The Pure and Applied Mathematics 3 (1996), No 1, pp. 59–65
J. Korea Soc. of Math. Edu. (Series B)

SOME PROPERTIES OF CS–SEMISTRATIFIABLE SPACES

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ABSTRACT. In this paper, we study spaces admitting cs-semistratification and cs - semistratifications with (CF) property. The class of cs-semistratifiable spaces lies between the class of k-semistratifiable spaces and that of semistratifiable spaces which lie between the class of semi-metric spaces and the class of spaces in which closed sets are G_σ and really differs from the classes of stratifiable spaces.

1. Introduction

All spaces are assumed to be T_1-topological spaces. The letter τ denotes the topology of a space X. We denote by the letter ω the sets of all positive integers.

In his paper [10], Michael introduced the class of semistratifiable spaces, which lies between the class of semi-metric spaces and the class of spaces in which closed set are G_δ. On the other hand, in [8], Lutzer introduced the class of k-semistratifiable spaces, which lies between the class of stratifiable spaces in the sense of Borges [3] and Ceder [5] and the class of semistratifable spaces introduced by Michael [9] and studied by Creede [6].

In this paper, a class of spaces called cs-semistratifiable spaces is introduced, and we consider the limited classes of cs-semistratifiable spaces with (CF) property defined below. We show some properties of cs-semistratifiable space. We also show that this class of spaces is invariant with respect to taking countable products, closed maps, and closed unions; A semistratifiable space is F_σ-screenable; A first countable cs-semistratifiable space is stratifiable; The first countable and cs-semistratifiable is

This research is supported by Kyonggi University Research Grant in 1995.

equivalent to the first countable and k-semistratifiable; If a space X has a σ-HCP cs-network, then X is a cs-semistratifiable space with (CF) property.

Throughout this paper, σ-spaces are spaces with a σ-discrete network(or σ-closure-preserving network) and $\aleph$-space are spaces with a σ-locally finite k-network. At the same time that stratifiable spaces are spaces with the stratification. Cs - semistratifiable spaces are spaces with the cs-semistratification.

2. Properties of Cs-semistratifiable Spaces

We state the original definitions of the class of stratiable spaces, k-semistratifiable spaces and introduce cs-semistratifiable space in term of these definitions;

Definition 2.1. [Michael [9]] A space X is called a semistratifiable space if there exists a function $S: \omega \times \tau \to \{\text{closed subsets of } X\}$ such that;

S_1. For each $U \in \tau$, $U = \cup_{n=1}^{\infty}\{S_n(U) | n \in \omega\}$

S_2. If U, $V \in \tau$ and $U \subset V$, then $S_m(U) \subset S_m(V)$ for each $m \in \omega$

In this case, S is called a semistratification of X.

Definition 2.2. [Borges[3]] A space X is called a stratifiable space if there exists a function S of Definition 2.1 satisfying;

S_3. For each $U \in \tau$, $U = \cup S_n(U)^0$.

Definition 2.3. [Lutzer[8]]. A space X is called a k-semistratifiable space if there exists a function S satisfying S_1, S_2 and S_4;

S_4. If $C \subset U \in \tau$ with C compact, then $C \subset S_n(U)$ for some n.

Definition 2.4. A space X is called a cs-semistratifiable space if there exists a function(called a cs-semistratification of X) satisfying S_1, S_2 and S_5;

S_5. For each convergent sequence $x_n \to x$ and $U \in \tau$, containing x, there is a $k \in \omega$ such that $x \in S_k(U)$ and $\{x_n\}$ is eventually in $S_k(U)$.

From these definitions, it is clear that X is stratifiable $\Rightarrow$ k-semistratifiable $\Rightarrow$ cs-semistratifiable$\Rightarrow$ semistratifiable.

Theorem 2.5. *Every subspace of cs-semistratifiable space is also cs-semistratifiable.*

Proof. Let (X, τ) be a topological space, $A \subset X$ and $(A,\ \tau_A)$ a subspace of X.

Define a function $S_A : \omega \times \tau_A \rightarrow$ {closed subsets of A} by $S_A(n,\ U \cap A) = S(n,\ U) \cap A$, $U \in \tau$, where S is a cs-semistratification of X. Then it is sufficient to show that S_A is a cs-semistratification of A.

