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UNIFORMLY LOCALLY UNIVALENT FUNCTIONS

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ABSTRACT. A holomorphic function f on $D = \{z : |z| < 1\}$ is called uniformly locally univalent if there exists a positive constant ρ such that f is univalent in every hyperbolic disk of hyperbolic radius ρ. We establish a characterization of uniformly locally univalent functions and investigate uniform local univalence of holomorphic universal covering projections.

1. Introduction

We begin with a brief introduction to hyperbolic geometry on the open unit disk $D = \{z : |z| < 1\}$. For a general discussion of hyperbolic geometry on D we refer Beardon [2], Farkas and Kra [4]. The hyperbolic distance on D induced by the hyperbolic metric $\lambda_D(z)|dz| = 2|dz|/(1-|z|^2)$ is

$$d_h(a,b) = 2\tanh^{-1}\left|\frac{a-b}{1-a\bar{b}}\right|.$$

The hyperbolic disk in D with center $a \in D$ and hyperbolic radius $\rho, 0 < \rho \leq \infty$, is defined by

$$D_h(a,\rho) = \{z : d_h(z,a) < \rho\}.$$

Suppose f is a holomorphic function on D. For $z \in D$, let $\rho(z,f)$ be the hyperbolic radius of the largest hyperbolic disk in D centered at z in which f is univalent. Set

$$\rho(f) = \inf\{\rho(z,f) : z \in D\}.$$

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A holomorphic function f on D is called *uniformly locally univalent* (in the hyperbolic sense) if $\rho(f) > 0$. The quantity $\rho(f)$ is called the hyperbolic radius of uniform univalence for f.

We note that

$$z \in D_h(a,\rho) \Leftrightarrow |z - a1 - \bar{a}z| < \tanh\frac{\rho}{2}.$$

Let $R = \tanh\frac{\rho}{2}$. Then

$$z \in D_h(a,\rho) \Leftrightarrow |z-a|^2 < R^2\,|1-\bar{a}z|^2.$$

From the inequality $|z-a|^2 < R^2|1-\bar{a}z|^2$, we obtain

$$\left| z - \frac{(1-R^2)a}{1-R^2\ a^{\ 2}} \right|^2 < \frac{R^2(1-|a|^2)^2}{(1-R^2|a|^2)^2}.$$

Thus, the hyperbolic disk $D_h(a,\rho)$ is a euclidean disk $D(c,r) = \{z : |z-c| < r\}$ where

$$c = \frac{1-(\tanh\frac{\rho}{2})^2}{1-(\tanh\frac{\rho}{2})^2|a|^2}a, \quad r = (\tanh\frac{\rho}{2})\frac{1-|a|^2}{1-(\tanh\frac{\rho}{2})^2|a|^2}.$$

In particular, we have $D_h(0,\rho) = D(0,\tanh\frac{\rho}{2})$.

In this paper we investigate some properties of uniformly locally univalent functions. We establish a characterization of uniformly locally univalent functions and give a necessary and sufficient condition for a holomorphic universal covering projection to be uniformly locally univalent.

2. Uniform local univalence

Becker [3] proved that if f is holomorphic and locally univalent in D, and if

$$(1-|z|)^2\left|\frac{f''(z)}{f'(z)}\right| \le 1$$

for all z in D, then f is univalent in D. Let $\delta(z) = \delta_D(z) = \mathrm{dist}(z,\partial D) = 1-|z|$. Note that $\delta(z)$ is the radius of the largest disk in D with center z. The following result is a slight modification of Becker's univalence criterion.

Lemma 1. *Suppose f is holomorphic and locally univalent in D. If $|f''(z)/f'(z)| \leq M\lambda_D(z)$ for all $z \in D$, where $M \geq \frac{1}{4}$, then f is univalent in $D(a, \delta(a)/4M)$ for each $a \in D$.*

Proof. Let $a \in D$ and $\delta = \delta(a)$. Define $h(z) = (f \circ g)(z)$, $z \in D$, where $w = g(z) = \delta z/4M + a$. Then

$$h'(z) = \frac{\delta}{4M} f'(w), \quad h''(z) = \left(\frac{\delta}{4M}\right)^2 f''(w),$$

so that

$$\begin{aligned}(1 - |z|^2)\left|\frac{h''(z)}{h'(z)}\right| &= \left(1 - \left|\frac{4M}{\delta}(w-a)\right|^2\right) \frac{\delta}{4M} \left|\frac{f''(w)}{f'(w)}\right| \\ &= \frac{1}{4M\delta}(\delta^2 - (4M)^2|w-a|^2)\left|\frac{f''(w)}{f'(w)}\right|\end{aligned} \tag{1}$$

for all z in D. Note that $\delta + 4M|w-a| < 2\delta$ and $\delta - 4M|w-a| \leq \delta - |w-a| \leq 1 - |w| \leq 1 - |w|^2$. By hypothesis, $(1 - |w|^2)|f''(w)/f'(w)| \leq 2M$. So, by (1), we have $(1 - |z|)^2 |h''(z)/h'(z)| \leq 1$. Hence, by Becker's univalence criterion, h is univalent in D, so $f = h \circ g^{-1}$ is univalent in $D(a, \delta/4M)$. □

