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THE EXTENSION OF THE SUFFICIENT CONDITION FOR UNIVALENCE

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1. Introduction

In this paper we shall consider function $p(z)$ analytic in the open unit circle D and the solutions $y(z)$ of the differential equation

$$y''(z) + p(z)y(z) = 0. \tag{1.1}$$

The ratio $f(z) = u(z)/v(z)$ of any two independent solutions $u(z)$ and $v(z)$ of (1.1) will be function $f(z)$, meromorphic in D with only simple poles, and such that $f'(z) \neq 0$. We shall say that a meromorphic function which satisfies these two condition belongs to the restricted class. The Schwarzian derivative of $f(z)$,

$$S_f(z) = \varphi'(z) - \frac{1}{2}\varphi_f^2(z),\ \varphi_f(z) = f''(z)/f'(z)$$

is connected with $p(z)$ by

$$S_f(z) = 2p(z). \tag{1.2}$$

We know that $f(z)$ is univalent in D if no solution of (1.1) has more than one zero in D. Conversely, every univalent function $f(z)$ in D can be written as the ratio of two independent solutions of the (1.1) where $p(z)$ is defined by (1.2). These connections were first stated by Nehari in [1].

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2. A bound for the Euclidean distance

It is known that the existence of a common positive lower bound for the non-Euclidean distance of two zeros is equivalent to the assumption

$$p(z) = 0\left(1/(1-|z|^2)^2\right).$$

Clearly, conditions on $p(z)$ which ensure similarly the existence of a common bound for the Euclidean distance must be restrigent.

Lemma 1. *For $0 < \rho \leq t \leq a < 1, \rho < a$, We have*

$$\frac{1}{1-t^2} < \frac{a-\rho}{a+\rho}\frac{2}{(a-\rho)^2-(t-\rho)^2}. \tag{2.1}$$

Proof. For $a = t$, the right hand side become infinite so that we may assume $0 < \rho \leq t < a < 1$. Since $a^2 - t^2 < 1 - t^2$ it will suffice to show that

$$\frac{1}{a^2-t^2} \leq \frac{2(a-\rho)}{(a+\rho)[(a-\rho)^2-(t-\rho)^2]} = \frac{2(a-\rho)}{(a+\rho)(a+t-2\rho)(a-t)},$$

i.e., that

$$\frac{1}{a+t} \leq \frac{2(a-\rho)}{(a+\rho)(a+t-2\rho)}.$$

This inequality is equivalent to

$$(a+\rho)(a+t-2\rho) \leq 2(a-\rho)(a+t),$$

which by computing, we have

$$(a-\rho)^2 + \rho^2 + a\rho + at - 3\rho t \geq 0.$$

To prove the last inequality it will be enough to show that, for fixed ρ and a ($0 < \rho < a < 1$) and for all t, $\rho \leq t \leq a$, the function

$$f(t) = \rho^2 + a\rho + at - 3\rho t$$

is positive. However, $f(t)$ is positive at the endpoints of the interval $[\rho, a]$ and positive inside the interval. Thus (2.1) is proved. This completes the proof. □

Theorem 2. *Let $p(z)$ be analyitc in $|z| < 1$, and set*

$$M(t) = Max\{|p(z)| : |z| = t \text{ and } 0 \leq t < 1\}.$$

Assume that

$$(1 - t^2)M(t) \leq 1 \text{ for } r \leq t < 1,\ 0 < r < 1. \tag{2.2}$$

let $y(z)$ be any nontrivial solution of (1.1) and assume that $y(z_1) = y(z_2) = 0$, $z_1 \neq z_2$, $|z_1| < 1, |z_2| < 1$. Then

$$|z_1 - z_2| \geq 2\sqrt{1 - r^2}.$$

Proof. We assume that there exists a solution $y(z)$ of (1.1) such that

$$|z_1 - z_2| = \delta < 2\sqrt{1 - r^2} = d. \tag{2.3}$$

Multiplying (1.1) by $\overline{y}dz$ and integrating by parts from z_1 to z_2 along a path in D we obtain

$$[\overline{y}y']_{z_1}^{z_2} - \int_{z_1}^{z_2} |y'|^2\overline{dz} + \int_{z_1}^{z_2} p|y|^2 dz = 0.$$

Using now $y(z_1) = y(z_2) = 0$ and choosing as path the segement $[z_1, z_2]$ (whose length element we denote the $d\sigma$) we obtain

$$\int_{z_1}^{z_2} |y'|^2 d\sigma \leq \int_{z_1}^{z_2} |p||y|^2 d\sigma. \tag{2.4}$$

We shall reach the desired contradiction by three consecutive transformations of this inequality.

