi}(f(\cdot, y)) \\
&= \sum_i f(\xi_i, y)\Delta_i x \\
&= \sum_{\substack{\xi_i \ is \\ irrational}} f(\xi_i, y)\Delta_i x + \sum_{\xi_i \in S} f(\xi_i, y)\Delta_i x + \sum_{\substack{\xi_i \ is \ rational \\ \xi_i = \frac{m}{n} \\ n \ge N}} f(\xi_i, y)\Delta_i x \\
&\le 0 + \sum_{\frac{m}{n} \in S} \frac{1}{n}\Delta_i x + \sum_{\substack{\xi_i \ is \ rational \\ \xi_i = \frac{m}{n} \\ n \ge N}} \frac{1}{n}\Delta_i x \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

where $\Delta_i x$ is the lingth of the intervals Δ_i. Hence $\int_0^1 f(x,y)dx = 0$.

Let $\epsilon > 0, N \in N, S, \delta$ be the same as above.

If $\Delta = \{\Delta_{ij} | i = 1, 2, \cdots, s,\ j = 1, 2, \cdots, t\}$ is a partition of Q into finite rectangles $\|\Delta\| = \max_{i,j} dia\Delta_{ij} < \delta$, and (ξ, η) is a choice function on Δ then

$$\begin{aligned}
0 \le S_{\Delta\xi\eta}(f) &= \sum_{ij} f(\xi_{ij}, \eta_{ij})\Delta_{ij}A \\
&= \sum_{\substack{at\ least \\ one\ of\ \xi_{ij}, \eta_{ij} \\ is\ irrational}} f(\xi_{ij}, \eta_{ij})\Delta_{ij}A + \sum_{\substack{\eta_{ij}, irrational \\ \xi_{ij} \in S}} f(\xi_{ij}, \eta_{ij})\Delta_{ij}A \\
&+ \sum_{\substack{\eta_{ij}, irrational \\ \xi_{ij}, irrational \\ \xi_{ij} \notin S}} f(\xi_{ij}, \eta_{ij})\Delta_{ij}A \le 0 + \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Hence $\int_Q f(x,y)d(x,y) = 0$.

For a fixed rational $x = \frac{m}{n}$, $gcd(m,n) = 1$.

$$f(x,y) = \begin{cases} 0 & \text{if } y \text{ is irrational} \\ \frac{1}{n} & \text{if } y \text{ is rational} \end{cases}$$

$$\overline{\int_0^1} f(x,y)dy = \frac{1}{n}, \quad \underline{\int_0^1} f(x,y)dy = 0.$$

Hence $\int_0^1 f(x,y)dy$ does not exist for rational x.

II. Multiple Lebesgue Integrals

[1] Show that $f \in L(R^2)$ by the monotone convergence theorem and calculate the double integral $\int_{R^2} f(x,y)d(x,y)$

Example : For fixed $c, 0 < c < 1$, define f on R^2 as follows;

$$f(x,y) = \begin{cases} (1-y)^c/(x-y)^c & \text{if } 0 \le y < x, \ 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution : Define $f_n(x,y) = \begin{cases} \frac{(1-y)^2}{(x-y)^c} & \text{if } 0 \le y < x - \frac{1}{n}, \ 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Then f_n is Riemann integrable and

$$\int\int_{R^2} f_n(x,y)d(x,y) = \int_0^{1-\frac{1}{n}} \int_{\frac{1}{n}}^{1-v} \frac{(1-v)^c}{u^c} dudv$$
$$= \frac{1}{1-c}\left(\left(1-\frac{1}{n}\right) - \frac{1}{2}\left(1-\frac{1}{n}\right)^2\right)$$
$$+ \frac{1}{1-c}\cdot\frac{1}{n^{1-c}}\cdot\left(\frac{1}{1+c}\cdot\frac{1}{n^{1+c}}\right) \to_{n\to\inf} \frac{1}{2(1-c)}$$

where $x-y=u,\ y=v,\ x=u+v,\ y=v.$

By the monotone convergence theorem, $\int_{R^2} f(x,y)d(x,y) = \frac{1}{2(1-c)} < \infty$.

Hence $f \in L(R^2)$ and $\int_{R^2} f(x,y)d(x,y) = \frac{1}{2(1-c)}$.

[2] Show the both iterated integrals $\int_R \left[\int_R f(x,y)dx\right] dy$ and $\int_R \left[\int_R f(x,y)dy\right] dx$ exist and are equal, but that the double integral of f over R^2 does not exist.

Counterexample: Let $f(x,y) = e^{-xy}\sin x \sin y$ if $x \geq 0, y \geq 0$ and let $f(x,y) = 0$ otherwise.

Solution : $|f(x,y)| \leq e^{-xy}$ and for each y, the function $x \mapsto e^{-xy}$ is integrable by the bounded convergence theorem with integral $\frac{1}{y}$ if $y > 0$. Hence $x \mapsto f(x,y)$ is integrable. If $y = c$ then $f(x,y) = 0$ for all x so that $f(x,y)$ is integrable with integral 0. Hence for all $y \geq 0$, and calculate the double integral,

$$\int_R \left[\int_R f(x,y)dx\right] dy = \int_R \left[\int_R f(x,y)dy\right] dx = \frac{\pi}{2}.$$

By the Tonelli-Hobson test, $\int_{R^2} f(x,y)d(x,y)$ does not exist.

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