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PSEUDOLINDELÖF SPACES AND HEWITT REALCOMPACTIFICATION OF PRODUCTS

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ABSTRACT. The concept of pseudoLindelöf spaces is introduced. It is shown that the followings are equivalent:

(a) for any two disjoint zero-sets in X, at least one of them is Lindelöf,
(b) $|\upsilon X - X| \leq 1$, and
(c) for any space T with $X \subseteq T$, there is an embedding $f : \upsilon X \to \upsilon T$ such that $f(x) = x$ for all $x \in X$ and that if $X \times Y$ is a z-embedded pseudoLindelöf subspace of $\upsilon X \times \upsilon Y$, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.

1. Introduction

For any Tychonoff space X, βX denotes the Stone-Čech comactification of X and υX denotes the Hewitt realcompactification of X. Glicksberg [5] showed that for any infinite spaces X and Y, $\beta X \times \beta Y = \beta(X \times Y)$ if and only if $X \times Y$ is pseudocompact. An important open question in the theory of Hewitt realcompactifications of Tychonoff spaces concerns when the equality $\upsilon X \times \upsilon Y = \upsilon(X \times Y)$ is valid (cf. [6]). Comfort [3] showed that if $X \times Y$ is C^*-embedded in $\upsilon X \times \upsilon Y$, then $\upsilon X \times \upsilon Y = \upsilon(X \times Y)$ and that if card(X) or card(Y) is non-measurable and $X \times Y$ is C^*-embedded in $X \times \beta Y$, then $\upsilon X \times \upsilon Y = \upsilon(X \times Y)$. McArthur [7] has shown that $X \times Y$ is C^*-embedded in $X \times \beta Y$ if and only if the projection $\pi_X : X \times Y \to X$ is z-closed.

In this paper, we introduce the concept of pseudoLindelöf spaces and show that for a pseudoLindelöf space X, the followings are equivalent(cf. Theorem 2.6):

(a) $|\upsilon X - X| \leq 1$.
(b) For any two disjoint zero-sets in X, at least one of them is Lindelöf.

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(c) For any space T with $X \subseteq T$, $vX \subseteq vT$.

Moreover, we will show that if $X \times Y$ is a z-embedded pseudoLindelöf subspace of $vX \times vY$, then $v(X \times Y) = vX \times vY$ and that if $X \times Y$ is a pseudoLindelöf space such that card (X) or card (Y) is non-measurable and X is a P-space, then $v(X \times Y) = vX \times vY$ if and only if the projection $\pi_X : X \times Y \to X$ is z-closed. For the terminology, we refer to Gillman-Jerison [4] and Porter-Woods [8].

2. PseudoLindelöf spaces

All topological spaces discussed in this paper are assumed to be Tychonoff spaces. For a space X, $C(X)$ denotes the ring of all continuous real-valued functions on X and $C^*(X)$ denotes the subring of bounded functions. A subspace S of a space X is said to be C-mbedded in X if every function in $C(S)$ extends to a function in $C(X)$. C^*-embedding is defined analogously. For a space X, βX denotes the Stone-Čech compactification of X, which is characterized as a compact space in which X is densely C^*-embedded and vX denotes the Hewitt realcompactification of X, which is characterized as a realcompact space in which X is densely C-embedded. Both of the spaces βX and vX are unique up to a homeomorphism which extends the identity on X.

Definition 2.1. A space X is called *pseudoLindelöf* if vX is Lindelöf.

Every Lindelöf space is pseudoLindelöf. A separable space X is pseudoLindelöf if and only if every base is complete (cf. [2]). If X is a pseudocompact space, then $vX = \beta X$ and hence X is a pseudoLindelöf space. PseudoLindelöf spaces are not productive and a C-embedded subspace of a pseudoLindelöf space is again pseudoLindelöf.