[First transformation] Choose a, $0 < a < 1$, such that $|z_1| < a, |z_2| < a$ and such that, setting $\rho = \sqrt{a^2 - \delta^2/4}$, by (2.3) we have

$$r < \rho. \tag{2.5}$$

We moved the segement $[z_1, z_2]$ in a way that it became a chord in $|z| = a$ and it is obvious that during this motion the distance of each point from $z = 0$ increased. The distance of this chord from the origin is ρ. If we denote the length coordinate of the chord, measured from it centre, by $s(-\sqrt{a^2-\rho^2} \leq s \leq \sqrt{a^2-\rho^2})$, then the distance of the point with the coordinate s from $z = 0$ will be $\sqrt{\rho^2 + s^2}$.

We define $y_1(s)$ on the chord by giving that function the same values which $y(z)$ took at the corresponding points of the segement $[z_1, z_2]$; similarly we define $p_1(s)$ by the values of $p(z)$ on $[z_1, z_2]$. $y_1(s)$ is therefore analytic for $-\sqrt{a^2-\rho^2} \leq \sqrt{a^2-\rho^2}$ and $y_1(\pm\sqrt{a^2-\rho^2}) = 0$. As $M(t)$ is, by the maximum principle, a non-decreasing function of t, it follows from the above remark about the increasing distance from the origion that

$$|p_1(s)| \leq M(\sqrt{\rho^2+s^2}), \qquad 0 \leq \pm s \leq \sqrt{a^2-\rho^2}.$$

(2.4) implies therefore

$$\int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} \left|\frac{dy_1}{ds}\right|^2 ds \leq \int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} M(\sqrt{\rho^2+s^2})|y_1(s)|^2 ds. \tag{2.6}$$

[Second transformation] We maps $0 \leq s \leq \sqrt{a^2-\rho^2}$ onto $\rho \leq t \leq a$ and $-\sqrt{a^2-\rho^2} \leq s \leq 0$ onto $-a \leq t \leq -\rho$. These transformations are given by

$$t = \pm\rho + \frac{a-\rho}{\sqrt{a^2-\rho^2}} s \text{ for} 0 \leq \pm s \leq \sqrt{a^2-\rho^2}. \tag{2.7}$$

It is easily seen that

$$\sqrt{\rho^2+s^2} \le \rho \pm \frac{a-\rho}{\sqrt{a^2-\rho^2}} s \text{ for } 0 \le \pm s \le \sqrt{a^2-\rho^2},\ 0<\rho<a,$$

where the equality holds only for $s = 0, \pm\sqrt{a^2-\rho^2}$. By (2.7) this shows that under this second transformation the distance of each point from the origin again increases, except for the points $s = 0, \pm\sqrt{a^2-\rho^2}$ whose distance remains constant.

The function $Y(t)$ defined by

$$Y(t) = Y\left(\pm\rho + \frac{a-\rho}{\sqrt{a^2-\rho^2}} s\right) = y_1(s)$$

will thus have the following properties;

(1) $Y(t)$ is analytic on the segments $-a \le t \le -\rho$ and $\rho \le t \le a$

(2) $Y(a) = Y(-a) = 0$

(3) $Y(t)$ and all its derivative take the same values at $t = \rho$ and $t = -\rho$.

Defining

$$M(t) = M(-t) \qquad \text{for } -1 < t < 0, \tag{2.8}$$

and observing that the distance from the origin do not decrease under this second transformation, we obtain from (2.6)

$$\int_{-a}^{-\rho} \left|\frac{dY}{dt}\right|^2 dt + \int_{\rho}^{a} \left|\frac{dY}{dt}\right|^2 dt \le \frac{a+\rho}{a-\rho}\left\{\int_{-a}^{-\rho} M(t)|Y(t)|^2 dt + \int_{\rho}^{a} M(t)|Y(t)|^2 dt\right\}. \tag{2.9}$$

