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APPROXIMATE CONTROLLABILITY FOR NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

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1. Introduction

Our objective is to investigate approximate controllability of a class of partial integrodifferential systems. This work continuous the investigations of [8]. As a model for this class one may take the equation

$$\frac{\partial y(t,\xi)}{\partial t} = \frac{\partial}{\partial \xi}(a(t,\xi)\frac{\partial y(t,\xi)}{\partial \xi}) + F(t, y(t-r,\xi), \int_0^t k(t,s,y(s-r,\xi))ds) + b(\xi)u(t),$$
$$0 \le \xi \le 1, \quad 0 \le t \le T$$

with initial-boundary conditions

$$y(t,0) = y(t,1) = 0, 0 \le t \le T,$$
$$y(t,\xi) = \phi(t,\xi), 0 \le \xi \le 1, -r \le t \le 0.$$

Above equation can be formulated abstractly as

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t)x(t) + F(t, x_t(\phi:u), \int_0^t k(t,s,x_s(\phi:u))ds) + Bu(t), \ 0 \le t \le T \\ x(\phi:u)(\theta) = \phi(\theta) \quad \theta \in (-r, 0] \end{cases} \tag{1}$$

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where F is a nonlinear function, r is a positive number and ϕ is a given initial function.

Before proceeding we shall set forth some notation and terminology that will be used throughout the paper.

The state $x(t)$ of the system lies in a Hilbert space X and the control function $u(\cdot)$ is from $L^2(0,T:V)$, V is assumed to be another Hilbert space. For each $t \in [0,T]$, linear operator $A(t) : X \to X$ generates a strongly continuous evolution system $\{U(t,s)\}$. and B is a bounded linear operator from $L^2(0,T:V) \to L^2(0,T:X)$. Let $C = C(-r,0:X)$ be a Banach space of all continuous functions from an interval of the form $I = [-h,0]$ to X with the norm defined by supremum. The norm of the space X and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$, respectively. If a function x is continuous from $I \cup [0,T]$ to X, then x_t is an element in C which has point-wise definition:

$$x_t(\theta) = x(t+\theta) \qquad \text{for} \qquad \theta \in I.$$

We assume that $k : [0,T]\times[0,T]\times C \to X$, $F : [0,T]\times C\times X \to X$ are nonlinear function.

The nonlinear integrodifferential equation considered serves as an abstract formulation of many partial integrodifferential equations which arise in the problems with heat flow in material with memory, viscoelasticity, and many other physical phenomena.(See Refs.[2],[3],[6])

2. Approximate Controllability

The norm of the space $L^2(0,T:X)$ and $L^2(0,T:V)$ are denoted by $\|\cdot\|_{L^2(0,T:X)}$ and $\|\cdot\|_{L^2(0,T:V)}$, respectively.

We consider a unique milde solution of (1), for each u in $L^2(0,T:V)$,

$$\begin{cases} x_t(\phi:u)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s,x_s(\phi:u),\int_0^s k(s,\tau,x_\tau(\phi:u))d\tau) \\ \qquad\qquad + Bu(s)\}ds, \quad 0 < t \leq T, \\ x(\phi:u)(\theta) = \phi(\theta), \qquad \theta \in (-r,0] \end{cases} \tag{2}$$

with $\phi \in C(-r,0:X)$.

We assume the following hypotheses;

(A) There exist positive constant M such that

$$\|U(t,s)\| \leq M, \quad 0 \leq s \leq t \leq T.$$

(F) There exists a constant $\beta > 0$ such that

$$\|F(t,\psi_1,x_1) - F(t,\psi_2,x_2)\| \leq e^{-\beta t}(\|\psi_1 - \psi_2\|_C + \|x_1 - x_2\|),$$
$$\|k(t,s,\psi_1) - k(t,s,\psi_2)\| \leq e^{-\beta(t-s)}\|\psi_1 - \psi_2\|_C,$$
$$\psi_1,\psi_2 \in C, \quad x_1,x_2 \in X, \quad 0 \leq s \leq t \leq T$$

and

$$F(t,0,0) \equiv 0, \quad k(t,s,0) \equiv 0.$$

Lemma 1([1]). *Let $a(t), b(t)$ and $c(t)$ be real valued nonegative continuous functions defined on R^+, for which the inequality*

$$c(t) \leq C_0 + \int_0^t a(s)c(s)ds + \int_0^t a(s)[\int_0^s b(\tau)c(\tau)d\tau)]ds,$$

holds for all $t \in R^+$, where C_0 is a nonnegative constant. Then

$$c(t) \leq C_0[1 + \int_0^t a(s)exp[\int_0^s (a(\tau) + b(\tau))d\tau]ds],$$

for all $t \in R_+$.