Example 2.2. Let w_1 be the first uncountable ordinal and $D(w_1)$ the discrete space of cardinality w_1. Let $S = D(w_1) \cup \{p\}$, topologized as follows. Each point of $D(w_1)$ is isolated, and a subset G of S that contains p is open in S if and only if $|S - G| \leq \aleph_0$. Then S is a zero-dimensional Hausdorff space and hence Tychonoff. Let $N^* = N \cup \{w\}$ denote the one-point compactification of N and $X = S \times N^* - \{(p, w)\}$. Then X is called *Dieudoneé plank* and $vX = S \times N^*$ (cf. [8]). Since S is Lindelöf, X is pseudoLindelöf. But X is neither Lindelöf nor pseudocompact.

For a space X and $f \in C(X), f^{-1}(0)$, denoted by $Z(f)$, is called a *zero-set* in X and $X - f^{-1}(0)$ is called a *cozero-set* in X. It is well-known that for any $f \in C(X)$, $\mathrm{cl}_{vX}(Z(f)) = Z(f^v)$, where f^v is the extension of f to vX (cf. [4]).

Lemma 2.3. *Let X be a pseudoLindelöf space and A a zero-set in X. Then A is closed in vX if and only if A is Lindelöf.*

Proof. Suppose that A is Lindelöf. Let $p \in vX - A$. If $p \in X$, then $p \notin \mathrm{cl}_{vX}(A)$. Suppose that $p \notin X$. For any $a \in A$, there is a cozero-set neighborhood C_a of a in vX such that $p \notin C_a$. Since A is Lindelöf, there is a countable subfamily $\mathcal{U}$ of $\{C_a : a \in A\}$ with $A \subseteq \bigcup \mathcal{U}$. Let $C = \bigcup \mathcal{U}$ and $Z = vX - C$. Then $p \in Z$, Z is a zero-set in vX and $A \cap Z = \emptyset$.

Since X is C^*-embedded in vX, $\mathrm{cl}_{vX}(A) \cap \mathrm{cl}_{vX}(Z \cap X) = \emptyset$ and since $\mathrm{cl}_{vX}(Z \cap X) = Z$, $p \notin \mathrm{cl}_{vX}(A)$ and hence $A = \mathrm{cl}_{vX}(A)$. The converse is trivial. □

For a space X, $Z(X)$ denotes the set of zero-sets in X. A non-empty subfamily $\mathcal{F}$ of $Z(X)$ is called a *z-filter* on X if

(i) $\emptyset \notin \mathcal{F}$,
(ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
(iii) if $Z \in \mathcal{F}$ and $Z \subseteq A \in Z(X)$, then $A \in \mathcal{F}$.

A maximal z-filter on X is called a *z-ultrafilter* on X and a z-ultrafilter on X is called *real* if it has the countable intersection property.

Definition 2.4. Let X be a dense subspace of a space T, $\mathcal{F}$ a z-filter on X and $p \in T$. Then $\mathcal{F}$ *converges* to the limit p if every neighborhoods of p in T contains a member of $\mathcal{F}$.

Lemma 2.5 [4]. *Let X be a dense subspace of T. The following are equivalent:*

(a) *X is C-embedded in T.*
(b) *Every point of T is the limit of a unique real z-ultrafilter on X.*
(c) *$vX = vT$, that is, there is a homeomorphism $h : vX \to vT$ such that $h(x) = x$ for all $x \in X$.*

For any space X and $\mathcal{F} \subseteq 2^X$, $\bigcap \mathrm{cl}_X(\mathcal{F})$ denotes the set $\bigcap\{\mathrm{cl}_X(F) : F \in \mathcal{F}\}$.

Theorem 2.6. *Let X be a pseudoLindelöf space. Then the following are equivalent:*

(a) *For any two disjoint zero-sets in X, at least one of them is Lindelöf.*

(b) $|\upsilon X - X| \leq 1$.

(c) *For any space T with $X \subseteq T$, there is an embedding $f : \upsilon X \to \upsilon T$ such that $f(x) = x$ for all $x \in X$.*

Proof. (a) $\Rightarrow$ (b) Suppose that $2 \leq |\upsilon X - X|$. Pick $p, q \in \upsilon X - X$ with $p \neq q$. By Lemma 2.5, there are z-ultrafilters $\mathcal{A}^p$ and $\mathcal{A}^q$ on X such that p (q, resp.) is a limit of $\mathcal{A}^p$ ($\mathcal{A}^q$, resp.) and since $p \neq q, \mathcal{A}^p \neq \mathcal{A}^q$ and hence there are disjoint zero-sets A, B in X such that $A \in \mathcal{A}^p$ and $B \in \mathcal{A}^q$. Note that $p \in \mathrm{cl}_{\upsilon X}(A)$ and $q \in \mathrm{cl}_{\upsilon X}(B)$. We may assume that A is Lindelöf. By Lemma 2.3, A is closed in υX and hence $p \notin A = \mathrm{cl}_{\upsilon X}(A)$. This is a contradiction.

