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AN IMPLEMENTATION OF WEIGHTED L_∞ – METRIC PROGRAM TO MULTIPLE OBJECTIVE PROGRAMMING

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1. Introduction

Multiple objective programming has been a popular research area since 1970. The pervasiveness of multiple objective in decision problems have led to explosive growth during the 1980's. Several approaches (interactive methods, feasible direction methods, criterion weight space methods, Lagrange multiplies methods, etc) have been developed for solving decision problems having multiple objectives. However there are still many mathematically challengings including multiple objective integer, nonlinear optimization problems which require further mathematically oriented research.

In this paper, we give a method that is similar to the interval criterion weight/ vector maximum procedure. But, instead of using weighting vectors, we use weighted L_∞ - metrics. This method enable us to compute unsupported and improperly nondominated criterion vectors. Also, the results can be applied for an efficient interactive procedure in multiple objective programming.

A formulation of multiple objective programming (MOP) is given by

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(MOP) \qquad \begin{array}{ll} \max & f_1(X) = z_1 \\ & f_2(X) = z_2 \quad \text{or} \quad \max \quad z = f(X) \\ & \cdots\cdot \qquad\qquad\quad \text{subject to} \quad x \in s \\ & f_k(X) = z_k \\ & \text{subject to} \quad x \in S \end{array}
$$

where the objectives f_i is real valued function, S is the decision space that is a subset of R^n. Here, the f_i need not be linear and S need not be convex. But it is assumed that each f_i is bounded over S and that there does not exist a point in S at which all objectives are simultaneously maximized.

To utilize the weighted L_∞ - metric approach, we first compute a z^{**} **ideal criterion** vector for (MOP). The ith component z_i^{**} of z^{**} is given by

$$z_i^{**} = z_i^* + \epsilon_i$$

where $z_i^* = \sup\{f_i(X) | x \in S\}$ and $\in_i \geq 0$. In general, it suffices for each ϵ_i to be positive. However, it is permissible for one or more of the ϵ_i's to be zero. The only time a given ϵ_i must be positive is either there is more than one criterion vector that maximizes f_i or there is only one criterion vector that maximizes f_i, but this criterion vector also maximizes one of the other objectives.

In the following developments, Z denotes the criterion space and N denotes the set of nondominated criterion vectors of (MOP).

2. Basic definitions

In this section, we quote some definitions which will be needed.

Definition 2.1. Let $\lambda \in \bigwedge = \{(\lambda_1, \lambda_2, \cdots, \lambda_k \in R^k | \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1\}$ be a given weight vector. **The weighted** $L_\infty - metric$ with respect to λ is given by

$$||z - w||^\lambda = \max_{1 \leq i \leq k} \{\lambda |z_i - w_i|\}$$

for $z = (z_1, z_2, \cdots, z_K), w = (w_1, w_2, \cdots, w_K) \in R^k$.

Note that if z^{**} is an ideal criterion vector and Z is any vector in the criterion space Z, only the lower left - hand portions of $||z^{**} - z||^\lambda$ contour can intersect Z. Therefore, since we will only be using this metric to find points in Z closest to z^{**}, we can drop the absolute value sign in the above definition.

Definition 2.2.

(1) Let $\bar{z} \leq z^{**}$ and $\lambda \in \bigwedge$. Then $\bar{z}$ is a **definition point** of the $||z^{**} - \bar{z}||^\lambda$ contour if and only if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ is defined by

$$\lambda_i = \begin{cases} \frac{1}{z_i^{**} - \bar{z}_i} \left(\sum_{j=1}^k \frac{1}{z_j^{**} - \bar{z}_j} \right)^{-1} & \text{if } \bar{z}_j \neq z_j^{**} \text{ for all } j \\ 1 & \text{if } \bar{z}_i = z_i^{**} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(2) Let $\bar{z} < z^{**}$. Then, $\bar{z}$ is a **vertex** of a give $||z^{**} - \bar{z}||^\lambda$ contour if and only if $\bar{z}$ is an extreme point of the closed convex set in R^k whose boundary is the contour.

(3) Let $\bar{z}$ be a definition point of a $||z^{**} - \bar{z}||^\lambda$ contour. Then the line segment connecting z^{**} with $\bar{z}$ is the **diagonal** of the contour. A direction $\bar{z} - z^{**}$ is called a **diagonal direction** of the contour.