We can define the solution mapping

$$W : L^2(0, T : V) \to C(-r, t_1 : X)$$

can be defined by

$$(Wu)(t) = x_t(\phi : u).$$

Theorem 1([8]). *Let $u(\cdot) \in V$ and $\phi \in C$. Then under Hypotheses (A) and (F) the solution mapping $(Wu)(t) = x_t(\phi : u)$ satisfies*

$$\|x_t(\phi : u)\|_C \leq C(M\|\phi\|_C + M_B\|u\|_{L^2(0,T:V)}\sqrt{t_1})e^{\beta t_1}, \quad 0 \leq t \leq t_1,$$

where C dependes on β, M and t_1.

Lemma 2. *Let $v_1(\cdot)$ and $v_2(\cdot)$ in V. Then under hypothesis (F) the solution mapping $(Wv)(t) = x_t(\phi : v)$ of (2) satisfies*

$$\begin{aligned} &\|x_t(\phi : v_1) - x_t(\phi : v_2)\|_C \\ \leq &KM\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T:X)}\sqrt{T}e^{\beta T}. \end{aligned}$$

Proof.

$$\begin{aligned}
&\|x(\phi : v_1)(t+\theta) - x(\phi : v_2)(t+\theta)\| \\
\leq & M \int_0^{t+\theta} (F(s, x_s(\phi : v_1), \int_0^s K(s, \tau, x_\tau(\phi : v_1))d\tau) \\
& - F(s, x_s(\phi : v_2), \int_0^s k(s, \tau, x_\tau(\phi : v_2))d\tau))ds \\
& + M \int_0^{t+\theta} \|Bv_1(s) - Bv_2(s)\|_{L^2(0,T:X)} ds \\
\leq & M \int_0^{t+\theta} e^{-\beta s}(\|x_s(\phi : v_1) - x_s(\phi : v_2)\|_C \\
& + \int_0^s e^{-\beta(s-\tau)} \|x_\tau(\phi : v_1) - x_\tau(\phi : v_2)\|_C d\tau)ds \\
& + M \int_0^{t+\theta} \|Bv_1(s) - Bv_2(s)\|_{L^2(0,T:X)} ds.
\end{aligned}$$

By using Grouwall's inequality

$$\begin{aligned}
&\|x_t(\phi : v_1) - x_t(\phi : v_2)\|_C \\
\leq & M\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T,:X)}\sqrt{T} \\
& \times [1 + \int_0^T Me^{-\beta s} \exp(\int_0^s Me - \beta\tau + e^{-\beta(s-\tau)})d\tau)ds] \\
\leq & KM\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T:X)}\sqrt{T}e^{\beta T}.
\end{aligned}$$

We define the reachahble set $K(F)$ in $C(0,T:X)$ by

$$\begin{aligned}
K(F) = & \{x_t(\phi : u)(0) \in C(0,T:X); x_t(\phi : u)(0) = U(t,0)\phi(0) \\
& + \int_0^t U(t,s)[F(s, x_s(\phi : u), \int_0^s k(s, \tau, x_\tau(\phi : u))d\tau) \\
& \qquad + (Bv)(s)]ds, \quad v \in L^2(0,T:V)\}.
\end{aligned}$$

Similarly, for its corresponding linear system (in case $F \equiv 0$), $K(0)$ can be defined.