(b) $\Rightarrow$ (a) Suppose that $\upsilon\ X - X = \{p\}$. Take any disjoint zero-sets A and B in X. Then $\mathrm{cl}_{\upsilon X}(A) \cap \mathrm{cl}_{\upsilon X}(B) = \emptyset$ and hence $p \notin \mathrm{cl}_{\upsilon X}(A)$ or $p \notin \mathrm{cl}_{\upsilon X}(B)$. So $\mathrm{cl}_{\upsilon X}(A) = A$ or $\mathrm{cl}_{\upsilon X}(B) = B$. Hence A is Lindelöf or B is Lindelöf.

(b) $\Rightarrow$ (c) Suppose that $\upsilon X - X = \{p\}$. Take any space T with $X \subseteq T$. Then there is a continuous map $f : \upsilon X \to \upsilon T$ such that $f(x) = x$ for all $x \in X$ (cf. [4]). Let $q = f(p)$ and $Y = X \cup \{q\}$. Then X is a dense subspace of Y. Let g be the corestriction of f to Y, then $g : \upsilon X \to Y$ is one-to-one, onto, and continuous.

We will show that g is a homeomorphism. Since υX is Lindelöf, Y is Lindelöf and so Y is a realcompactification of X. Since X is C-embedded in υX, there is a unique real z-ultrafilter $\mathcal{A}^p$ on X such that p is a limit point of $\mathcal{A}^p$. Take any neighborhood V of q in Y. Then $g^{-1}(V)$ is a neighborhood of p in υX. Since p is a limit point of $\mathcal{A}^p$, there is $A \in \mathcal{A}^p$ with $A \subseteq g^{-1}(V)$ and so $g(A) = A \subseteq V$. Hence q is a limit point of $\mathcal{A}^p$.

Suppose that $\mathcal{F}$ is a real z-ultrafilter on X such that q is a limit point of $\mathcal{F}$. If $\bigcap \mathcal{F} \neq \emptyset$, then $\bigcap \mathcal{F} = \{x\}$ for some $x \in X$. Since $x \neq q$, there are disjoint zero-set neighborhoods C of x and D of q in Y. Since $C \cap X \in \mathcal{F}$ and $(C \cap X) \cap (D \cap X) = \emptyset, q$ is not a limit point of $\mathcal{F}$. Hence $\bigcap \mathcal{F} = \emptyset$. Since $\mathcal{F}$ is real, $\mathrm{cl}_{\upsilon X}(\mathcal{F}) = \{\mathrm{cl}_{\upsilon X}(F) : F \in \mathcal{F}\}$ is a z-filter on υX with the countable intersection property and since υX is Lindelöf, $\bigcap \mathrm{cl}_{\upsilon X}(\mathcal{F}) \neq \emptyset$. Hence $\bigcap \mathrm{cl}_{\upsilon X}(\mathcal{F}) = \{p\}$. Let $F \in \mathcal{F}$ and suppose that $F \notin \mathcal{A}^p$. Then there is $B \in \mathcal{A}^p$ with $F \cap B = \emptyset$ and so $\mathrm{cl}_{\upsilon X}(F) \cap \mathrm{cl}_{\upsilon X}(B) = \emptyset$. Since $p \in \mathrm{cl}_{\upsilon X}(B)$, this is a contradiction. Hence $\mathcal{F} = \mathcal{A}^p$. Thus every point of Y is the limit of a unique real z-ultrafilter on X. By Lemma 2.5, X is C-embedded in Y and therefore, g is a homeomorphism.

