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AN UNFOLDING OF DEGENERATE EQUILIBRIA WITH LINEAR PART $x' = y,\ y' = 0$.

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ABSTRACT. In this paper, we study the dynamics of a two-parameter unfolding system $x' = y, y' = \beta y + \alpha f(x,\alpha) \pm xy + yg(x)$, where $f(x,\alpha)$ is a second order polynomial in x and $g(x)$ is strictly nonlinear in x. We show that the higher order term $yg(x)$ in the system does not change qulitative structure of the Hopf bifurcations near the fixed points for small α and β if the nontrivial fixed point approaches to the origin as α approaches zero.

1. Introduction

Consider

$$\begin{aligned} x' &= y + F(x,y) \\ y' &= \qquad G(x,y), \end{aligned} \tag{1.1}$$

where F and G are strictly nonlinear in x and y. Then the system (1.1) has a double zero eigenvalue and has a linear part

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

at the origin. (i.e., The origin is a nilpotent singularity). Takens [4] has shown that this class of systems can be put into the normal form:

$$\begin{aligned} u' &= v + \sum_{n=2} b_n u^n \\ v' &= \sum_{n=2} a_n u^n. \end{aligned} \tag{1.2}$$

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Bogdanov [2], Guckenheimer and Holmes [3] and Arnold [1] have chosen the alternate normal form for this problem:

$$\begin{aligned} U' &= V \\ V' &= \sum_{n=2} a_n U^n + V \sum_{n=2} n b_n U^{n-1}, \end{aligned} \tag{1.3}$$

which is obtainable from (1.2) via the near-identity transformation

$$\begin{aligned} u &= U \\ v &= V - \sum_{n=2} b_n U^n. \end{aligned}$$

Assume that a_n in (1.3) vanish for all n and further assume that b_2 does not vanish. Then the system (1.3) has the following type of system:

$$\begin{aligned} x' &= y \\ y' &= y(ax + g(x)), \end{aligned} \tag{1.4}$$

where $g(x)$ is strictly nonlinear in x. The system (1.4) has a line of fixed points $y = 0$ for all x.

Now, consider the following unfolding system:

$$\begin{aligned} x' &= y \\ y' &= \beta y + \alpha f(x, \alpha) + axy + yg(x). \end{aligned} \tag{1.5}$$

The fixed points depend only on the term $\alpha f(x, \alpha)$. Clearly the number of fixed points can be infinite and hence the unfolding system has codimension infinity and therefore is impossible to analyze. In order to mimic this problem, we simplify the problem by assuming that after unfolding, we have in addition to the origin only one nontrivial fixed point. Therefore we think of the unfolding parameter α as controlling the fixed point while β introduces linear dissipation. With a suitable rescaling, for any $a \neq 0$, the possible cases can be reduced to two ([5]):

$$a = \pm 1.$$

Fix $a = -1$. (There is a similar analysis for the case of $a = 1$). Then the system (1.5) becomes

$$\begin{aligned} x' &= y \\ y' &= \beta y + \alpha f(x, \alpha) - xy + yg(x), \end{aligned} \tag{1.6}$$

where $f(x,\alpha)$ is a second order polynomial in x and $g(x)$ is strictly nonlinear in x. In this paper, we show that if the nontrivial fixed point approaches the origin, then the higher order term $yg(x)$ in (1.6) does not change the qualitative structure of the Hopf bifurcations near the fixed points for small α and β.

2. Unfolding Analysis

Assume that $(0,0)$ and $(x^*,0)$ are fixed points of the system (1.6). Then one of the fixed points is always a saddle and the other fixed point might undergo a Hopf bifurcation on some values of α and β. Note that $yg(x)$ in (1.6) does not change above property. Suppose that $(0,0)$ is always a saddle. Then depending on the form of $f(x,\alpha)$, we have one of the following cases:

(1) As α goes to zero, x^* goes to zero.
(2) x^* does not depend on α. (i.e., x^* is a constant).
(3) As α goes to zero, x^* goes to infinity.

Theorem 2.1. *For the case of* (1), *the higher order term* $yg(x)$ *in* (1.6) *does not change the qualitative structure of the Hopf bifurcations near the fixed points for small* α *and* β.