Definition. The system (2) is called approximately controllable on $[0,T]$ if for any given $\epsilon > 0$ and $\xi_T \in L^2(0,T:X)$ there exists some control $v(\cdot) \in L^2(0,T:V)$ such that

$$\|\xi_T - U(T,0)\phi - \tilde{S}F(\cdot, x_.(\phi : v), \int_0^{\cdot} k(.,\tau, x(\phi : v))d\tau)) - \tilde{S}Bv\| < \epsilon.$$

We assume the following hypotheses:

(B) For any given $\epsilon > 0$ and $p(\cdot) \in L^2(0,T:X)$ there exists some $v(\cdot) \in L^2(0,T,:V)$ such that

(B1) $\|\tilde{S}p - \tilde{S}Bv\| < \epsilon$

(B2) $\|B_{(0,T)}v(\cdot)\|_{L^2(0,T:X)} \leq q_1 \|p(\cdot)\|_{L^2(0,T:X)}$

(B3) $q_1(1 + \frac{1}{\beta}(1 - e^{-\beta T}))KM\sqrt{T} < 1.$

Theorem 2. *Under hypothesis (B), the system (2) is ϵ-approximately controllable on $[0,T]$.*

Proof. Since the domain $D(A)$ is dense in $L^2(0,T:X)$ it is sufficient to prove

$$D(A) \subset \overline{K(F)}$$

i.e. for any given $\epsilon > 0$ and $\xi_T \in D(A)$ there exists $v(\cdot) \in V$ such that

$$\|\xi_T - U(t,0)\phi - \tilde{S}F(s, x_s(\phi : v), \int_0^s k(s,\tau, x_\tau(\phi : v))d\tau) - \tilde{S}Bv\| < \epsilon$$

where

$$x_t(\phi : v) = U(t,0)\phi + \int_0^t U(t,s)F(s, x_s(\phi : v), \int_0^s k(s,\tau, x_\tau(\phi : v))d\tau) + Bv(s)ds.$$

As $\xi_T \in D(A)$ there exists some $p(\cdot) \in C(0,T:X)$ such that

$$\tilde{S}p = \xi_T - U(T,0)\phi$$

Assume $v_1(\cdot) \in V$ is arbitrary given. By hypothesis (B1) there exists some $v_2(\cdot) \in L^2(0,T:X)$ such that

$$\|\xi_T - U(T,0)\phi - \tilde{S}F(\cdot, x_\cdot(\phi : v_1), \int_0^\cdot k(\cdot,\tau,x_\tau(\phi : v_1))d\tau) - \tilde{S}Bv_2\| < \frac{\epsilon}{2^2}.$$

For $v_2(\cdot)$ thus obtained,we determined $w_2(\cdot) \in V$ by hypothesis (B1) and (B2) such that

$$\begin{aligned}&\|\tilde{S}[F(\cdot, x_\cdot(\phi : v_2), \int_0^\cdot k(\cdot,\tau,x_\tau(\phi : v_2))d\tau)\\ &\quad - F(\cdot, x_\cdot(\phi : v_1), \int_0^\cdot k(\cdot,\tau,x_\tau(\phi : v_1))d\tau)] - \tilde{S}Bw_2\| < \frac{\epsilon}{2^3}\end{aligned}$$

and by Lemma 2

$$\begin{aligned}\|Bw_2(\cdot)\| \leq& q_1\|F(s, x_s(\phi : v_2), \int_0^s k(s,\tau,x_\tau(\phi : v_2))d\tau)\\ &- F(s, x_s(\phi : v_1), \int_0^s k(s,\tau,x_\tau(\phi : v_1))d\tau)\|\\ \leq& q_1 e^{-\beta s}(\|x_s(\phi : v_2) - x_s(\phi : v_1)\|_C\\ &+ \int_0^s e^{-\beta(s-\tau)}\|x_\tau(\phi : v_2) - x_\tau(\phi : v_1)\|_C d\tau)\\ \leq& q_1 e^{-\beta s}\|x_s(\phi : v_2) - x_s(\phi : v_1)\|_C(1 + \int_0^s e^{-\beta(s-\tau)}d\tau)\\ \leq& q_1 e^{-\beta s}(1 + \frac{1}{\beta}(1 - e^{-\beta s}))\|x_s(\phi : v_2) - x_s(\phi : v_1)\|_C\\ \leq& q_1 e^{-\beta s}(1 + \frac{1}{\beta}(1 - e^{-\beta s})KM\sqrt{T}e^{\beta T}\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T:X)}.\end{aligned}$$