Let S be the class of normalized schlicht functions in D. That is, $f \in S$ if f is univalent and holomorphic in D with the development $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$. It is well known that if $f \in S$, then $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. Let f be a locally univalent holomorphic function on D and let $T \in Aut(D)$, the group of conformal automorphisms of D. The Koebe transform f_T of f is defined by

$$f_T(z) = \frac{f(T(z)) - f(T(0))}{f'(T(0))T'(0)}.$$

For $T(z) = (z + a)/(1 + \bar{a}z)$, we have

$$f_T(0) = 0, \; f_T'(0) = 1, \; f_T''(0) = \frac{f''(0)}{f'(0)}(1 - |a|^2) - 2\bar{a}.$$

We now establish a characterization of uniformly locally univalent functions.

Theorem 2. *A locally univalent holomorphic function f defined on D is uniformly locally univalent if and only if there exists a constant $M \geq \frac{1}{4}$ such that $|f''(z)/f'(z)| \leq M\lambda_D(z)$ for all $z \in D$.*

Proof. First, suppose that f is uniformly locally univalent. Then there exists a constant $\rho > 0$ such that f is univalent in each hyperbolic disk $D_h(a, \rho), a \in D$. Let

$r = \tanh(\rho/2)$. Then

$$d_h\left(\frac{rz+a}{1+\bar{a}rz}, a\right) = 2\tanh^{-1} r|z| < 2\tanh^{-1} r = \rho, \ z \in D.$$

This yields $T(rz) \in D_h(a,\rho), z \in D$, where $T(z) = (z+a)/(1+\bar{a}z)$. Let $g(z) = \frac{1}{r} f_T(rz)$, where f_T is the Koebe transform of f. Then g is univalent in $D, g(0) = 0, g'(0) = 1$, and

$$g''(0) = r(1-|a|^2)\frac{f''(a)}{f'(a)} - 2\bar{a}r.$$

Since $g \in S$, the class of normalized schlicht functions, we obtain $|g''(0)| \leq 2$. Therefore, we have

$$\left|\frac{f''(a)}{f'(a)}\right| = \left|\frac{g''(0)+2\bar{a}r}{r(1-|a|^2)}\right| \leq M\lambda_D(a),$$

where $M = (1+r)/r \geq 1$. Next, suppose that there exists a constant $M \geq \frac{1}{4}$ such that $|f''(z)/f'(z)| \leq M\lambda_D(z)$ for all $z \in D$. Then, by Lemma 1, f is univalent in $D(a, \delta(a)/4M)$ for each $a \in D$. Let $s \in (0, \frac{1}{8M})$ and $\rho = 2\tanh^{-1} s$. Then we have

$$s\frac{1-|c|^2}{1-s^2|c|^2} < \frac{2s(1-|c|)}{1-s^2|c|^2} < \frac{(1-|c|)(1+s^2|c|)}{4M(1-s^2|c|^2)} = \frac{1}{4M}\delta\left(\frac{1-s^2}{1-s^2|c|^2}c\right)$$

for each $c \in D$. This yields

$$D_h(c,\rho) \subset D\left(\frac{1-s^2}{1-s^2|c|^2}c, \ \frac{1}{4M}\delta\left(\frac{1-s^2}{1-s^2|c|^2}c\right)\right), \ c \in D.$$

This implies that f is univalent in $D_h(c,\rho)$ for each $c \in D$. □

3. Holomorphic universal covering projections

A region Ω in the complex plane $\mathbf{C}$ is called *hyperbolic* if the complement of Ω with respect to $\mathbf{C}$ contains at least two points. If a region Ω is hyperbolic, then, by the uniformization theorem [1, p. 142], there exists a holomorphic universal covering projection φ of D onto Ω. The collection of all holomorphic universal covering projections of D onto Ω consists of functions $\varphi \circ T$, where $T \in Aut(D)$. The density $\lambda_\Omega(z)$ of the hyperbolic metric $\lambda_\Omega(z)\ |dz|$ on a hyperbolic region Ω is obtained from

$$\lambda_\Omega(\varphi(z))|\varphi'(z)| = \lambda_D(z),$$

where φ is any holomorphic universal covering projection of D onto Ω. The density of the hyperbolic metric is independent of the choice of the holomorphic universal covering projection since

$$\frac{2|T'(z)|}{1-|T(z)|^2} = \frac{2}{1-|z|^2}$$

for any $T \in Aut(D)$.

