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ON THE ELLIPTIC EQUATION $\Delta u + H(x)e^u = 0$ ON COMPACT MANIFOLDS

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1. Introduction

In this paper, we consider the existence of a solution to the elliptic nonlinear partial differential equation

$$\Delta u + H(x)e^u = 0 \qquad (H \neq 0) \tag{1}$$

on a compact manifold without boundary. This equation is related to the problem of a pointwise conformal deformation of metrics on two dimensional compact connected manifolds.

Let (M, g) be a Riemannian manifold of dimension 2 and $K(x)$ be a given function on M. Then one may ask the following question: Can we find a new metric g_1 on M such that $K(x)$ is the Gaussian curvature of g_1 and g_1 is pointwise conformal to g (i.e., there exists a function $u(x)$ on M such that $g_1 = e^{2u}g$)? If M admits $k(x)$ as the Gaussian curvature of g, then this is equivalent to the problem of solving the elliptic equation

$$\Delta u - k(x) + K(x)e^{2u} = 0, \tag{2}$$

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where Δ is the Laplacian in the g-metric (cf.[Av.], [C.L.], [H.], [H.T.], [K.W.2,3], [M.2], etc.).

Let v be a solution of $\Delta v = k(x) - \overline{k(x)}$ (cf.Lemma 1), where $\overline{k(x)} = \frac{1}{vol(M)} \int_M k(x)dV$ is the average of $k(x)$, and let $w = 2(u - v)$. Then w satisfies

$$\Delta w = 2\overline{k(x)} - (2Ke^{2v})e^w. \tag{3}$$

It turns out that the equation (3) is easier to analyze if we free it from geometry and consider instead

$$\Delta u - c + H(x)e^u = 0, \tag{4}$$

where $H(x)$ is some prescribed function and c is a constant. The equation (1) is the case with $c = 0$.

In case of noncompact 2-dimensional manifolds, it follows from the uniformization theorem that every noncompact Riemannian surface is conformal to a complete Riemannian surface of constant Gaussian curvature $k = 0$ or $k = -1$. In the case that (M, g) is conformal to the flat Euclidean plane R^2, the equation (1) has been studied extentively by P.Aviles ([Av.]), K.S.Cheng and W.M.Ni ([C.N.]), R.McOwen ([M.1,2]), D.H.Sattinger ([S.]), etc. In the case that (M, g) is conformal to the hyperbolic disc H^2 with $k = -1$, the equation (4) with $-c > 0$ has been studied by J.Bland and M.Kalka ([B.K.]), D.Hulin and M.Troyanov ([H.T.]), etc.

In case of compact 2-dimensional manifolds, it follows from Gauss-Bonnet theorem that the c-value admits negative, zero, or positive. According to the c-values, the equation (4) has been studied by J.L.Kazdan and F.W.Warner ([K.W. 2,3]), X.Xu and P.C.Yang ([X.Y.]), K.S.Cheng and J.A.Smoller([C.S.]), W.Chen and W.Ding ([C.D.]), Z.C. Han ([H.]), etc. In case of compact 2-dimensional manifolds, the above almost authors studied the case that c is nonzero, in particular, on the usual sphere S^2.

In [K.W.3], J.L.Kazdan and F.W.Warner have studied the necessary and sufficient conditions of the solvability of (1) on two dimensional compact connected manifolds. That is, a solution of (1) exists if and only if both $\overline{H} < 0$ and H changes sign (in case that $H \neq 0$). These necessary conditions must still be satisfied in the $n(\geq 3)$ dimensional case, too (cf. Theorem 3). J.L.Kazdan and F.W.Warner conjectured that these two necessary conditions on H for the solvability of (1) on M of dimension $n(\geq 3)$ would be sufficient, much as in Theorem 5.3 of [K.W.1], which is still an open problem (Also see some open problems in [K.],p.47).

In this paper, we shall prove that they are also sufficient, i.e., the sufficiency extends to dim $M \geq 3$ on a compact manifold without boundary. In case of noncompact $n(\geq 3)$- dimensional manifolds, the equation (1) has been studied by K.S.Cheng and J.T.Lin ([C.L.]), K.Nagasaki and T.Suzuki ([N.S.]), H.Bellout ([B.]), etc.

And although throughout this paper, we will assume that all data (M, metric g, and curvature, the given function, etc.) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs, we need only assume that the given function $K(x)$ is Holder continuous. And for the existence of solutions of the given equations, the above almost authors use the method of calculus of variations. However, in this paper, for basic existence theorems, we use the method of upper and lower solutions (See [K.W.1,2,3] or [C.H.], pp.370-371). The merit of this method is that we can overcome the difficulty of critical Sobolev exponent , which implies the gap of Yamabe's proof (cf.[A., p56,Ex.237,Ex.238],[T.],[J.]).