Proof. Assume that x^* goes to zero as α goes to zero. Consider the simplified system

$$\begin{aligned} x' &= y \\ y' &= \beta y + \alpha f(x,\alpha) - xy. \end{aligned} \tag{2.1}$$

The Jacobian matrix J is the following:

$$J = \begin{pmatrix} 0 & 1 \\ \alpha f_x(x,\alpha) - y & \beta - x \end{pmatrix}.$$

So,

$$J\,|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \alpha f_x(0,\alpha) & \beta \end{pmatrix}$$

and

$$J\mid_{(x^*,0)}=\begin{pmatrix} 0 & 1 \\ \alpha f_x(x^*,\alpha) & \beta - x^* \end{pmatrix}.$$

Therefore

$$\det J\mid_{(0,0)}=-\alpha f_x(0,\alpha), \quad \text{tr} J\mid_{(0,0)}=\beta,$$

and

$$\det J\mid_{(x^*,0)}=-\alpha f_x(x^*,\alpha), \quad \text{tr} J\mid_{(x^*,0)}=\beta - x^*. \tag{2.2}$$

Note that $(0,0)$ and $(x^*,0)$ are the only fixed points for the systems (1.6) and (2.1), and that the function $f(x,\alpha)$ is a second order polynomial in x, which implies

$$f_x(0,\alpha)=-f_x(x^*,\alpha).$$

Therefore $\det J\mid_{(0,0)}$ and $\det J\mid_{(x^*,0)}$ have the same value but different sign, so one of the fixed points is always a saddle. Since we have already assume that $(0,0)$ is always a saddle, we find that

$$\det J\mid_{(0,0)}=-\alpha f_x(0,\alpha)<0.$$

Hence

$$\det J\mid_{(x^*,0)}=-\alpha f_x(x^*,\alpha)>0.$$

Because of (2.2), we might expect that $\beta = x^*$ is a bifurcation curve on which $(x^*,0)$ for the system (2.1) undergoes a Hopf bifurcation. Actually, the eigenvalues associated with the linearization of $(x^*,0)$ for the system (2.1) are given by

$$\lambda_{1,2}=\frac{(\beta-x^*)\pm\sqrt{(\beta-x^*)^2+4\alpha f_x(x^*,\alpha)}}{2}$$

and so those on the curve $\beta = x^*$ are given by

$$\lambda_{1,2}=\pm i\sqrt{-\alpha f_x(x^*,\alpha)}.$$

If we view β as a parameter, then we obtain

$$\frac{d}{d\beta}Re\lambda_{1,2}\mid_{\beta=x^*}=\frac{1}{2}\neq 0.$$

Hence it appears that a Hopf bifurcation occurs on $\beta = x^*$ for the system (2.1). Therefore the fixed point $(x^*,0)$ for the system (2.1) has the following properties:

(1) If $\beta - x^* > 0$, then $(x^*, 0)$ is unstable.
(2) If $\beta - x^* < 0$, then $(x^*, 0)$ is stable.
(3) If $\beta - x^* = 0$, then the fixed point $(x^*, 0)$ undergoes a Hopf bifurcation.

Next, consider the system (1.6). The Jacobian matrix is the following:

$$J = \begin{pmatrix} 0 & 1 \\ \alpha f_x(x,\alpha) - y + y g_x(x) & \beta - x + g(x) \end{pmatrix}.$$

So,

$$J \mid_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \alpha f_x(0,\alpha) & \beta \end{pmatrix}$$

and

$$J \mid_{(x^*,0)} = \begin{pmatrix} 0 & 1 \\ \alpha f_x(x^*,\alpha) & \beta - x^* + g(x^*) \end{pmatrix}.$$

Since $(0,0)$ is a saddle, we might expect that $\beta = x^* - g(x^*)$ is a bifurcation curve on which $(x^*, 0)$ for the system (1.6) undergoes a Hopf bifurcation. Again the eigenvalues associated with the linearization of $(x^*, 0)$ for the system (1.6) are given by

$$\lambda_{1,2} = \frac{(\beta - x^* + g(x^*)) \pm \sqrt{(\beta - x^* + g(x^*))^2 + 4\alpha f_x(x^*,\alpha)}}{2}$$

and so those on the curve $\beta = x^* - g(x^*)$ are given by

$$\lambda_{1,2} = \pm i\sqrt{-\alpha f_x(x^*,\alpha)}$$

and

$$\frac{d}{d\beta} Re\lambda_{1,2} \mid_{\beta = x^* - g(x^*)} = \frac{1}{2} \neq 0.$$

Therefore the fixed point $(x^*, 0)$ for the system (1.6) also undergoes a Hopf bifurcation on $\beta = x^* - g(x^*)$ and has the following properties:

(1) If $\beta - x^* - g(x^*) > 0$, then $(x^*, 0)$ is unstable.
(2) If $\beta - x^* - g(x^*) < 0$, then $(x^*, 0)$ is stable.
(3) If $\beta - x^* - g(x^*) = 0$, then the fixed point $(x^*, 0)$ undergoes a Hopf bifurcation.

