J. Korea Soc. Math. Ed. Series B: The Pure and Applied Mathematics
4, No. 2 (1997) pp. 145–149

CONTINUOUS EXTENDIBILITY OF THE SZEGŐ KERNEL

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ABSTRACT. Suppose Ω is a bounded n-connected domain in $\mathbb{C}$ with C^2 smooth boundary. Then we prove that the Szegő kernel extends continuously to $\bar{\Omega} \times \bar{\Omega}$ except the boundary diagonal set.

1. Introduction

Let Ω be an n-connected, bounded, planar domain with C^∞ boundary components.

Bell proved in [1; p.100] that the Szegő kernel associated to Ω extends smoothly to $\bar{\Omega} \times \bar{\Omega}$ except the boundary diagonal set by using the Kerzman-Stein formula (see [1; p.11]) which states that $P[I + (C - C^*)] = C$, where P denotes the Szegő projection and C^* denotes the L^2 adjoint of the Cauchy transform C. In this note, we prove that the Szegő kernel associated to a domain Ω extends continuously to $\bar{\Omega} \times \bar{\Omega}$ except the boundary diagonal set when Ω is an n-connected, bounded, planar domain with C^2 boundary components. We use the transformation formula for the Szegő kernel under biholomorphic map between two bounded domains in $\mathbb{C}$. Before we proceed to prove our results, it seems necessary to describe some basic definitions and list useful properties.

Let Ω be an n-connected, bounded, planar domain with C^2 boundary components and let $b\Omega$ denote the boundary of Ω.

Let $L^2(b\Omega)$ denote the space of square integrable complex valued functions on $b\Omega$ with the inner product given by $< u, v >= \int_{b\Omega} u\bar{v}ds$ where ds denotes the arc length measure.

Received by the editors Oct. 2, 1997 and, in revised form Nov. 20, 1997.
1991 *Mathematics Subject Classification.* 30C40.
Key words and phrases. Szegő kernel, transformation formula, Riemann mapping.

Let $H^2(b\Omega)$ denote the closed subspace of $L^2(b\Omega)$ consisting of boundary values of holomorphic functions on Ω.

The orthogonal projection $P : L^2(b\Omega) \to H^2(b\Omega)$ is called the Szegő projection and is represented by the Szegő kernel $S_\Omega(z,w)$ on $\Omega \times \overline{\Omega}$ via

$$P\varphi(z) = \int_{b\Omega} S_\Omega(z,w)\varphi(w)\,ds_w$$

for φ in $L^2(b\Omega)$ and z in Ω. Here we have identified $P\varphi \in H^2(b\Omega)$ with its unique holomorphic extension to Ω. The Szegő kernel is holomorphic in z, antiholomorphic in w, and $S_\Omega(z,w) = \overline{S_\Omega(w,z)}$.

The Szegő kernel is intimately tied to the behavior of conformal mapping. For example, the transformation formula for the Szegő kernel under a biholomorphic map gives rise to a nice formula for the Riemann map (see [1;p.46]) and the Ahlfors map is written by the Szegő kernel associated to a multiply-connected domain (see [1;p.106]).

Since Ω is an n-connected domain with C^2 boundary components, there are C^2 complex-valued functions $z_i(t)$ of $t \in [0,1] \subset \mathbb{R}$, $i = 1,\ldots,n$, which parametrize the n boundary curves of Ω in the standard sense. If $z \in b\Omega$, then the complex unit tangent function T is defined by the formula $T(z'(t)) = z'(t)/|z'(t)|$. Hence $dz = T ds$ when ds represents the differential of the arc-length.

2. Results

The following lemma is a well-known fact, but it is worthwhile to give a proof (see [6;p.341]).

Lemma 1. *A bounded n-connected domain Ω_1 with C^∞ smooth boundary is biholomorphically equivalent to a bounded n-connected domain Ω_2 with real analytic boundry.*

Proof. Let the complement of Ω_1 be the union of C_i, $i = 1,\ldots,n$, where C_n is the outer component containing ∞ and $C_1,\ldots,C_{n-1}$ are bounded. Then $\mathbb{C} \cup \{\infty\} - C_n$ is a simply connected domain. By the Riemann mapping theorem, there exists a biholomorphic map f_1 of $\mathbb{C} \cup \{\infty\} - C_n$ onto the unit disc U_1. Pick $w_1 \in f_1(C_1)$. Take a mapping $\tilde{f}_1(z) = \frac{1}{z-w_1}$ which is a biholomorphic map of $U_1 - \cup_{i=1}^{n-1} f_1(C_i)$

onto its image and maps w_1 to ∞ and bU_1 onto a simple real analytic curve. Repeat the construction to get a biholomorphic map f_2 of $\mathbb{C}\cup\{\infty\}-\tilde{f}_1\circ f_1(C_1)$ onto the unit disc U_2. Pick $w_2 \in f_2\circ\tilde{f}_1\circ f_1(C_2)$, etc.. So Ω_1 is biholomorphic to a bounded domain Ω_2 whose boundary consists of n non-intersecting simple real analytic curves. □

We know that the Szegő kernel functions transform via certain formulas. Recently we proved the transformation formula for the Szegő kernel under proper holomorphic map between two bounded domains in $\mathbb{C}$ (see [3, 5]). But, now we only need the following transformation rule for the Szegő kernel under biholomorphic map between two bounded domains in $\mathbb{C}$ which can be found in [1;p.44, 2;p.110] and we will give a proof for convenience.