For each $H \in \tau_A$, there exists $U \in \tau$ such that $S_A(n, H) = S_A(n,\ A \cap U) = S(n\ , U) \cap A = U \cap A = H$. And if $H_1,\ H_2 \in \tau$ and $H_1 \subset H_2$, then there are some open subsets $U_1,\ U_2$ of X with $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$. Since $U_1 \cap U_2 \subset U_2$ and $H_1 = U_1 \cap U_2 \cap A$, $S_A(n,\ H_1) = S(n,\ U_1 \cap U_2) \cap A \subset S(n,\ U_2) \cap A = S_A(n,\ H_2)$.

In the last place, let $\{x_n\}$ converges to $x \in A$ and let H be an open subset of A containing x. Since S is a cs-semistratification of X, there exists a $k \in w$ and an open subset U of X, such that $x \in S_A\ (k,\ U)$ and $H = U \cap A$ and $\{x_n\}$ is eventually in $S(k,\ U)$. Then $x \in S(k,\ U) \cap A = S_A(k,\ H)$ that is, $\{x_n\}$ is eventually in $S_A(k,\ H)$, and thus S_A is a cs-semistratification of A □

Lemma 2.6. *X is cs-semistratifiable if and only if there is a semistratifiable function $g : wxX \rightarrow$ {open sets of X} with an additional condition:*

Given a convergent sequence $x_n \rightarrow x$ and an open subset U containing x, there is a $k \in \omega$ such that $x \notin \cup_{x \in X \setminus U} g_k(x)$ and $\{n \in \omega | x_n \in \cup_{x \in X \setminus U g_k}(x)\}$ is finite. In this case, g is called a cs-semistratifiable fuction.

Proof. Let a cs-semistratification S be given. For each $n \in \omega$ and $x \in X$, define $g_n(x) = X \setminus S(n,\ X \setminus Cl\{x\})$.

Creede proved g is a semistratifiable function for X in [6]. To show that g satisfies the additional condition above, consider the following $\cup_{x \notin V} g_k(x) = \cup_{x \notin V}\{X \setminus S(k,\ X \setminus Cl\{x\}) = X \setminus \cap_{x \notin V} S(k,\ X \setminus Cl\{x\})$ which is contained in $X \setminus S(k, V)$. If $\{x_n\}$ is eventually in $S(k,\ V)$, $\{n \in \omega | x_n \in \cup_{x \notin V} g_k(x)\}$ is finite.

For the converse, let $S(n,\ U) = X \setminus \cup_{x \notin X \setminus U} g_n(x)$; then S is a cs-semistratification for X □

Theorem 2.7. *The countable product of cs-semistratifiable* space is cs-semistratifiable.

Proof. For each $i \in N$, let X_i be a space with cs-semistratifiable function g_i, let $X = \Pi_{i=1}^{\infty} X_i$ and let π_i be the projection of X onto X_i.

For each i, $j \in \omega$ and $x \in X$, let $h_{ij}(x) = g_{ij}(\pi_i(x))$ if $i \leq j$ and $h_{ij}(x) = X_i$ if $i > j$. Now let $g_j(x) = \Pi_{i=1}^{\infty} h_{ij}(x)$ for each j and x. It is easily verified that g is a cs-semistratifiable function for the space X with the aid of Lemma 2.6.

To show g satisfies the condition of Lemma 2.6, let $\{x_n\}$ be a sequence converging to z and let $z \in U \in \tau$. Take a basic open neighborhood V of z, $V = \Pi_{i \in F} V_i \times \Pi_{i \in \omega \setminus F} X_i \subset U$, where F is a finite subset of ω. For each i, $\{\pi_i(x_n) | n \in \omega\}$ is a sequence converging to $\pi_i(z)$, and $\pi_i(V)$ is open in X_i and contains $\pi_i(z)$, thus there is a $k_i \in \omega$ such that $\{n \in \omega | \pi_i(x_n) \in \cup\{g_{ik}(y) | y \in X_i \setminus \pi_i(V)\}$ is finite for each $i \in F$. Let $k = \max\{k_i | i \in F\}$. But $x_n \in \cup_{x \in X \setminus V} g_k(x)$ if and only if there is an $x \in X \setminus V$ such that $x_n \in g_k(x)$ if and only if there is an $x \in X$ such that $\pi_i(x) \in X_i \setminus \pi_i(V)$ for some $i \in F$ and $x_n \in g_k(x)$ if and only if there is an $x \in X$ such that $\pi_i(x) \in X_i \setminus \pi_i(V)$ for some $i \in F$ and $\pi_i(x_n) \in g_{ik}(\pi_i(x))$. This implies $\pi_i(x_n) \in \cup\{g_{ik}(y) | y \in X_i \setminus \pi_i(V)\}$. Thus $\{n \in \omega | x_n \in \cup_{x \in X \setminus U} g_k(x)\}$ is finite since $V \subset U$ □