Since x^* goes to zero as α goes to zero and $g(x)$ is strictly nonlinear in x, both Hopf bifurcation curves $\beta = x^*$ and $\beta = x^* - g(x^*)$ pass through $(\alpha, \beta) = (0, 0)$. Thus the curve $\beta = x^* - g(x^*)$ for the system (1.6) is tangent to the curve $\beta = x^*$ for the system (2.1) for small α and β. Next, we introduce the following theorem to check the stability of bifurcating periodic orbits.

Theorem 2.2. ([3, 6]) *In a two-dimensional system of the form*

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y,0) \\ G(x,y,0) \end{pmatrix}, \tag{2.3}$$

where $F(x,y,0)$ and $G(x,y,0)$ are strictly nonlinear in x and y, the stability of the bifurcating periodic orbit is determined by

$$\begin{aligned} \gamma = {} & \frac{1}{16}[F_{xxx} + F_{xyy} + G_{xxy} + G_{yyy}] \\ & + \frac{1}{16\omega}[F_{xy}(F_{xx} + F_{yy}) - G_{xy}(G_{xx} + G_{yy}) \\ & - F_{xx}G_{xx} + F_{yy}G_{yy}], \end{aligned} \tag{2.4}$$

where all partial derivatives are evaluated at the bifurcating point $(x, y, \mu) = (0, 0, 0)$. If $\gamma > 0$, then the bifurcating periodic solution is unstable and if $\gamma < 0$, then the bifurcating periodic solution is asymtotically stable.

Recall from the above theorem that this involves putting the equation into a normal form and then computing a coefficient, γ, which is given by derivatives of functions occuring in this normal form. First we transform the fixed point to the origin via

$$\begin{aligned} \bar{x} &= x - x^* \\ \bar{y} &= y \end{aligned}$$

and then apply Taylor series expansion of f and g with respect to $\bar{x}$ about $\bar{x} = 0$, so that (1.6) has the following system:

$$\begin{aligned} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}' = {} & \begin{pmatrix} 0 & 1 \\ \alpha f_{\bar{x}}(\bar{x} + x^*, \alpha) & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ \frac{\alpha f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)}{2}\bar{x}^2 + (g_{\bar{x}}(\bar{x} + x^*) - 1)\bar{x}\bar{y} \\ +\bar{y}(\frac{g_{\bar{x}\bar{x}}(\bar{x}+x^*)}{2}\bar{x}^2 + h(\bar{x})) \end{pmatrix} \end{aligned} \tag{2.5}$$

at $\beta - x^* + g(x^*) = 0$, where all derivatives are evaluated at $\bar{x} = 0$ and $h(\bar{x})$ is a third or higher order polynomial in $\bar{x}$. We put the linear part of (2.5) in normal form via the linear transformation:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sqrt{-\alpha f_{\bar{x}}(\bar{x} + x^*, \alpha)}\,|_{\bar{x}=0} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

under which (2.5) is transformed into the following system:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & -\sqrt{-\alpha f_{\bar{x}}(\bar{x}+x^*,\alpha)} \\ \sqrt{-\alpha f_{\bar{x}}(\bar{x}+x^*,\alpha)} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \frac{\alpha f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)}{2\sqrt{-\alpha f_{\bar{x}}(\bar{x}+x^*,\alpha)}} v^2 + (g_{\bar{x}}(\bar{x}+x^*)-1)uv \\ +u(\frac{g_{\bar{x}\bar{x}}(\bar{x}+x^*)}{2}v^2 + \hat{h}(v)) \\ 0 \end{pmatrix}, \tag{2.6}$$

where again all derivatives are evaluated at $\bar{x} = 0$ and $\hat{h}(v)$ is a third or higher order polynomial in v. Note that the system (2.6) is exactly in the form of (2.3). And we can easily check that the term $\hat{h}(v)$ in (2.6) does not affect the normal form coefficient γ.