2. Main results

Let M be a compact connected $n(\geq 3)$-dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure g. We denote the volume element of this metric by dV, the gradient by ∇, and the associated Laplacian by Δ.

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in $L_p(M)$. The norm on $H_{s,p}(M)$ will be denoted by $\| \quad \|_{s,p}$. The usual norm $L_2(M)$ inner product will be written $\| \quad \|$.

Lemma 1. *Let (M,g) be a compact Riemannian manifold. There exists a weak solution $w \in H_{1,2}(M)$ of $\Delta w = f$ if and only if $\overline{f} = 0$. The solution w is unique up to a constant. Moreover, if f is smooth, then w is also smooth.*

Proof. See Theorem 4.7 in [A.].

Lemma 2. *Let $H \in L_p(M)$ for some $p > n = dimM$. If there exist functions $u_+, u_- \in H_{2,p}(M)$ such that*

$$\Delta u_+ + H(x)e^{u_+} \leq 0, \qquad \Delta u_- + H(x)e^{u_-} \geq 0,$$

with $u_- < u_+$, then there is a $u \in H_{2,p}(M)$ satifying (1) and $u_- < u < u_+$. Moreover, u is smooth in any open set in which $H(x)$ is smooth.

Proof. For detail, see Lemma 9.3 in [K.W.3] or Lemma 2.6 in [K.W.1] or a standard argument in pp.370-371 in [C.H.].

Here u_+ and u_- are called upper and lower (or super and sub) solutions of (1), respectively.

Theorem 3. *If a solution u of (1) exists and $H \not\equiv 0$, then H must change sign and $\overline{H} < 0$.*

Proof. By integrating both sides of (1) over M, we obtain that $\int_M He^u dV = 0$. Since $H \not\equiv 0$, H must change sign. To obtain the second result, observe that $u \equiv constant$ cannot be a solution. Multiplying (1) by e^{-u} and using an integration by parts over M, we find that

$$\int_M H dV = -\int_M e^{-u}\Delta u dV = -\int_M e^{-u}|\nabla u|^2 dV < 0.$$

Theorem 4. *If (1) has a solution for given H and if $H_1 = mH$ for some constant $m > 0$, then (1) has a solution for H_1.*

Proof. If u is a solution of (1) for H and $m = e^r$, then $v = u - r$ is a solution of (1) for $H_1 = mH$.

Theorem 5. *[Existence of an upper solution] Let $H(\neq 0)$ belong to $C^\infty(M)$ such that H changes sign and $\overline{H} < 0$. Then there exists an upper solution $u_+ > 0$ of (1), that is,*

$$\Delta u_+ + H(x)e^{u_+} \leq 0.$$

Proof. Lemma 1 implies that there exists a solution w of $\Delta w = \overline{H} - H$. We can pick $b > 0$ so small that $|e^{bw} - 1| \leq \frac{-\overline{H}}{4\|H\|_\infty}$. Let $e^r = b$. Put $v = bw + r$. Then

$$\begin{aligned}
\Delta v + He^v &= \Delta(bw + r) + He^{bw+r} \\
&= b\Delta w + bHe^{bw} \\
&= b\overline{H} + bH(e^{bw} - 1) \\
&\leq b\overline{H} + b\|H\|_\infty |e^{bw} - 1| \\
&\leq b\overline{H} - b\frac{\overline{H}}{4} \\
&= b\frac{3\overline{H}}{4} < 0.
\end{aligned}$$

Thus $u_+ = bw + r$ is an upper (weak) solution of (1).

From the above theorem, if $\overline{H} < 0$, then we can always have an upper solution of (1). Hence in order to show that (1) has a solution, it suffices to find a lower (weak) solution u_- such that $u_- < u_+$ and

$$\Delta u_- + H(x)e^{u_-} \geq 0.$$

Now we consider the first eigenvalue of the differential operator $Lu = -\Delta u - Hu$, that is,

$$\begin{aligned} \lambda_1 &= inf._{v \neq 0, v \in H_{1,2}(M)} \frac{\|\nabla v\|^2 - \int Hv^2 dV}{\|v\|^2} \\ &= inf.(\|\nabla v\|^2 - \int Hv^2 dV) \quad on\{v \in H_{1,2}(M), \|v\|^2 = 1\}. \end{aligned}$$

Note that the eigenfunction is never zero and smooth. In fact, since $|\nabla v| = |\nabla |v||$ almost everywhere (See Proposition 3.69 in [A.]), the variational characterization of λ_1 shows that one can take $v \geq 0$, while the strong maximum principle shows that $v > 0$. Thus the eigenspace has dimension 1 and we can assume that the eigenfunction is positive.