By our assumption (2.2), and in view of (2.5) and (2.8), it follows that

$$(1-t^2)M(t) \le 1 \qquad \text{for } \rho \le \pm t \le a. \tag{2.10}$$

(2.9),(2.10) and Lemma 1 yield

$$\int_{-a}^{-\rho}\left|\frac{dY}{dt}\right|^2 dt+\int_{\rho}^{a}\left|\frac{dY}{dt}\right|<\int_{-a}^{-\rho} g(t)|Y(t)|^2dt+\int_{\rho}^{a} g(t)|Y(t)|^2dt, \qquad (2.11)$$

where

$$g(t)=\frac{2}{(a-\rho)^2-(t\mp\rho)^2} \qquad \rho\le \pm t\le a.$$

[Third transformation] We translate the two segement $[-a,-\rho]$ and $[\rho,a]$ of the real axis until they meet at the origin, i.e., we introduce the variable x by $x=t\mp\rho$ for $\rho\le\pm t\le a$. With the notation $a-\rho=b$, it follows that x varies between $-b$ and b. Definding now $g_1(x)=g_1(t\mp\rho)=g(t)$, we have $(b^2-x^2)g_1(x)=2, -b\le x\le b$. Similarly, we define $Y_1(x)=Y_1(t\mp\rho)=Y(t)$ and it follows that $Y_1(x)$ is analytic for $-b\le x\le b$ and $Y_1(\pm b)=0$. (2.11) transforms into

$$\int_{-b}^{b}\left|\frac{dY_1}{dx}\right|^2 dx<2\int_{-b}^{b}\frac{|Y_1(x)|^2}{b^2-x^2}dx. \qquad (2.12)$$

We use the integral inequality

$$2\int_{-b}^{b}\frac{u^2}{b^2-x^2}dx\le\int_{-b}^{b}u'^2dx,\ \ u=u(x), \qquad (2.13)$$

which holds for continuously differentiable real functions $u(x)$ having at $x=\pm b$ zeros of the first order [3,p.193]. (2.13) follows from the semi-definiteness of the integral

$$\int_{-b}^{b}\left(u'+\frac{2xu}{b^2-x^2}\right)^2dx.$$

Expanding and integrating by parts, we obtain

$$\int_{-b}^{b}u'^2dx+2\left.\frac{xu^2}{b^2-x^2}\right|_{-b}^{b}-2\int_{-b}^{b}\frac{(b^2+x^2)u^2}{(b^2-x^2)^2}dx+4\int_{-b}^{b}\frac{x^2-u^2}{(b^2-x^2)^2}dx\ge 0.$$

u being $0(b-x)$ and $0(b+x)$ and $x = b$ and $x = -b$ respectively, the integrals exist and the integrated part vanishes, which proves (2.13). Writing now $Y_1(x) = u(x) + iv(x)$ and applying (2.13) to both $u(x)$ and $v(x)$, we obtain the desired contradiction to (2.12). This completes the proof □

We remark that without any modification our proof holds also in the case $r = 0$. Assumption (2.2) becomes then

$$(1-t^2)M(t) \leq 1 \qquad \text{for } 0 \leq t < 1.$$

and the conclusion is that no solution $y(z)$ of (1.1) has more than one zero in $|z| < 1$. But this is clearly a consequence of the sufficient part of Theorem I of [1] and also of a criterion announced by Pokornyi [2], stating that

$$(1-t^2)M(t) \leq 2 \qquad \text{for } 0 \leq t < 1, \tag{2.14}$$

is sufficient to ensure the same conclusion. In view of the geometrical meanning of d and r (length of chord and its distance from the origin) it seems natural not to change definition (2.3). We have the following statement.

Corollary 3. *No condition of the form*

$$(1-t^2)^\mu M(t) \leq c, \qquad \mu > 1,\ c > 0,\ r \leq t < 1, \tag{2.15}$$

is, for all r $(0 \leq r < 1)$, sufficient to ensure that

$$|z_1 - z_2| \geq d. \tag{2.3}$$

Proof. Let $p(z) = c_1, c_1 > c$. The distance d' between neighboring zeros of any solution of (1.1) is then $d' = \pi/\sqrt{c_1}$. (2.15) holds for $r \leq t < 1$. where r is given by $(1-r^2)^\mu = c/c_1$. The bound d, given by (2.3), becomes

$$d = 2\sqrt{1-r^2} = 2\sqrt{(c/c_1)^{1/\mu}}.$$

As $\mu > 1$, the lower bound d would, for large c_1, be larger than the actual distance d' and we have proved the above statement □

Remark. We mentioned that $f = 0$, condition (2.14) is sharp. It follows that in Theorem 2, (2.2) cannot be replaced by a condition of the form

$$(1-t^2)M(t) \le c, \qquad r \le t < 1,\ 0 \le r < 1,\ \text{with } c > 2.$$

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