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MAPPING THEOREMS ON $X_1 \oplus X_2$

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ABSTRACT. We show that if $f_i : X_i \to Y$ is strongly continuous(resp. weakly continuous, set connected, compact, feebly continuous, almost-continuous, strongly θ-continuous, θ-continuous, g-continuous, V-map), then $F : X_1 \oplus X_2 \to Y$ is strongly continuous(resp.weakly continuous, set connected, compact, feebly continuous, almost-continuous, strongly θ-continuous, θ-continuous, g-continuous, V-map).

1. Introduction

Let $\{X_s | s \in S\}$ be a family of disjoint topological spaces, i.e., let $X_s \cap X_{s'} = \emptyset$ for $s \neq s'$. Let us consider the set $X = \cup_{s \in S} X_s$ and let us assume that sets $U \subset X$, such that the intersection $U \cap X_s$ is open in X_s for every $s \in S$, are open in X. The open sets defined in this manner satisfy the conditions for a topology. The set X with this topology is called the sum of spaces $\{X_s | s \in S\}$ [3], and is denoted by the symbol $\oplus_{s \in S} X_s$ or $X_1 \oplus \cdots \oplus X_k$ if $S = \{1, 2, \cdots, k\}$. It is well known that $V \subset X$ is closed iff $V \cap X_s$ is closed in X_s for all $s \in S$ [3].

In this paper, we restrict S to $\{1, 2\}$. The author obtained the hints of this paper from following:

Theorem [3]. *$F : \oplus_{s \in S} X_s \to Y$ is continuous iff for every $s \in S$, $f_s : X_s \to Y$ is continuous.*

We shall show that

if $f_i : X_i \to Y$ is strongly continuous(resp. weakly continuous, set connected, compact, feebly continuous, almost-continuous, strongly θ-continuous, θ-continuous, g-continuous, V-map), then $F : X_1 \oplus X_2 \to Y$ is strongly continuous(resp.weakly

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2. Preliminaries

$F : X_1 \oplus X_2 \to Y$ is defined by $F(x) = f_i(x)$ if $x \in X_i$, where $f_i : X_i \to Y$, $i = 1, 2$. Let A be a subset of a topological space X. We shall denote the closure of A and interior of A in X by $\overline{A}$ and $\mathrm{Int}(A)$ respectively. By $f : X \to Y$ we denote a function f from a topological space X into a topological space Y. No separation axioms on a topological space is assumed unless stated.

$\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\mathrm{Int}(A) \cup \mathrm{Int}(B) \subset \mathrm{Int}(A \cup B)$ are very useful in this paper. Sets X, Y and X_i are topological spaces.

3. Definitions

Definition 1 [5]. X is said to be *connected* between A and B if there exists no closed-open set F of X such that $A \subset F$ and $F \cap B = \emptyset$.

Definition 2 [2]. $B \subset X$ is *g-closed* if $\overline{B} \subset G$, whenever $B \subset G$ and G is open.

A function $f : X \to Y$ is said to be

Definition 3 [7]. *strongly-continuous* if for every subset $A \subset X$, $f(\overline{A}) \subset f(A)$.

Definition 4 [8]. *weakly-continuous* if for each $x \in X$, and each open set H containing $f(x)$, there is an open set G containing x such that $f(G) \subset \overline{H}$.

Definition 5 [6]. *set-connected provided* that if X is connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology.

Definition 6 [1]. *compact* if the inverse image of every compact subset of Y is compact subset of X.

Definition 7 [10]. *feebly-continuous* if for each open set U of Y, $f^{-1}(U) \neq \emptyset$ implies $\mathrm{Int}(f^{-1}(U)) \neq \emptyset$.

Definition 8 [11]. *almost-continuous*(resp. *strongly θ-continuous, θ-continuous*) if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Int}(\overline{V})$ (resp. $f(\overline{U}) \subset V$, $f(\overline{U}) \subset \overline{V}$).

Definition 9 [2]. *g-continuous* if the inverse image of closed set in Y is g-closed in X.

Definition 10 [4]. *V-map* if it satisfies: Given any open cover β of Y, $\{\mathrm{Int}(f^{-1}(U)) | U \in \beta\}$ is an open cover of X.