Theorem 2.8. *The finite union of closed cs-semistratifiable space is cs-semistratifiable.*

Proof. Let $X = \cup_{i=1}^{n} X_i$ where X_i is cs-semistratifiable for each $i (i = 1, 2, \cdots, n)$. For each $i (i = 1, 2, \cdots, n)$, let S_i be a cs-semistratification for X_i and let τ be the topology of X. Define $S; w \times \tau \to$ {closed subsets of X} by $S_k(U) = \cup_{i=1}^{n} S_i(k,\ U \cap X_i)$. Each $\{S_i(k,\ U \cap X_i) | i = 1, 2, \cdots, n\}$ is finite, in special it has the closure preserving property stated in the next section later. This insures each $S_n(U)$ is closed in X. We can show that S has the required additional condition and so is it a cs-semistratification for X □

Definition 2.9. A topological space is F_σ-screenable if every open cover has a σ-discrete closed refinement which covers the space.

Theorem 2.10. *A cs-semistratifiable space is F_σ- screenable.*

Proof. Let X be a cs-semistratifiable space with a cs-semistratification, $U = \cup_{n=1}^{\infty} S_n$ (U).

For each $n \in w$ and $x \in X$, define $g_n(x) = X \setminus S_n(X\ Cl\{x\})$. where g is a semistratifiable function of X satisfying the additional condition of Lemma 2.6. Let $\{V_\alpha | \alpha \in I\}$ be an open cover of X and let I be well-ordered. For each natural

$n \in \omega$, define $H_{1n} = (V_1)_n$ and for each $\alpha > 1$, $H_{\alpha n} = (V_\alpha)n \setminus \cup\{V_\beta | \beta \in I,\ \beta < \alpha\}$. For each $n \in w$, let $\mathcal{A}_n = \{H_{\alpha n} | \alpha \in I\}$. Then $\mathcal{A}$ is a discrete collection of closed sets. By the well-ordering on I, $A = \cup_{n=1}^{\infty} \mathcal{A}_n$ covers X □

3. Mapping and Stratifiable Spaces

Theorem 3.1. *The closed continuous image of a cs-semistratiable space is cs-semistrtifiable.*

Proof. Let f be a closed continuous function from a cs-semistratifiable space X onto a topological space Y. Let S be a cs-semistratification for X. For each open V of Y and for each $n \in \omega$, let $T_n(V) = f[S_n(f^{-1}(V))]$. Then T: $W \times T_Y \to \{closed\ subsets\ of\ Y\}$ is a semistratification for Y.

Now let $y_n \to y_0$ be a convergent sequence in Y, since the function $f : X \to Y$, continuous and so it is sequentially condinuous at any point $x \in X$, there exists a convergent sequence $x_n \to x_0$ in such that $f(x_n) = y_n$ for $n = 0, 1, 2, \cdots$. Hence there is an $n_0 \in \omega$ such that $\{x_n | x_n \in \omega\}$ is eventually in $S_{n_0}(f^{-1}(V))$ for any open $V(y_0)$ in Y. Thus $\{y_n\}$ is eventually in $T_{n_0}(V) = f[S_{n_0}(f^{-1}(V))]$ □

Lutzer [8] proved that a first countable k-semistratifiable space is stratifiable. The proof of his insures the following.

Theorem 3.2. *A first countable cs-semistratifiable space is stratifiable.*

Proof. Let S be a cs-semistratification for X. Suppose $p \in V$ where V is open. Let $\{W(n) | n \in \omega\}$ be a local base of a neighborhoods for p such that $V \supset W[1] \supset W[2] \supset \cdots$. If $W[n] \subset S_n(V)$ for each $n \in \omega$, choose points $y(n) \in W[n] \setminus S_n(V)$ for each $n \in \omega$. The sequence $\{y(n) | n \in \omega\}$ converges to p, and so there is an $n_0 \in \omega$ such that $\{y(n) | n \in \omega\}$ is eventually in $S_{n_0}(V)$. Therefore, for some $n \in \omega$, $W(n) \subset S_n(V)$, that is $p \in S_n(V)^0$ □

Corollary 3.3. *X is first countable and k-semistratifiable if and only if X is first countable and cs-semistratifiable.*

4. Cs-semistratifiable with (CF) property

Definition 4.1. [10, Definition 3.1] A family $\mathcal{A}$ of subsets of a space X is called finite on compact subsets of X, briefly CF in X, if $\mathcal{A}|K$ is a finite family for any compact subsets K of X.