Theorem 3. *Let φ be a holomorphic universal covering projection of D onto a hyperbolic region Ω. If Ω is simply connected, then $\rho(\varphi) = \infty$.*

Proof. By the Riemann Mapping Theorem, there exists a univalent mapping f of D onto Ω. Since the collection of all holomorphic universal covering projections of D onto Ω consists of the functions $\varphi \circ T$, where $T \in Aut(D)$, it follows that $f = \varphi \circ T$ for some $T \in Aut(D)$. This yields $\varphi = f \circ T^{-1}$; hence φ is univalent in D. Thus, we have

$$D\left(0, \tanh\frac{\rho(\varphi)}{2}\right) = D_h(0, \rho(\varphi)) = D.$$

This yields $\rho(\varphi) = \infty$. □

Uniform local univalence is closely related to the concept of uniform perfectness. For a hyperbolic region Ω and $z \in \Omega$ let $\delta_\Omega(z)$ denote the euclidean distance from z to the boundary of Ω. For an arbitrary hyperbolic region Ω there is a simple relationship between $\lambda_\Omega(z)$ and $\delta_\Omega(z)$, namely $\lambda_\Omega(z)\delta_\Omega(z) \le 2$ [5, p.45]. On the other hand, for any simply connected hyperbolic region Ω, we have $\frac{1}{2} \le \lambda_\Omega(z)\delta_\Omega(z)$ [5, p. 45]. For an arbitrary hyperbolic region Ω there does not exist a positive constant $c = c(\Omega)$ such that $c \le \lambda_\Omega(z)\delta_\Omega(z)$. For instance, if $\Omega = \{z : 0 < |z| < R\}$, then

$$\lambda_\Omega(z) = \frac{1}{|z|\log(R/|z|)}$$

and $\delta_\Omega(z) = |z|$ for $0 < |z| \le \frac{R}{2}$ so that $\lambda_\Omega(z)\delta_\Omega(z) \to 0$ as $z \to 0$. A hyperbolic region Ω is called *uniformly perfect* if there exists a positive constant $c = c(\Omega)$ such that $c \le \lambda_\Omega(z)\delta_\Omega(z)$ for all z in Ω. Pommerenke [7] proved that a hyperbolic region Ω is uniformly perfect if and only if every holomorphic universal covering projection $\varphi : D \to \Omega$ is uniformly locally univalent.

The gradient $\nabla g(z)$ is the complex vector $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ and its squared length is $|\nabla g|^2 = \left|\frac{\partial g}{\partial x}\right|^2 + \left|\frac{\partial g}{\partial y}\right|^2$. If g is a real-valued differentiable function, then we have $|\nabla g| = 2\left|\frac{\partial g}{\partial z}\right|$, where $\frac{\partial}{\partial z}$ is the differential operator

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \mathrm{z} = x + iy.$$

Osgood [6] proved that a hyperbolic region Ω is uniformly perfect if and only if there exists a constant $c > 0$ such that $|\nabla \log \lambda_\Omega(z)| \leq c\lambda_\Omega(z)$ for all z in Ω.

From the uniform perfectness criterion of Osgood [6] and Pommerenke [7], we obtain the following theorem. We establish a new proof of this theorem using Theorem 2.

Theorem 4. *Let $w = \varphi(z)$ be a holomorphic universal covering projection of D onto a hyperbolic region Ω. Then φ is uniformly locally univalent if and only if there exists a constant $c > 0$ such that $|\nabla \log \lambda_\Omega(w)| \leq c\lambda_\Omega(w)$ for all w in Ω.*

Proof. From the identity $\log \lambda_\Omega(w) = \log \lambda_D(z) - \log|\varphi'(z)|$, we obtain

$$\begin{aligned} \frac{\partial}{\partial w}\log \lambda_\Omega(w) &= \frac{1}{\varphi'(z)}\left(\frac{\partial}{\partial z}\log \lambda_D(z) - \frac{\varphi''(z)}{2\varphi'(z)}\right) \\ &= \frac{1}{2\varphi'(z)}\left(\bar{z}\lambda_D(z) - \frac{\varphi''(z)}{\varphi'(z)}\right). \end{aligned} \tag{2}$$

If φ is uniformly locally univalent, then, by Theorem 2, there exists a constant $M \geq \frac{1}{4}$ such that $\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \leq M\lambda_D(z)$ for all z in D. From (2), we obtain

$$\begin{aligned} |\nabla \log \lambda_\Omega(w)| &= 2\left|\frac{\partial}{\partial w}\log \lambda_\Omega(w)\right| \\ &= \frac{1}{|\varphi'(z)|}\left|\bar{z}\lambda_D(z) - \frac{\varphi''(z)}{\varphi'(z)}\right| \\ &= (1+M)\frac{\lambda_D(z)}{|\varphi'(z)|} = (1+M)\lambda_\Omega(w), \quad w \in \Omega. \end{aligned}$$

Next, suppose that there exists a constant $c > 0$ such that $|\nabla \log \lambda_\Omega(w)| \leq c\lambda_\Omega(w)$ for all w in Ω. From (2), we obtain

$$\begin{aligned} \left|\frac{\varphi''(z)}{\varphi'(z)}\right| &= \left|\bar{z}\lambda_D(z) - 2\varphi'(z)\frac{\partial}{\partial w}\log \lambda_\Omega(w)\right| \\ &\leq \lambda_D(z) + |\varphi'(z)||\nabla \log \lambda_\Omega(w)| \\ &\leq \lambda_D(z) + c|\varphi'(z)|\lambda_\Omega(w) = (1+c)\lambda_D(z), \mathrm{z} \in D. \end{aligned}$$

Hence, by Theorem 2, φ is uniformly locally univalent. □

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