4. Theorems

Lemma 1 [1]. *$f : X \to Y$ is strongly continuous iff $f^{-1}(y)$ is open in X for each $y \in Y$.*

Lemma 2 [8]. *$f : X \to Y$ is weakly continuous iff for each open set $V \subset Y$, $f^{-1}(V) \subset Int(f^{-1}(\overline{V}))$.*

Theorm 3. *If f_i are weakly continuous, so is F.*

Proof. Let $V \subset Y$ be open. Then $F^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V) \subset \mathrm{Int}(f_1^{-1}(\overline{V})) \cup \mathrm{Int}(f_2^{-1}(\overline{V})) \subset \mathrm{Int}(f_1^{-1}(\overline{V}) \cup f_2^{-1}(\overline{V})) = \mathrm{Int}(F^{-1}(\overline{V}))$. Hence by Lemma 2, F is weakly-continuous.

Lemma 4 [9]. *If a surjection $f : X \to Y$ is weakly-continuous, then f is set-connected.*

Corollary 5. *If F is surjective, and f_i are weakly-continuous, then F is set connected.*

Proof. Use Theorem 3 and Lemma 4.

Theorem 6. *If f_i are compact, so is F.*

Proof. Let V be a compact subset of Y. Then $f_i^{-1}(V)$ are compact. Hence $F^{-1}(V) = \cup f_i^{-1}(V)$ is compact.

Theorem 7. *If f_i are feebly continuous, so is F.*

Proof. Let V be an open set in Y. Then $f_i^{-1}(V) \neq \emptyset$ implies $\mathrm{Int}(f_i^{-1}(V)) \neq \emptyset$.

Let $F^{-1}(V) \neq \emptyset$. Then $\emptyset \neq \cup\text{Int}(f_i^{-1}(V)) \subset \text{Int}(\cup f_i^{-1}(V)) = \text{Int}(F^{-1}(V))$. Hence F is feebly continuous.

Theorem 8. *If f_i are almost-continuous, so is F.*

Proof. If $x \in X_1 \oplus X_2$, then we have $x \in X_1$ or $x \in X_2$. Let $x \in X_1$. Since f_1 is almost-continuous, for each open set V containing $F(x) = f_1(x)$, there exists a neighborhood U of x in X_1 such that $F(U) = f_1(U) \subset \text{Int}(\overline{V})$. In the case of $x \in X_2$, the proof is similar to that of $x \in X_1$. This completes the proof.

The proofs of the following Theorems 9 and 10 are similar to that of Theorem 8. Hence we omit the proofs.

Theorem 9. *If f_i are strongly θ-continuous, so is F.*

Theorem 10. *If f_i are θ-continuous, so is F.*

Theorem 11. *If f_i are g-continuous, so if F.*

Proof. Let C be closed in Y. Let G be a neighborhood of $F^{-1}(C) = \cup f_i^{-1}(C)$ in $X_1 \oplus X_2$. Since $G \cap X_i$ is a neighborhood of $f_i^{-1}(C)$ in X_i, we have $\overline{f_i^{-1}(C)} \subset G \cap X_i$. Thus we have $\overline{F^{-1}(C)} = \overline{\cup f_i^{-1}(C)} = \cup \overline{f_i^{-1}(C)} \subset \cup(G \cap X_i) = G \cap (\cup X_i) = G$. Hence F is g-continuous.

Theorem 12. *If f_i are V-maps, so is F.*

Proof. Let β be an open cover of Y. Then $\{\text{Int}(f_i^{-1}(U)) | U \in \beta\}$ is an open cover of X_i. Since $\text{Int}(F^{-1}(U)) = \text{Int}(\cup f_i^{-1}(U)) \supset \cup\text{Int}(f_i^{-1}(U))$, $\{\text{Int}(F^{-1}(U)) | U \in \beta\}$ is an open cover of $X_1 \oplus X_2$. Hence F is a V-map.

Theorem 13. *If f_i are strongly continuous, so is F.*

Proof. Since $f_i^{-1}(y)$ is open in X_i for each $y \in Y$ by Lemma 1, $F^{-1}(y) = \cup f_i^{-1}(y)$ is open in $X_1 \oplus X_2$. Hence F is strongly continuous.

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