Thus we may define $v_3(\cdot) = v_2(\cdot) - w_2(\cdot)$ in V which has the following property;

$$\begin{aligned}
&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi : v_2), \int_0^s k(s,\tau,x_\tau(\phi : v_2))d\tau) - \tilde{S}Bv_3\| \\
=&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi : v_1), \int_0^s k(s,\tau,x_\tau(\phi : v_1))d\tau) - \tilde{S}Bv_2 + \tilde{S}Bw_2 \\
&- \tilde{S}[F(s, x_s(\phi : v_2), \int_0^s k(s,\tau,x_\tau(\phi : v_2))d\tau) \\
&\qquad - F(s, x_s(\phi : v_1), \int_0^s k(s,\tau,x_\tau(\phi : v_1))d\tau)]\| \\
<&(\frac{1}{2^2} + \frac{1}{2^3})\epsilon.
\end{aligned}$$

By induction, it is proved that there exists a sequence $v_m(\cdot)$ in V such that

$$\begin{aligned}
&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi : v_n), \int_0^S k(s,\tau,x_\tau(\phi : v_n))d\tau) - \tilde{S}Bv_{n+1}\| \\
<&(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\epsilon, \quad n = 1,2,3,\cdots
\end{aligned}$$

and

$$\begin{aligned}
&\|Bv_{n+1}(\cdot) - Bv_n(\cdot)\|_{L^2(0,T:X)} \\
\leq& q_1(1 + \frac{1}{\beta}(1 - e^{-\beta T}))KM\sqrt{T}\|Bv_n(\cdot) - Bv_{n-1}(\cdot)\|_{L^2(0,T:X)}.
\end{aligned}$$

By hypothesis (B3) the sequence $\{Bv_n; n = 1,2,\cdots\}$ is a Cauchy sequence in the Banaha space $L^2(0,T : X)$ and there exists some $v_1(\cdot)$ in $L^2(0,T : X)$ such that

$$\lim_{n\to\infty} Bv_n(\cdot) = u(\cdot) \text{ in } L^2(0,T : X).$$

Therefore, for any given $\epsilon > 0$ there exists some integer N_ϵ such that

$$\|\tilde{S}Bv_{N_\epsilon+1} - \tilde{S}Bv_{N_\epsilon}\| < \frac{\epsilon}{2}$$

and

$$
\begin{aligned}
&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi : v_{N_\epsilon}), \int_0^s k(s,\tau,x_\tau(\phi : v_{N_\epsilon}))d\tau) - \tilde{S}Bv_{N_\epsilon}\| \\
\leq&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi : v_{N_\epsilon}), \int_0^s k(s,\tau,x_\tau(\phi : v_{N_\epsilon})d\tau) - \tilde{S}Bv_{N_\epsilon+1}\| \\
&+ \|\tilde{S}Bv_{N_\epsilon+1} - \tilde{S}Bv_{N_\epsilon}\| \\
\leq&(\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\epsilon + \frac{\epsilon}{2} \\
\leq&\epsilon.
\end{aligned}
$$

Thus the nonliner system (2) is approximately controllable on $[0,T]$.

REFERENCES

1. Dhakne, M. B. and Pachpatte, B. G., *On a general class of abstract functional integrodifferential equations*, Indian J. Pure Appl. Math. **19(8)** (1988), 728–746.
2. Fitzgibbon, W. E., *Semilinear integrodifferential equations in Banach space*, Nonlinear Analysis Theory, Methods and Applications **4** (1980), 745–760.
3. Heard, M. L., *An abstract semilinear Hyperbolic volterra integrodifferential equation*, J. Math. Anal. Appl. **80** (1981), 175–202.
4. Henry, D., *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin, Germany, 1981.
5. Kwun, Y. C., Park, J. Y. and Ryu, J. W., *Approximate controllability and controllability for delay Volterra systems*, Bull. Korean Math. Soc. **28** (1991), 131–145.
6. Miller, R. K., *Volterra integral equations in a Banach space*, Funkcialaj Ekvacioj **18** (1975), 163–193.
7. Naito, K. and Park, J. Y., *Approximate controllability for trajectories of a delay Volterra control system*, J. of Optimization Theory and Applications **61** (1989), 271–279.
8. Park, J. Y., Kwun, Y. C. and Jeong, J. M., *Existence and approximate controllability for nonlinear integrodifferential equations, preprint.*
9. Ryu, J. W., Park, J. Y. and Kwun, Y. C., *Approximate controllability of delay volterra control system*, Bull. Korean Math. Soc. **30(2)** (1993), 277–284.

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