

EXTENSION OF GANELIUS' THEOREM

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1. Introduction

In this paper, we extend Ganelius' lemma in Anderson [1]. In the Ganelius' original version several of the a_k are equal to 1, but in our extension theorem we have the a_k distinct and all unequal to 1. Then our theorem can be used to introduce an indefinite quadrature formula for $\int_{-1}^1 f(x)dx$, $f \in H^p$, with $p > 1$.

We will also correct an error in the proof of Ganelius' theorem provided in Ganelius [2].

2. Extension theorem

Ganelius' theorem [2] sharpens and extends Newman [5]'s result on rational approximation of $|x|$. When we try to derive an indefinite quadrature formula using the derivatives of integrand, we can not use the Ganelius' lemma in Anderson [1] because of the several multiple points equal to 1. Here we have an extension theorem of Ganelius' lemma that does not have any multiple points. Moreover, all points in the extension theorem are unequal to 1.

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Theorem. For $r > 0$ and N a positive integer, there are distinct numbers a_1, a_2, \dots, a_N in $(0, 1)$ such that

$$\max_{x \in [0,1]} x^r \prod_1^N \left| \frac{x - a_k}{x + a_k} \right| \leq C \exp(-\pi\sqrt{Nr}),$$

where C is an absolute constant.

Proof. We begin by rewriting the inequality in theorem as

$$\sum_{k=1}^N \log \left| \frac{x + a_k}{x - a_k} \right| \geq \pi\sqrt{Nr} + r \log x - C \quad (1)$$

and observing that the sum may be written as

$$\int_0^1 \log \left| \frac{x + y}{x - y} \right| d\nu(y),$$

where $d\nu$ is a discrete measure with unit masses at the points a_k . In order to determine such a_k we introduce the continuous measure

$$d\tilde{\mu}(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq 1/\varphi(n); \\ \frac{2}{\pi^2} \log(y\varphi(n))^r y^{-1} dy & \text{if } 1/\varphi(n) < y < 1. \end{cases}$$

Here n is a positive integer which we shall relate to N below, and $\varphi(n) = \exp\left(\pi\sqrt{\frac{n}{r}}\right)$.

We shall show that

$$\int_0^1 \log \left| \frac{x + y}{x - y} \right| d\tilde{\mu}(y) \geq \pi\sqrt{nr} + r \log x - \frac{\pi x}{2} \sqrt{nr} - C. \quad (2)$$

We then introduce the discrete measure $d\mu(y)$ that has unit masses at the $n + 1$ points

$$y_k = \varphi(k)/\varphi(n), \text{ for } k = 0, 1, 2, \dots, n - 1, \text{ and } y_n = \varphi(n - \frac{1}{2})/\varphi(n),$$

and show that

$$\int_0^1 \log \left| \frac{x+y}{x-y} \right| d\mu(y) - \int_0^1 \log \left| \frac{x+y}{x-y} \right| d\tilde{\mu}(y) \geq -C. \quad (3)$$

Finally, we shall add n' additional points y_k , for $k = n+1, n+2, \dots, n+n'$, with $n' = O(\sqrt{n})$, to remove the $\frac{\pi x}{2} \sqrt{nr}$ term from (2). This will establish (1), with $N = n+1+n'$ and the a_{k+1} being the y_k .

Proof of (2) : we have

$$\int_0^1 \log \left| \frac{x+y}{x-y} \right| d\tilde{\mu}(y) = \frac{2}{\pi^2} \int_{1/\varphi(n)}^1 \log \left| \frac{x+y}{x-y} \right| \log(y\varphi(n))^r y^{-1} dy.$$