Theorem 6. *If $H(x)$ changes sign, then the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for some large $m > 0$.*

Proof. Since H changes sign and M is a compact manifold, there exists a smooth nonnegative function $u(x)$ on M such that $u(x)$ is positive on some open ball in $\{x \in M | H(x) > 0\}$ and $u \equiv 0$ otherwise. Then, for sufficiently large $m > 0$, $\|\nabla u\|^2 - m \int Hu^2 dV < 0$. Thus the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for sufficiently large $m > 0$.

Theorem 7. *If the eigenvalue of $Lu = -\Delta u - mHu$ is negative for some $m > 0$, then there exists a solution of (1) for mH, so by Lemma 4, there exists a solution of (1) for H.*

Proof. We have sevaral steps to prove. They are similar to the proof in Theorem 7 in [J.]. So for detail, cf. Theorem 7 in [J.] and here we sketch the outline of the proof.

Step 1. Since $m\overline{H} < 0$, Theorem 5 implies that there exists an upper solution u_+ of (2) for mH.

Step 2. Now we have only to show that there exists a lower solution $u_- < u_+$ of (1) for mH. Let $f > 0$ be a corresponding eigenfunction of L, that is,

$$\Delta f + mHf = -\lambda_1 f, f > 0. \tag{5}$$

Since cf is also an eigenfunction of (5), we can assume that $f > 1$. Now put $v = b(f^{r^2} - e^{-r})^{1+\frac{1}{r}} + t$, where r is a sufficiently small positive real number and b and t are chosen suitably so that our conditions are satisfied. Then

$$\nabla v = b(r^2 + r)(f^{r^2} - e^{-r})^{\frac{1}{r}} f^{r^2-1} \nabla f$$

and

$$\Delta v = b(r^2 + r)(f^{r^2} - e^{-r})^{\frac{1}{r}} f^{r^2-1} [\Delta f + \{\frac{r^2 - 1}{f} + \frac{r}{f^{r^2} - e^{-r}} \frac{f^{r^2}}{f}\} |\nabla f|^2].$$

Hence for $e^t = b(r^2 + r)(1 - e^{-r})^{\frac{1}{r}}$,

$$\begin{aligned}
&\Delta v + mHe^v \\
&= b(r^2 + r)(f^{r^2} - e^{-r})^{\frac{1}{r}} f^{r^2-1} [\Delta f + \{\frac{r^2 - 1}{f} + \frac{r}{f^{r^2} - e^{-r}} \frac{f^{r^2}}{f}\} |\nabla f|^2 \\
&+ mHe^{b(f^{r^2} - e^{-r})^{1+\frac{1}{r}}} f^{1-r^2} (\frac{1 - e^{-r}}{f^{r^2} - e^{-r}})^{\frac{1}{r}}] \\
&= b(r^2 + r)(f^{r^2} - e^{-r})^{\frac{1}{r}} f^{r^2-1} [-\lambda_1 f + \{\frac{r^2 - 1}{f} + \frac{r}{f^{r^2} - e^{-r}} \frac{f^{r^2}}{f}\} |\nabla f|^2 \\
&+ mHf\{e^{b(f^{r^2} - e^{-r})^{1+\frac{1}{r}}} f^{-r^2} (\frac{1 - e^{-r}}{f^{r^2} - e^{-r}})^{\frac{1}{r}} - 1\}]
\end{aligned}$$

For sufficiently small $r > 0$, $(\frac{1-e^{-r}}{f^{r^2}-e^{-r}})^{\frac{1}{r}} \to 1$ as $r \to +0$ and $|\frac{r^2-1}{f} + \frac{r}{f^{r^2}-e^{-r}}\frac{f^{r^2}}{f}| < \frac{-\lambda_1}{2}$. Therefore, pick $b > 0$ so small that $u_- < u_+$ and

$$|e^{b(f^{r^2}-e^{-r})^{1+\frac{1}{r}}} f^{-r^2}(\frac{1-e^{-r}}{f^{r^2}-e^{-r}})^{\frac{1}{r}} - 1| \le \frac{-\lambda_1}{2m\|H\|_\infty}.$$

Then u_- is our desired lower solution of (1).

[Remark]. [1] When M is of dimension 2, Kazdan and Warner proved the existence of solution of (1) using the calculus of variation (See Theorem 5.3 in [K.W.1]). As we see, their proof depends on the dimension of the given manifold. However, our proof does not depend on the dimension of the given manifold. So this result can be applied to the case of dimension 2.

[2] By the proofs of Theorem 5 and Theorem 7, we can see that there exist many solutions of (1) if $H(x)$ changes sign and $\overline{H} < 0$. In particular, we can also see that when the given manifold of dimension 2 admits zero total Gaussian curvature, there exist many conformal metrics with $K(x)$ as the Gaussian curvature if $K(x)$ changes sign and $\overline{K} < 0$.

Corollary 8. *On compact manifolds, a solution of $\Delta u + H(x)e^{cu} = 0$, where $H \neq 0$ and c is a positive constant, exists if and only if both $\overline{H} < 0$ and H changes sign.*

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