Note that if $\bar{z}$ is a definition point and $\bar{z} < z^{**}$. then $\bar{z}$ is the vertex of the contour. A $||z^{**} - \bar{z}||^\lambda$ contour can have at most one vertex. when $k > 2$ and the contour does not have a vertex, the contour has an infinite number of diagonal directions. However, when the contour has a vertex, its unique diagonal direction is given by

$$-\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_k}\right).$$

3. Weighted L_∞ - metric Programs

We now formulate programs for finding the points in criterion space Z closest to z^{**} ideal criterion vector according to the weighted L_∞ - metric. The weighted L_∞ - metric program $(WP)_\lambda$ is given by

$$(WP)_\lambda \qquad \begin{array}{ll} \min \alpha & \\ \text{subject to} & \alpha \geq \|z^{**} - z\|^\lambda \\ & f(x) = z \\ & x \in S \end{array}$$

The augmented weighted L_∞ - metric program $(AWP)_{\lambda,\rho}$ is given by

$$(AWP)_{\lambda,\rho} \qquad \begin{array}{ll} \min\{\alpha + \rho e^T(z^{**} - z)\} & \\ \text{subject to} & \alpha \geq \|z^{**} - z\|^\lambda \\ & f(x) = z \\ & x \in S \end{array}$$

where $\rho > 0$ and e^T is the sum vector of ones.

Any solution to $(WP)_\lambda$ or $(AWP)_{\lambda,\rho}$ is a vector of the form $(\bar{x}, \bar{z}, \bar{\alpha}) \in R^{n+k+1}$ where $\bar{z}$ is a closest criterion vector and $\bar{x}$ is its inverse image in the decision space.

For the sake of simplicity, let the decision space S be finite. Then, the set of criterion space Z and, hence the set of nondominated criterion vectors N are finite, Even if S is polyhedral set or infinite - discrete region, the arguments in the followings are still valid.

We will show how each of the element in N is uniquely computable via the $(AWP)_{\lambda,\rho}$, that is , for each $\bar{z} \in N$, we can find a $\lambda \in \bigwedge$ such that $\bar{z}$ is the unique optimal solution of the $(AWP)_{\lambda,\rho}$.

The next theorem shows that among the optimal solution of the $(WP)_\lambda$, we can find at least one nondominated criterion vector.

Theorem 3.1. *Let $M = \{z \in Z | (x, z, \alpha)$ is a minimal solution of $(WP)_\lambda\}$ Then $M \cap N \neq \phi$.*

Proof. Since Z is finite, $N \neq \phi$. Let $(\bar{x}, \bar{z}, \bar{\alpha})$ be a solution of $(WP)_\lambda$. O_n the contrary, suppose $M \cap N = \phi$. Let $\hat{z}$ be an element of M that is not dominated by another element of M. If $\hat{z} \notin N$, then there exists a $z' \in N$ such that. $z' \geq \hat{z}, z' \neq \hat{z}$. Since $\bar{\alpha}$ is optimal value of $(WP)_\lambda$, $z' \in M$, which is a contradiction.

For a given z^{**} ideal criterion vector for (MOP) and for $\alpha \geq 0$, define

$$\phi(\alpha) = \{z = (z_1, z_2, \cdots, z_K) \in R^k | z_i \in [z_i^{**} - \frac{\alpha}{\lambda_i}, \infty),\ \lambda_i > 0\}$$

Then, by finding the minimal $\alpha \in R$ of the $(WP)_\lambda$, we find the smallest (in a subset sense) $\phi(\alpha)$ in the nested family of sets $\{\phi(\alpha)\}_{\alpha \geq 0}$ that intersect the criterion space Z.

Lemma 3.2. *Let $Z^p \in N$ be a definition point of $||z^{**} - z^p||^\lambda$ contour. Then, there does not exist $z^q \neq z^p$ such that $z^q \in Z \cap \phi(\alpha_{pp})$, where*

$$\alpha_{pp} = ||z^{**} - z^p||^\lambda$$

Proof. First, suppose $z_j^p \neq z_j^{**}$ for each j. Substituting for λ_j in each of the k inequalities

$$\alpha_{pp} \geq \lambda_j(z_j^{**} - z_j^p), \quad 1 \leq j \leq k$$

we have

$$\begin{aligned} \alpha_{pp} &= \sum_{j=1}^{k} [\frac{1}{z_j^{**} - z_j^p}]^{-1} \\ &= \lambda_j(z_j^{**} - z_j^p), \quad 1 \leq j \leq k \end{aligned}$$

Hence

$$z_j^p = z_j^{**} - \frac{\alpha_{pp}}{\lambda_j}, quad1 \leq j \leq k$$

Since $z^p \in N$, there does not exist a $z^q \neq z^p$ such that $z^q \in \phi(\alpha_{pp})$ Now, assume that for some i, $z_i^p = z_i^{**}$. Then $\lambda_j = 0$ for all $j \neq i$. Thus, $\alpha_{pp} = 0$. In this case,

$$\phi(\alpha_{pp}) = \{z \in R^k | z_i \in [z_i^p, \infty)\}$$

Since $z^p \in N$, there does not exist a $z^q \neq z^p$ such that $z^q \in \phi(\alpha_{pp})$.