By setting $\xi = x\varphi(n)$ and $u = y/x$ the last expression takes the form

$$\begin{aligned} & \frac{2r}{\pi^2} \int_{\frac{1}{\xi}}^{\frac{1}{x}} \log \left| \frac{1+u}{1-u} \right| \log \xi \cdot \frac{1}{u} du + \frac{2r}{\pi^2} \int_{\frac{1}{\xi}}^{\frac{1}{x}} \log \left| \frac{1+u}{1-u} \right| \log u \cdot \frac{1}{u} du \\ &= \frac{2r}{\pi^2} \int_{\frac{1}{\xi}}^{\frac{1}{x}} \log \left| \frac{1+u}{1-u} \right| \log u \cdot \frac{1}{u} du + \frac{2r}{\pi^2} \int_0^{\infty} \log \left| \frac{1+u}{1-u} \right| \log \xi \cdot \frac{1}{u} du \\ & \quad - \frac{2r}{\pi^2} \int_0^{\frac{1}{\xi}} \log \left| \frac{1+u}{1-u} \right| \log \xi \cdot \frac{1}{u} du - \frac{2r}{\pi^2} \int_{\frac{1}{x}}^{\infty} \log \left| \frac{1+u}{1-u} \right| \log \xi \cdot \frac{1}{u} du \quad (4) \end{aligned}$$

The integrand in the first term of (4) is positive for $u > 1$, so the term is bounded below by

$$\frac{2r}{\pi^2} \int_0^1 \log \left| \frac{1+u}{1-u} \right| \log u \cdot \frac{1}{u} du$$

The second term of (4) is $\log \xi^r$ that is $\pi \sqrt{nr} + r \log x$, and if $\xi \geq 1$ i.e. if $x \geq 1/\varphi(n)$ the third term of (4) is bounded by

$$\frac{2r}{\pi^2 \xi} \int_0^1 \log \left| \frac{1+u}{1-u} \right| \log \xi \cdot \frac{1}{u} du$$

because

$$\frac{1}{u} \log \frac{1+u}{1-u}$$

is increasing on $(0, 1)$. Finally, by letting $u = 1/t$ the last term of (4) becomes

$$\frac{2r}{\pi^2} \int_0^x \log \left| \frac{1+t}{1-t} \right| \log \xi \cdot \frac{1}{t} dt.$$

Since

$$\begin{aligned} \int_0^x \log \left| \frac{1+t}{1-t} \right| \frac{1}{t} dt &= \int_0^x 2 \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots \right) dt \\ &\leq 2x \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{\pi^2 x}{4}, \end{aligned}$$

the last term of (4) is bounded below by $\frac{rx}{2} \log \xi$ that is $\frac{\pi x}{2} \sqrt{nr}$. Then we get (2) for $x \geq 1/\varphi(n)$. For $0 \leq x < 1/\varphi(n)$ it is obvious.

Proof of (3) : letting

$$g(x, y) = \log \left| \frac{x+y}{x-y} \right|,$$

we write the left hand side of (3) as

$$\begin{aligned} \sum_{k=0}^{n-2} \left[\frac{1}{2} g(\xi, \varphi(k)) + \frac{1}{2} g(\xi, \varphi(k+1)) - \int_k^{k+1} g(\xi, \varphi(u)) du \right] \\ + \frac{1}{2} g(\xi, \varphi(0)) + \frac{1}{2} g(\xi, \varphi(n-1)) \\ + \left[g(\xi, \varphi(n-1/2)) - \int_{n-1}^n g(\xi, \varphi(u)) du \right], \quad (5) \end{aligned}$$

where $\xi = x\varphi(n)$ as before. The second and third terms are positive and so may be ignored. We shall focus on proving that the first term - the sum - is bounded below,

and only remark on the (mostly similar) proof that the fourth term is bounded below. Each term in the sum is the error of a trapezoidal approximation to an integral, which is non-negative if the integrand is convex. Suppose k_0 is such that $y_{k_0} < x < y_k$ if $x \leq \varphi(n-1)/\varphi(n)$; if $x > \varphi(n-1)/\varphi(n)$ take $k_0 = n-1$. Ganelius [2] states that for $k \neq k_0$, $g(\xi, \varphi(u))$ is convex on $(k, k+1)$ in the proof of his theorem with $r = 1$. That is true for $k \neq 0$, but for $k = 0$ we need a small additional argument. It is sufficient to show that

$$\int_0^1 g(\xi, \varphi(u)) du = \int_0^1 \log \left| \frac{\xi + \varphi(u)}{\xi - \varphi(u)} \right| du$$

is bounded for $\xi > 1$. That is true since the integrand approaches 0 uniformly in u as ξ approaches infinity, so that the integral approaches 0 as ξ approaches infinity. And the integral exists for all ξ and is a continuous function of ξ .

