J. Korea Soc. Math. Ed. Series B: The Pure and Applied Mathematics
4, No. 2 (1997) pp. 105–110

BINOMIAL PROMOTION AND POISSON RECRUITMENT MODEL FOR MANPOWER DEVELOPMENT

U. H. ETUK

ABSTRACT. The distribution of staff in a hierachial organization has been studied in a variety of forms and models. Results here show that the promotion process follows a binomial distribution with parameters n and $\alpha = e^{-pt}$ and the recruitment process follows a poisson distribution with parameter λ. Futhermore, the mean time to promotion in the grade was estimated.

1. Introduction

In a graded organization, entry into the grades is by promotion from a lower grade and direct recruitment into the grades. The particular grade can absorb as many new entrants as possible from the recruitment process but cannot attract more than the number of staff in lower grade by the promotion process. Etuk and Nduka[2] presented two models of input process into the system-additive and multiplicative models. For the multiplicative model, the distribution of the input process was assumed to be the convolution of the distributions of the recruitment and promotion processes. Pollard[3] presented models with poisson recruitment and got the result that the number of new employees are mutually independent poisson variate at all times. Bartholomew[1] assumed that number of staff that move from one grade to the other follows a binomial distribution with a given initial stock of staff.

2. Assumptions and Model

Received by the editors May 14, 1996 and, in revised form May 9, 1997.
1991 *Mathematics Subject Classification.* 60H15.
Key words and phrases. Manpower development, generating function, binomial, poisson and exponential distributions, Kolmogorov's equation.

Let α be the probability that an individual in the grade gets promoted to next grade and β the probability that he withdraws from the system at the grade. Since the individual that withdraws leaves the system entirely and does not constitute an input into the next grade, we shall only consider the probabilities that an individual gets promoted from the grade or remains in the grade at that point in time. Hence, roughly, $1-\alpha$ is the probability that the staff remains in the grade. With these, the number of staff promoted can be seen to be a binomial random variable with parameter α. If i is the number of staff currently in the grade, the expected number of staff that will be promoted out of the grade is given by $i\alpha$. The recruitment process is a random process that can be assumed poisson(Pollard[3]). Assumed also is the fact that there is no bulk recruitment and that the recruitment population is infinite. According to Etuk and Nduka[2], each grade is termed a system with infinite number of servers and no queue build-up. The number of busy servers corresponds to the number of staff in the grade.

Let p be the number promoted out of the grade where there are i staff and r the number recruited into the grade. Hence, $n = i_0 + r + p$ is the number of staff in the grade after time t, where R and P are independent random variables and i_0 is the fixed initial number of staff in the grade. The Kolmogorov differential equations are given by

$$\frac{\partial p_n(t)}{\partial t} = -(R+NP)p_n(t) + Rp_{n-1}(t) + (N+1)Pp_{n+1}(t), \quad N \geq 1, \tag{1}$$

$$\frac{dp_0(t)}{dt} = -Rp_0(t) + Pp_1(t) \tag{2}$$

which satisfy the initial condition $P_n(0) = 1$ for $n = i$. By means of the generating function $G(u,t) = \sum_{n=0}^{\infty} p_n(t)u^n$, we have the partial differential equation

$$\frac{\partial G(u,t)}{\partial t} - P(1-u)\frac{\partial G(u,t)}{\partial u} = -R(1-u)G(u,t). \tag{3}$$

The general solution is given by

$$G(u,t) = \exp(Ru/P) f(\exp(-Pt)(1-u)). \tag{4}$$

From the initial conditions,

$$f(1-u) = u^i \exp(-Ru/p). \tag{5}$$

Letting $\theta = 1 - u$, equation (4) becomes

$$G(u,t) = \exp(-R/P)(1 - \exp(-Pt))^{1-u}(1 - (1-u)\exp(-Pt))^{i}. \tag{6}$$

Also, letting $\alpha = \exp(-Pt)$ and $\beta = 1 - \alpha$, equation(6) can be written as

$$G(u,t) = \exp(-R/P)\beta^{(1-u)}(1 + \frac{\alpha u}{\beta})^{i}\beta^{i}. \tag{7}$$

The mean $m(t)$ of the number of staff at time $t \neq 0$ is given by

$$m(t) = (i\alpha - \ln\beta)\exp(-r/p) \tag{8}$$

and the variance $v(t)$, is given by

$$v(t) = (i(i-1)\alpha^2 - (\ln\beta)^2)\exp(-r/p), \tag{9}$$

where α and β are functions of t.

Clearly, the generating function (7) is seen to be a product of two generating functions, namely, a poisson distribution with parameter $t\beta/p$ and a binomial distribution.

Theorem. *In an organization where the process of promotion and recruitment operate, the distribution of the process of promotion assumes the binomial from when there is no recruitment.*

Proof. From the system of equations defined in equations (1) and (2) and the resulting partial differential equations in (3),

$$\frac{\partial G(u,t)}{\partial t} - P(1-u)\frac{\partial G(u,t)}{\partial u} = -R(1-u)G(u,t). \tag{10}$$

Let $R = 0$, the auxilliary equations are given by

$$\frac{dt}{1} = \frac{du}{-P(1-u)} = \frac{dG}{0} \tag{11}$$

Solving these equations we get two systems of equations $(1-u)\exp(-Pt) = C_1$, constant and $G = C_2$, constant.