(c) $\Rightarrow$ (b) Suppose that there are $p, q \in \upsilon X - X$ with $p \neq q$. Let $Y = X \cup \{p, q\}$ and $R = \{(x, x) : x \in Y\} \cup \{(p, q), (q, p)\}$. Then R is an equivalence relation on Y.

Let K be the quotient space Y/R and $\pi : Y \to K$ the quotient map. Clearly, K is a Tychonoff space and X is a dense subspace of K. By the assumption, there is an embedding $f : vX \to vK$ such that $f(x) = x$ for all $x \in X$. Since X is dense in Y and $(j \circ \pi)|_X = f|_X$, $j \circ \pi = f|_Y$, where $j : K \to vK$ is the dense embedding. Since f is one-to-one and $p \neq q$, $f(p) \neq f(q)$ but $j(\pi(p)) = \pi(p) = [p] = [q] = \pi(q) = j(\pi(q))$. This is a contradiction. □

A subspace Y of a space X is z-embedded in X if for any zero-set A in Y, there is a zero-set Z in X with $A = Z \cap Y$. It is known that a space X is z-embedded in each of its compactifications if and only if for any two disjoint zero-sets in X, one of them is Lindelöf (cf. [1]). Using this, we have the following:

Corollary 2.7. *Let X be a pseudoLindelöf space. Then $|vX - X| \leq 1$ if and only if X is z-embedded in each of its compactifications.*

Recall that a space X is called *quasi-F* if every dense cozero-set in X is C^*-embedded in X, equivalently, every dense z-embedded subspace of X is C^*-embedded in X.

Corollary 2.8. *Let X be a pseudoLindelöf space. If $|vX - X| \leq 1$, then βX is the unique compactification of X which is quasi-F.*

3. Hewitt realcompactification of a product space.

The equality $v(X \times Y) = vX \times vY$ is to be interpreted to mean that $X \times Y$ is C-embedded in $vX \times vY$.

Lemma 3.1 [3]. *Let X and Y be spaces. Then $v(X \times Y) = vX \times vY$ if and only if $X \times Y$ is C^*-embedded in $vX \times vY$.*

Theorem 3.2. *Let X and Y be spaces such that $X \times Y$ is a pseudoLindelöf spaces. Then $X \times Y$ is z-embedded in $vX \times vY$ if and only if $v(X \times Y) = vX \times vY$.*

Proof. Suppose that $X \times Y$ is z-embedded in $vX \times vY$. Since $vX \times vY$ is a realcompact space, there is a continuous map $f : v(X \times Y) \to vX \times vY$ such that $f((x, y)) = (x, y)$ for all $(x, y) \in X \times Y$. Take any $(p, q) \in (vX \times vY) - (X \times Y)$. Then there is a z-ultrafilter $\mathcal{A}^p$ on X ($\mathcal{A}^q$ on Y, resp.) such that p (q, resp.) is the limit of $\mathcal{A}^p$ ($\mathcal{A}^q$, resp.) and hence

$$\{(p,q)\} = (\bigcap \text{cl}_{vX}(\mathcal{A}^p)) \times (\bigcap \text{cl}_{vY}(\mathcal{A}^q)).$$

Let $\mathcal{F}$ be the z-filter on $X \times Y$ generated by $\{A \times B : A \in \mathcal{A}^p, B \in \mathcal{A}^q\}$. Then $\mathcal{F}$ has the countable intersection property and $\bigcap \mathcal{F} = \emptyset$. Since $v(X \times Y)$ is Lindelöf, $\bigcap \text{cl}_{v(X\times Y)}(\mathcal{F}) \neq \emptyset$. Pick $x \in \bigcap \text{cl}_{v(X\times Y)}(\mathcal{F})$. Then for any $A \in \mathcal{A}^p$ and $B \in \mathcal{A}^q$,

$$\begin{aligned} f(x) \in f(\text{cl}_{v(X\times Y)}(A \times B)) &\subseteq \text{cl}_{(vX\times vY)}(f(A \times B)) \\ &= \text{cl}_{(vX\times vY)}(A \times B) \\ &= \text{cl}_{vX}(A) \times \text{cl}_{vY}(B). \end{aligned}$$

Hence $f(x) \in (\bigcap \text{cl}_{vX}(\mathcal{A}^p)) \times (\bigcap \text{cl}_{vY}(\mathcal{A}^q))$. So $f(x) = (p,q)$. Thus f is onto.