Definition 4.2. A cs-semistratification S of a space X is called to have (CF) property if the following condition (CF) is satisfied:

(CF) For each $n \in \omega$, $\{S(n, U)|U \in \tau\}$ is CF in X.

A space having S with(CF) property is called a cs-semistratifiable space with(CF) property.

Definition 4.3. A network(or net)[1] in a space X is a collection $\mathcal{B}$ of subsets of X such that given any open subset $U \subset X$ and $x \in U$, there is a member B of $\mathcal{B}$ such that $x \in B \subset U$. A k-network(called a pseudo base by Michael [9]) is a collection $\mathcal{B}$ of subsets of X such that given any compact subset K and any open subset U of X containing K, there is a $B \in \mathcal{B}$ such that $K \subset B \subset U$. A cs-network [6] is a collection $\mathcal{B}$ of subsets of X such that given any convergent sequence $x_n \to x$ and any open U containing x, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$ and $\{x_n\}$ is eventually in B.

Definition 4.4. $\mathcal{B}$ is closure-preserving(or σ-closure-preserving) if $\mathcal{B}$ can be represented as a union of countably many closure-preserving subcollections, that is,

$$\cup\{\bar{B}|B \in \mathcal{B}\} = \overline{\cup\{B|B \in \mathcal{B}\}}.$$

Theorem 4.5. *If a space X has a σ-HCP(=hereditarily closure-preserving) cs-network, then X is a cs-semistratifiable space with(CF) property.*

Proof. Let $\cup\{\mathcal{B}_n|n \in \omega\}$ be a cs-network for X, where for each n, $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ and B_n is an HCP family of closed subset of X. For each $(n, U) \in \omega \times \tau$, let

$$S_n(U) = \cup\{B \in \mathcal{B}_n|B \subset U\}$$

Then it is easily seen from [10, Proposition 3.2] that S is a cs-semistratification with (CF) property □

Example 4.6. *There exists a stratifiable, cs-semistratifiable space with (CF) property, but does not have a σ-HCP k-network.*

Solution. Let Y be a non-metrizable Lašnev space which has no σ-locally finite k-network.(For example, let Y be the quotient space obtained from $\oplus\{S_\alpha|\alpha < \omega_1\}$ by identifying all the limit points, where each S_α is the convergent sequence with its limit point.) Then by S.Lin in [7] the product space $X = Y \times [0, 1]$ has no σ -HCP k-network.

X is obviously a stratifiable space. By Theorem 4.6 stated below, X is a cs - semistratifiable space with(CF) property □

Theorem 4.6. *If a space X is embedded into a countable product of Lašnev spaces, then X is a cs-semistratifiable with (CF) property.*

Proof. By the same method as in [9, Lemma 5.1 and Proposition 6.1] and by [10, Proposition 3.3], we can show that X has a σ-closure-preserving, CF family $\cup_n^\infty B_n$ of closed subsets of X, which forms a k-network for X. For each $(n, U) \in \omega \times \tau$, let $S_n(U) = \cup\{B \in \cup_{t\leq n}\mathcal{B}_t | B \subset U\}$. Then S is a cs-semistratification with (CF) property □

References

1. H.R. Bennett and H.W. Martin, *A note on certain spaces with bases(modk)*, Can.J. Math. **27** (1975), 469-474.
2. R.H. Bing, *Metrization of topological spaces*, Can.J. Math. **3** (1951), 175-186.
3. C.J.R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1-16.
4. ________, *On metrizability of topological spaces*, Can.J. Math. **20** (1968), 795-804.
5. J.G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105-125.
6. G.D. Creede, *Concerning semi-stratifiable space*, Pacific J. Math. **32 No.1** (1970), 47-54.
7. S. Lin, *On a problem of K.Tamano*, ibid (1988), 99-102.
8. D.J. Lutzer, *Semistratifiable and stratifiable spaces*, General Topology and its Appl. 1 (1971), 43-48.
9. E.A. Michael, $\aleph_0$*-spaces*, Journal Math. Mech. **15** (1966), 983-1002.
10. T. Mizokami, *On CF families and hyperspaces of compact subsets*, Topology and its Appl. **1** (1974), 182-201.

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