The general solution is given by

$$G(u,t) = f((1-u)\exp(-Pt)), \tag{12a}$$

where f is any arbitrary function. From the initial conditions, $P_n(0) = 1$, for $u = i$ and $G(u,0) = u^i$, we have $f(1-u) = u^i$, for all $|u| < 1$. Hence, for all values θ such that $|1-\theta| < 1$, $f(\theta) = (1-\theta)^i$.
Replacing θ by $(1-u)\exp(-Pt)$ in equation (12a), we have

$$G(u,t) = [1-(1-u)\exp(-Pt)]^i \tag{12b}$$

which is clearly the binomial generation function.

Corollary. *If there are no promotions in every finite value of t, then the distribution is poisson in form.*

Proof. It is clear from equation (6).

By the expansion of equation (7), the coefficient of u^n is

$$P_n(t) = \sum_{k=0}^{n} \binom{t}{k} \frac{1}{(n-k)!} \exp(-t\beta/p)\alpha^{(n-k)}\beta^{(n-i+2k)}(rp^{-1})^{n-k}. \tag{13}$$

This is clearly the probability that n staffs are in the grade at time t.

With the direct use of the result that the number of staff promoted from a grade follows a binomial distribution and the number recruited directly into the grade follows the poisson distribution, another form of equation (13) can thus be got. It can be noted that the processes of promotion and recruitment are independent since the staff that is promoted is not the staff that is recruited. Thus, let R be the number/rate of staff recruited and P the number/rate of staff promoted from lower grade into the grade in time t. Then $N = R + P$ is the total input of staff into the grade. From statistical theory, the joint density of $N = R + P$ is the product of marginal densities of R and P. Since the product of Fourier transforms is the Fourier transform of the convolution of the two functions(Spiegel[5]), we have

$$\begin{aligned}
f^+_{R+P}(n) &= f^+_R(r) f^+_P(p) \\
&= \int_{-\infty}^{\infty} f^*_R(r) f^*_P(n-r)dr \\
&= \int_{-\infty}^{\infty} \sum_{j=0}^{i} \binom{i}{j} \alpha^i (1-\alpha)^{i-j} \delta(n-r-j) \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \delta(r-k)dr \\
&= \sum_{j=0}^{i} \sum_{k=0}^{\infty} \binom{i}{j} \frac{\alpha^j (1-\alpha)^{i-j} \lambda^k e^{\lambda} \delta(n-j-k)}{k!}
\end{aligned} \tag{14}$$

where δ-function is the linear functional that assigns to each test function $f(x)$ the real number $f(0)$, Schuss[4], and α the probability of being promoted. Clearly,

$$P(R+P=N)=\sum_{j+k=n}^{\infty}\frac{e^{-\lambda}\binom{i}{j}\alpha^{j}(1-\alpha)^{i-j}\lambda^{k}}{k!}. \tag{15}$$

This is the probability that n new staffs go into the grade at time t.

3. Mean Time to Promotion in the Grade

In any of the grades, and at any point in time t, the staff in the grade gets promoted with rate p or withdraws with w. Let S be the length of stay in the grade. Let $1-P(S<t)=P(S\geq T)=G(t)$. Then $G(t+h)=G(t)G(h)=G(t)(1-(p+w)h)+o(h)$.

$$\frac{dG(t)}{dt}=-(p+w)G(t)$$
$$G(t)=Ae^{-(p+w)t}$$

where $P(S\geq 0)=1$. Clearly, S follows an exponential distribution with mean $(p+w)^{-1}$. This confirms the assumption made in Etuk and Nduka[2] that the length of stay in any grade is exponentially distributed.

Hence, the probability that a staff gets promoted to the next grade is $p/(p+w)$ and the probability that the staff withdraws is $w/(p+w)$.

4. Conclusion

A statistical analysis of a graded organization is presented and the number of new entrants into any grade is seen to comprise the entrants through promotion from a lower grade and entrants by direct recruitment into the grade. It is shown that the corresponding distributions are binomial and poisson respectively. Furthemore, the length of stay in the grade follows exponential distribution.

REFERENCES

1. D. J. Bartholomew, *The control of grade structure in the university,* In J. G. Andrew and R. R. Melone (Eds.), Mathematical Modelling, Butherworth and Co., London, 1976, 161.
2. U. H. Etuk and E. C. Nduka, *A Stochastic Model for manpower development in an organization; a series queue approach*, Internat. J. Math. Ed. Sci. Tech. **24(2)** (1993), 177-181.
3. J. H. Pollard, *Hierarchial population models with poisson recruitment*, J. Appl. Probab. 4 (1967), 209-213.
4. Z. Schuss, *Theory and applications of stochastic differential equations*, John Wiley and Sons, New York (1980).
5. M. R. Spiegel, *Fourier Analysis*, McGraw-Hill Book Company, New York (1974).

DEPARTMENT OF MATHEMATICS/STATISTICS, UNIVERSITY OF PORT HARCOURT, PORT HARCOURT, NIGERIA.