Take any two zero-sets A and B in $X \times Y$ with $A \cap B = \emptyset$. Then there are zero-sets C and D in $vX \times vY$ with $A = C \cap (X \times Y)$ and $B = D \cap (X \times Y)$. Since $f^{-1}(C \cap D) \cap (X \times Y) = \emptyset$ and $f^{-1}(C \cap D)$ is a zero-set in $v(X \times Y)$, $f^{-1}(C \cap D) = \emptyset$ and since f is onto, $C \cap D = \emptyset$. So $\text{cl}_{(vX\times vY)}(A) \cap \text{cl}_{(vX\times vY)}(B) = \emptyset$. By Urysohn's extension theorem, $X \times Y$ is C^*-embedded in $vX \times vY$. By Lemma 3.1, $v(X \times Y) = vX \times vY$. The converse is trivial. □

Definition 3.3. Let X and Y be spaces. Then $f : X \to Y$ is called *z-closed* if for any zero-set Z in X, $f(Z)$ is closed in Y.

Recall that a space X is called a *P-space* if every G_δ-set in X is open in X.

Remark 3.4. (1) If the projecton $\pi_X : X \times Y \to X$ is z-closed, then X is a P-space or Y is a pseudocompact space (cf. [8]).
(2) The projecton $\pi_X : X \times Y \to X$ is z-closed if and only if $X \times Y$ is C^*-embedded in $X \times \beta Y$ (cf. [3]).
(3) If card(X) or card(Y) is non-measurable and $X \times Y$ is C^*-embedded in $X \times \beta Y$, then $v(X \times Y) = vX \times vY$ (cf. [3]).

Theorem 3.5. *Let X be a P-space and $X \times Y$ a pseudoLindelöf space. If $v(X \times Y) = vX \times vY$, then the projecton $\pi_X : X \times Y \to X$ is z-closed.*

Proof. Take any zero-set A in $X \times Y$ and $x \notin \pi_X(A)$. Then $(\{x\} \times Y) \cap A = \emptyset$. We will show that $\{x\} \times Y$ is C-embedded in $X \times Y$. Take any continuous map $f : \{x\} \times Y \to R$. Note that the map $h : Y \to \{x\} \times Y$, defined by $h(y) = (x,y)$, is a homeomorphism. Let $k = f \circ h$ and define a map $\underline{0} : X \to R$ by $\underline{0}(x) = 0$

for all $x \in X$. Then the map $l : X \times Y \to R$, defined by $l((z, y)) = \underline{0}(z) + k(y)$, is continuous and $l|_{\{x\}\times Y} = f$. Hence $\{x\} \times Y$ is C-embedded in $X \times Y$. Thus $\{x\} \times Y$ and A are completely separated in $X \times Y$ (cf. [4]).

Since $v(X \times Y) = vX \times vY$, $(\{x\} \times vY) \cap \mathrm{cl}_{vX\times vY}(A) = \emptyset$. For any $y \in vY$, there are open neighborhoods C_y of x in X and D_y of y in Y such that $(C_y \times D_y) \cap A = \emptyset$. Since vY is Lindelöf, there is a sequence (y_n) in vY with $\{x\} \times vY \subseteq \bigcup\{C_{y_n} \times D_{y_n} : n \in N\}$. Let $Z = \bigcap\{C_{y_n} : n \in N\}$. Since X is a P-space, Z is open in X and $\{x\} \times vY \subseteq Z \times \bigcup\{D_{y_n} : n \in N\}$. Moreover, $(Z \times vY) \cap A = \emptyset$. Thus $Z \cap \pi_(A) = \emptyset$ and so $x \notin \mathrm{cl}_X(\pi_X(A))$. Therefore $\pi_X(A)$ is closed in X. □

Corollary 3.6. *Suppose that $X \times Y$ is a pseudoLindelöf space such that* card(X) *or* card(Y) *is non-measurable and X is a P-space. Then π_X is z-closed if and only if $v(X \times Y) = vX \times vY$.*

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