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A REMARK ON HALF-FACTORIAL DOMAINS

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ABSTRACT. An atomic integral domain R is a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. In this paper, we show that if $R[X]$ is an HFD, then $Cl_t(R) \cong Cl_t(R[X])$, and if G_1 and G_2 are torsion abelian groups, then there exists a Dedekind HFD R such that $Cl(R) = G_1 \oplus G_2$.

1. Introduction

Let R be an integral domain and $R^* = R - \{0\}$. A nonunit $r \in R^*$ is said to be *irreducible* if whenever $r = ab, a, b \in R$, either a or b is a unit of R. An integral domain$R$ is *atomic* if every nonzero nonunit of R can be factored as a product of irreducible elements of R. Following Zaks [13], we say that an atomic integral domain R is a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. It is well known that any Krull domain R with divisor class group $Cl(R) = \mathbb{Z}_2$ is a HFD, but not a UFD. It is classical that a ring of integers R of a number field is a UFD if and only if $Cl(R) = \{0\}$. The first arithmetic description of rings of integers with nontrivial divisor class groups was given in [8]. He proved that $|Cl(R)| \leq 2$ if and only if any two factorizations of an element of R into irreducible elements have the same number of factors. Thus a Dedekind domain R with the property that each nonzero ideal class contains a prime ideal is a HFD if and only if $|Cl(R)| \leq 2$. The ring of integers in a finte algebraic number field over the rationals is an example of a Dedekind domain which satisfies the condition of having a prime ideal in each ideal class. In order to measure how far an atomic integral domain R is from being a HFD, we define the *elasticity* of R as

$$\rho(R) = sup\{m/n | x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in R \text{ irreducible}\}.$$

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Thus $1 \leq \rho(R) \leq \infty$ and $\rho(R) = 1$ if and only if R is a HFD. This concept was introduced by Valenza [12], who studied $\rho(R)$ for R the ring of integers in an algebraic number field. In this paper, we show that if $R[X]$ is an HFD, then $Cl_t(R) \cong Cl_t(R[X])$, and if G_1 and G_2 are torsion abelian groups, then there exists a Dedekind HFD R such that $Cl(R) = G_1 \oplus G_2$.

For general references on factorization in integral domains, see [1], [3] or [5]

2. Half-factorial and locally half-factorial domains

Let R be an integral domain. We say that R is a *GCD-domain* if any two elements of R have a GCD in R. In [4], an integral domain R is said to be a *locally half-factorial domain* (LHFD) if each localization R_S of R is a HFD.

Given a Dedekind domain R, let $Cl(R)$ denote its divisor class group, and $[I]$ the ideal class of I in $Cl(R)$. If for a given abelian group G and subset $C \subseteq G - \{0\}$ there exists a Dedekind domain R such that $Cl(R) \cong G$ and $C = \{c | c \in G$ and c contains a nonprincipal prime ideal of $R\}$, then the pair $\{G, C\}$ is called *realizable* [11].

Let G be an abelian group and $C \subseteq G$. We say that C is an independent set in G if $n_1c_1 + \cdots + n_kc_k = 0, n_i \in \mathbb{Z}$, distinct $c_i \in C$, implies that each $n_ic_i = 0$. In fact, if R is a Dedekind domain with torsion divisor class group $\{Cl(R), A\}$ with A independent, then R is an LHFD [4, Theorem 2.5].

We start with following example:

Example 2.1. (1) Let R be a Dedekind domain with realizable pair $\{\mathbb{Z}, A\}$, where $A = \{-2, 1, 2, 3, \cdots\}$ [11, Theorem 2.4]. Then R is a HFD [7, Corollary 3.3]. Thus there exists a Dedekind HFD R' such that $\{\mathbb{Z} \bigoplus \mathbb{Z}, A \bigoplus A\}$ is a realizable pair, where

$$A \oplus A = \{(-2,0), (1,0), (2,0), \cdots (0,-2), (0,1), (0,2), \cdots\}$$

[6, Theorem 3.1]. Now, for each nonzero nonunit $f \in R'$, we have that $Cl(R'_f)$ is one of the followings:

$$\mathbb{Z} \oplus \mathbb{Z},\ \mathbb{Z}_2 \oplus \mathbb{Z},\ \{0\} \oplus \mathbb{Z},\ \mathbb{Z} \oplus \{0\},\ \mathbb{Z} \oplus \mathbb{Z}_2,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ \{0\}.$$

Suppose now that $Cl(R'_f) = \mathbb{Z}_2 \oplus \mathbb{Z}$. Then the prime ideals of R'_f are distributed in the classes $\{(1,0), (0,-2), (0,1), (0,2), \cdots\}$. If $Cl(R'_f) = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, then the prime ideals of R'_f are distributed in the classes $\{(1,0),(0,1)\}$. Since it is independent, R'_f is a HFD, and hence R' is a LHFD.

Example 2.2. Let R be a Dedekind domain with realizable pair $\{\bigoplus_{i=1}^{\infty} \mathbb{Z}_{n_i}, A\}$, where $A = \{(1,0,0,\cdots),(0,1,0,\cdots),\cdots\}$. Then A is independent, and hence R is a HFD and LHFD. On the other hand, if R is a Dedekind domain associated to the realizable pair $\{\mathbb{Z}_9, \{1,3\}\}$, then R is a HFD and LHFD. But $\{1,3\}$ is not an independent set.