Lemma 3.3. *Let $z^p \in N$ be a definition point of $||z^{**} - z^p||^\lambda$ contour. Let $z^q \in Z$ such that $z^q \neq z^p$. Let $\alpha_{pq} = ||z^{**} - z^q||^\lambda$. Then $\alpha_{pp} < \alpha_{pq}$.*

Proof. Since $z^p \in N$, by lemma 3.2, no other $z^q \in Z$ lies in $\phi(\alpha_{pp})$. Thus, all $z^q \in Z(z^q \neq z^p)$ lie in superset of $\phi(\alpha_{pp})$. Hence $\alpha_{pp} < \alpha_{pq}$.

Theorem 3.4. *Let $I_Z = \{i | z^i \in Z\}, I_N = \{i | z^i \in N\}$ and $z^p \in N$ be a definition point of $||z^{**} - z^p||^\lambda$ contour. Then, z^p uniquely minimizes $(AWP)_{\lambda,\rho}$ where*

$$0 < \rho < \min_{q \in I_z - \{p\}} \left\{ \frac{\alpha_{pq} - \alpha_{pp}}{\sum_{i=1}^{k}(z_i^q - z_i^p)} | \sum_{i=1}^{k}(z_i^q - z_i^p) > 0 \right\} \tag{2}$$

Proof. From lemma 3.3, it is clear that there exist a $\rho > 0$ as (2). Suppose $z^q \in Z, z^q \neq z^p$ minimizes $(AWP)_{\lambda,\rho}$. Then, a lower bound for the minimal value of the objective function is

$$\alpha_{pp} + \rho \sum_{i=1}^{k} (z_i^{**} - z_i^q).$$

The minimality of z^p is preserved if

$$\alpha_{pp} + \rho \sum_{i=1}^{k} (z_i^{**} - z_i^q) < \alpha_{pq} + \rho \sum_{i=1}^{k} (z_i^{**} - z_i^q).$$

Since, $\alpha_{pp} < \alpha_{pq}$, the optimality of z^p can be violated only when

$$\sum_{i=1}^{k} (z_i^{**} - z_i^q) < \sum_{i=1}^{k} (z_i^{**} - z_i^p)$$

or

$$\sum_{i=1}^{k} (z_i^p - z_i^q) > o.$$

To assure the optimality of z^p, we must have

$$\rho \sum_{i=1}^{k} (z_i^q - z_i^p) < \alpha_{pq} - \alpha_{pp}$$

for all $q \in I_z - \{p\}$.

Thus, it is suffices for ρ to be defined as (2) for z^p to uniquely minimize $(AWP)_{\lambda,\rho}$, since λ is defined as (1) in section 2.

Theorem 3.5. *Let $I_Z = \{i | z^i \in Z\}, I_Z = \{i | z^i \in N\}$ and let*

$$0 < \rho < \min_{i \in I_N} \left[\min_{j \in I_z - \{i\}} \left\{ \frac{\alpha_{ij} - \alpha_{ii}}{\sum_{l=1}^{k} (z_l^j - z_l^i)} | \sum_{l=1}^{k} (z_l^j - z_l^i) > 0 \right\} \right]$$

Then, $z^p \in N$ if and only if there exists a $\lambda \in \bigwedge$ such that z^p minimizes the $(AWP)_{\lambda,\rho}$.

Proof. Suppose $z^p \in N$. Let λ be defined as (1) in section 2. Then it follows from Theorem 3.4

To show the converse, suppose $z^p \notin N$ minimizes the $(AWP)_{\lambda,\rho}$ for some $\lambda \in \bigwedge$. Since z^p is dominated by some other element z^q,

$$(z^{**} - z^q) \leq (z^{**} - z^p)$$

so,

$$\rho \sum_{i=1}^{k} (z_i^{**} - z_i^q) < \rho \sum_{i=1}^{k} (z_i^{**} - z_i^p)$$

Hence, z^q would have a smaller objective function value than z^p, which is a contradiction. Thus, $z^p \in N$.

Remarks. *Theorem 3.4 and Theorem 3.5 are also valid even if the decision space S is a polyhedral set, since the contours of weighted L_∞ - metric are piecewise linear which implies N to be finite. For the case of nonlinear are infinite discrete feasible region, we can modify the weighted L_∞ - metric program slightly so that these theorems are still valid.*

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