Now to deal with the term in the sum of (5) for $k = k_0$, we write the integral in that term as

$$\frac{2r}{\pi^2} \int_{\varphi(k)}^{\varphi(k+1)} g(\xi, y) \frac{\log y}{y} dy$$

and note that replacing $\log y/y$ by $(\log \varphi(k))/\xi$ makes only a bounded difference. So all we need to deal with is

$$\frac{1}{2} \log \left| \frac{\xi + \varphi(k)}{\xi - \varphi(k)} \right| + \frac{1}{2} \log \left| \frac{\xi + \varphi(k+1)}{\xi - \varphi(k+1)} \right| - \frac{2r}{\pi^2} \int_{\varphi(k)}^{\varphi(k+1)} \log \left| \frac{\xi + y}{\xi - y} \right| \frac{\log \varphi(k)}{\xi} dy.$$

Setting $1 + \delta = \frac{\varphi(k+1)}{\xi}$ and $1 - \epsilon = \frac{\varphi(k)}{\xi}$, we have

$$\frac{2r}{\pi^2} \log \varphi(k) = \frac{1}{\epsilon + \delta} + O(1).$$

So the quantity we are dealing with is

$$\frac{1}{2} \log \frac{1}{\epsilon} + \frac{1}{2} \log \frac{1}{\delta} - \left(\frac{\epsilon}{\epsilon + \delta} \log \frac{1}{\epsilon} + \frac{\delta}{\epsilon + \delta} \log \frac{1}{\delta} \right) + O(1)$$

where $0 < \epsilon < \delta$. We can then see that this last quantity is bounded below.

As for the last term in (5), the difficulty is mainly for ξ between $\varphi(n-1)$ and $\varphi(n)$; it is dealt with by an argument similar to the one above for $k-k_0$. Hence (3) is established.

Our last step in proving theorem is the elimination of the $(\pi x/2)\sqrt{nr}$ term in (2). For that we use the fact that

$$\log \left| \frac{x+y}{x-y} \right| \geq 2x \text{ for } 0 \leq x \leq 1$$

if

$$y \geq \frac{e^2 - 1}{e^2 + 1} = 0.76 \dots$$

We set

$$n' = \left[\frac{\pi}{4} \sqrt{nr} \right] + 1$$

and take as $y_{n+1}, \dots, y_{n+n'}$, any distinct numbers in $[0.8, 1)$ that are different from all of $y_0, y_1, \dots, y_{n-1}, y_n$. \square

3. Application

By using the Ganelius' theorem [2], Anderson [1] estimates the upper bound of

$$\sigma_{q,n} = \inf_{B_n} \left(\int_{-1}^1 |B_n|^q dx \right)^{\frac{1}{q}}, \quad \text{with } q = p/(p-1),$$

obtaining from the error of definite optimal quadrature formula with n nodes for $\int_{-1}^1 f(x)dx$, $f \in H^p$, with $p > 1$ and

$$B_n(z) = \prod_{i=1}^n \left(\frac{z - z_i}{1 - z_i z} \right), \quad |z_i| < 1.$$

Transforming our extension theorem to the unit disk by

$$x = \frac{1 - z^2}{1 + z^2}, \quad b_k = \sqrt{\frac{1 - a_k}{1 + a_k}}, \quad \text{and } b_{-k} = -b_k,$$

we find that

$$\max_{z \in [-1, 1]} (1 - z^2)^r \prod_{-N}^N{}' \left| \frac{z - b_k}{1 - b_k z} \right| \leq C \exp(-\pi \sqrt{Nr}).$$

Here the prime on a product indicates that the index value $k = 0$ is excluded. Since all of the b_k used as nodes for the indefinite quadrature formula are not multiple, we can introduce derivatives of the integrand into the formula with $2N$ nodes for $\int_{-1}^1 f(x) dx$, $f \in H^p$, with $p > 1$. Then our indefinite quadrature formula will have simple form. In our quadrature formula abscissas and coefficients may be calculated easier than in Haber [4]'s formula and other formulas of that type that use the sin - integral which is not as easy to evaluate as the log. The convergence rate of our indefinite quadrature formula for functions in H^p will also be better than that of Stenger type formulas [6] by a factor of $\sqrt{2}$ in the constant of exponential.

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