Let

$$\begin{aligned} F = {} & \frac{\alpha f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{2\sqrt{-\alpha f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}} v^2 + (g_{\bar{x}}(\bar{x}+x^*)\mid_{\bar{x}=0} -1)uv \\ & + \frac{g_{\bar{x}\bar{x}}(\bar{x}+x^*,0)\mid_{\bar{x}=0}}{2} uv^2 \\ G = {} & 0. \end{aligned}$$

Then from (2.4),

$$\gamma = \frac{1}{16}(F_{uuu} + F_{uvv}) + \frac{1}{16\sqrt{-\alpha f_{\bar{x}}(\bar{x}+x^*,\alpha)}} F_{uv}(F_{uu} + F_{vv}),$$

which goes to

$$\frac{1}{16}(g_{\bar{x}\bar{x}}(\bar{x}+x^*,0) - \frac{g_{\bar{x}}(\bar{x}+x^*,0) f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)}{f_{\bar{x}}(\bar{x}+x^*,\alpha)} + \frac{f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)}{f_{\bar{x}}(\bar{x}+x^*,\alpha)})$$

as α goes to zero, where all derivatives are evaluated at $\bar{x} = 0$. A simple calculation shows that as α goes to zero,

$$g_{\bar{x}\bar{x}}(\bar{x}+x^*,0)\mid_{\bar{x}=0} - \frac{g_{\bar{x}}(\bar{x}+x^*,0)\mid_{\bar{x}=0} f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}} \longrightarrow \kappa,$$

where κ is a constant and

$$\frac{f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}} \longrightarrow \pm\infty.$$

Therefore

$$\gamma \longrightarrow \frac{1}{16}(\kappa + \frac{f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}) \quad \text{as} \quad \alpha \longrightarrow 0.$$

Denote

$$\frac{f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}} \equiv \tilde{\gamma}.$$

Then for small α and β, the stability of a periodic orbit for the system (1.6) is determined by $\tilde{\gamma}$. That means that if $\tilde{\gamma}> 0$, then the corresponding bifurcation is subcritical to an unstable periodic orbit. On the other hand, if $\tilde{\gamma} < 0$, then the corresponding bifurcation is supercritical to a stable periodic orbit. For the system (2.1), we can easily show that $\kappa = 0$. Therefore we have

$$\gamma \longrightarrow \frac{f_{\bar{x}\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}}{16 f_{\bar{x}}(\bar{x}+x^*,\alpha)\mid_{\bar{x}=0}} = \frac{\tilde{\gamma}}{16} \quad \text{as} \quad \alpha \longrightarrow 0.$$

Thus the stability of the periodic orbit of system (2.1) is determined by $\tilde{\gamma}$ too. Therefore we have the following conclusion:
If $(x^*, 0)$ undergoes a Hopf bifurcation for the system (2.1), then $(x^*, 0)$ undergoes a Hopf bifurcation for the system (1.6) and the stability of the corresponding periodic orbit for the system (1.6) is the same as the stability of the periodic orbit for the system (2.1) for small α and β. Note that for the cases of (2) and (3), we can easily show that there cannot be a Hopf bifurcation locally. □

From the Theorem 2.1, we give the following corollary.

Corollary 2.3. *Consider the system (2.1). Assume that (0,0) is a saddle and that $(x^*, 0)$ undergoes a Hopf bifurcation for some values of α and β. Then the followings are satisfied:*

(1) If $x^* > 0$, then the corresponding Hopf bifurcation is subcritical to an unstable periodic orbit.
(2) If $x^* < 0$, then the correspondig Hopf bifurcation is supercritical to a stable periodic orbit.

Note that if the coefficient of xy in the system (2.1) is $+1$, then the stability of a periodic orbit switches.

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REFERENCES

1. V. I. Arnold, *Lectures on Bifurcations in Versal Families*, Russian Math. Surveys **27** (1972), 54-123.
2. R. I. Bogdanov, *Versal Deformations of a Singular Point on the Plane in the case of Zero Eigenvalues*, Funtional Analysis and Its Applications **9** (1975), 144-145.
3. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.*, Applied Mathematical Sciences **42** (1983), Springer-Verlag.
4. F. Takens, *Unfolding of Certain Singularities of Vector Fields: Generalized Hopf Bifurcation*, J. Differential Equation **14** (1973), 476-493.
5. S. Wiggins, *Introduction to Applied Nonlinear Dynamical System and Chaos* (1990), Springer-Verlag.

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