A saturated multiplicative set S of R is called a *splitting multiplicative set* [2] if for each nonzero $d \in R$, $d = sa$ for some $s \in S$ and $a \in R$ with $s'R \cap aR = s'aR$ for all $s' \in R$. The set $T = \{0 \neq t | sR \cap tR = stR$ for all $s \in S\}$ is also a splitting set and we call T the *complementary multiplicative set* for S. A splitting multiplicative set S of R is said to be an *lcm-splitting set* if for each $s \in S$ and $d \in R$, $sR \cap dR$ is principal. S is a splitting set of R if and only if R_T is a GCD-domain, where T is the complementary multiplicative set [2, Proposition 2.4].

For an integral domain R, let $Cl_t(R)$ denote the t-class group of R, i.e., the group of t-invertible t-ideals of R modulo its subgroup of principal fractional ideals. For example, if R is an integral domain such that each nonunit element of R is a product of primary element, then $Cl_t(R) = \{0\}$. In particular, if R is atomic, then $\rho(R) = sup\{\rho(R_P) | htP = 1\}$ [6, Corollary 2.5].

Theorem 2.3. *Let R be an atomic domain and let S be an lcm-splitting multiplicative set. Then*

(1) $\rho(R) = \rho(R_S)$ *and* $Cl_t(R) \cong Cl_t(R_S)$.
(2) *If* $R[X]$ *is a HFD, then* $Cl_t(R) \cong Cl_t(R[X])$.

Proof. (1) Let T be the complementary multiplicative set of S. Then $\rho(R) = max\{\rho(R_S), \rho(R_T)\}$ [6, Theorem 2.3]. By [2, Proposition 2.4], R_T is a GCD-domain; so R_T is a UFD. Thus $\rho(R_T) = 1$ and hence $\rho(R) = \rho(R_S)$. By [2, Theorem 4.1], $Cl_t(R) \cong Cl_t(R_S)$. (2) Suppose now that $R[X]$ is a HFD. Then R is integrally closed [9, Theorem 2.2], and hence $Cl_t(R) \cong Cl_t(R[X])$ [10, Theorem 3.6]. □

With the notation in Theorem 2.3, S is an lcm-splitting multiplicative set if and only if S is generated by principal primes [2, Corollary 2.7], [6, Theorem 1.6]. In particular, R is a HFD if and only if R_S is a HFD.

Theorem 2.4. *Let G_1 and G_2 be torsion abelian groups. Then there exists a Dedekind HFD R such that $Cl(R) = G_1 \bigoplus G_2$.*

Proof. Let R_1, R_2 be a Dedekind HFDs with $Cl(R_i) = G_i$ [1, Theorem 3.2]. Suppose that R_i is associated to $\{G_i, A_i\}$ with $i = 1, 2$. Define $A = \{(a, 0), (0, b) | a \in A_1, b \in A_2\}$. Then A is realizable [11]. Let R be a Dedekind domain associated to $\{G, A\}$. Then $\rho(R) = max\{\rho(R_1), \rho(R_2)\} = 1$ [6, Theorem 3.1]. Thus R is a HFD. □

In view of the above theorem, we have:

Corollary 2.5. *Let $\{R_i\}$ be a finite family of Dedekind HFDs with torsion divisor class groups $\{Cl(R_i) = G_i\}$. Then there exists a Dedekind HFD R such that $Cl(R) \cong \bigoplus G_i$.* □

References

1. D.D. Anderson and D.F. Anderson, *Elasticity of factorization in integral domains*, J. Pure Appl. Algebra **80** (1992), 217–235.
2. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Splitting the t-class group*, J. Pure Appl. Algebra **74** (1991), 17–37.
3. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Factorization in integral domains II*, J. Algebra **152** (1992), 78–93.
4. D.F. Anderson, S. Chapman and W.W. Smith, *Overrings of half-factorial domains II*, Comm. Algebra **23** (1995), 3961–3976.
5. D.F. Anderson, S. Chapman and W.W. Smith, *Factorization sets and half-factorial sets in integral domains*, J. Algebra **178** (1995), 92–121.
6. D.F. Anderson, J. Park, G. Kim and H. Oh, *Splitting multiplicative sets and elasticity*, Comm. Algebra, to appear.
7. D.F. Anderson, S. Chapman and W.W. Smith, *Some factorization properties of Krull domains with infinite cyclic divisor class group*, J. Pure Appl. Algebra **96** (1994), 97–112.
8. L. Carlitz, *A characterization of algebraic number fields with class number two*, Pro. Amer. Math. Soc. **11** (1960), 391–392.
9. J. Coykendall, *A characterization of polynomial rings with the half-factorial property*, Lecture Notes in Pure and Applied Math. **189** (1997), 291–294.
10. S. Gabelli, *On divisorial ideals in polynomial rings over Mori domains*, Comm. Algebra **15** (1987), 2349- 2370.
11. A. Grams, *The distribution of prime ideals of a Dedekind domain*, Bull. Austral. Math. **11** (1974), 429- 441.
12. R.J. Valenza, *Elasticity of factorization in number fields*, J. Number Theory **36** (1990), 212–218.
13. A. Zaks, *Half-factorial domains*, Israel J. Math. **37** (1980), 218–302.

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