Lemma 2. *Let $f : \Omega_1 \to \Omega_2$ be a biholomorphic map between two bounded domains Ω_1 and Ω_2 in $\mathbb{C}$ with C^∞ smooth boundaries. Then the Szegő kernels associated to Ω_1 and Ω_2 transform via*

$$S_{\Omega_1}(z,w) = \sqrt{f'(z)}S_{\Omega_2}(f(z),f(w))\overline{\sqrt{f'(w)}}$$

for $z, w \in \Omega_1$.

Proof. Note that $f \in C^\infty(\bar{\Omega})$ and f' is nonvanishing on $\bar{\Omega}$(see [1; p. 42]). Assume that $f(z_i(t)) = \zeta_i(t)$ for $i = 1,\ldots,n$ where $z_i(t)$ and $\zeta_i(t)$ of $t \in [0,1] \subset \mathbb{R}$ be parametrizations of the boundary curves γ_i of Ω_1 and $\tilde{\gamma}_i$ of Ω_2 which is the image curve of γ_i under f, respectively in the standard sense. Then $f'(z_i(t))z_i'(t) = \zeta_i'(t)$ and

$$\frac{f'(z_i(t))}{|f'(z_i(t))|}\,\frac{z_i'(t)}{|z_i'(t)|} = \frac{\zeta_i'(t)}{|\zeta_i'(t)|}.$$

Let $T_1(z_i(t)) = \frac{z_i'(t)}{|z_i'(t)|}$ and $T_2(\zeta_i(t)) = \frac{\zeta_i'(t)}{|\zeta_i'(t)|}$. Then $\frac{f'(z_i(t))}{|f'(z_i(t))|}T_1(z_i'(t)) = T_2(\zeta_i(t))$.

Let $\Delta^i_{f'}$ denote the increase in arg $f'(z_i(t))$ as z_i traces out γ_i in the standard sense. The above identity reveals that $\Delta^i_{f'} \pm 2\pi = \pm 2\pi$ and therefore $\Delta^i_{f'} = 0, 4\pi, -4\pi$. It means that $\sqrt{f'(z_i)}$ doesn't change sign as z_i traces out γ_i and hence $\sqrt{f'}$ is well-defined and continuous on $\bar{\Omega}_1$. The arc-length magnification from $b\Omega_1$ to $b\Omega_2$ is $|f'(z)|$. If $\{\psi_j\}$ is a complete orthonormal system on $H^2(b\vee mega_2)$, then so is $\{\sqrt{f'(z)}\psi_j(f(z))\}$ on $H^2(b\Omega_1)$. Hence

$$\begin{aligned} S_{\Omega_1}(z,w) &= \sum_{j=1}^{\infty} \sqrt{f'(z)}\psi_j(f(z))\overline{\sqrt{f'(w)}}\,\overline{\psi_j(f(w))} \\ &= \sqrt{f'(z)}\sum_{j=1}^{\infty} \psi_j(f(z))\overline{\sqrt{f'(w)}}\,\overline{\psi_j(f(w))} \\ &= \sqrt{f'(z)}S_{\Omega_2}(f(z),f(w))\overline{\sqrt{f'(w)}} \end{aligned}$$

for $z \in \Omega_1$ and $w \in b\Omega_1$. With the identification of functions in $H^2(b\Omega_1)$ with their unique holomorphic extensions to Ω_1, we have

$$S_{\Omega_1}(z,w) = \sqrt{f'(z)}S_{\Omega_2}(f(z),f(w))\overline{\sqrt{f'(w)}}$$

for $z, w \in \Omega_1$. □

The following lemma is necessary to prove continuous extendibility of the Szegő kernel.

Lemma 3. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ with C^2 smooth boundary. For given $a \in \Omega$, let $f : \Omega \to U$ be the Riemann mapping function of Ω with the normalization $f(a) = 0$, $f'(a)$ real and positive where U is the unit disc in $\mathbb{C}$. Then f is in $C^{2-\epsilon}(\bar{\Omega})$ for any $0 < \epsilon < 1$.*

Proof. For fixed $a \in \Omega$, the Green's function $G_\Omega(z,a)$ associated to Ω for all $z \in \Omega$ is uniquely determined by the condition that $G_\Omega(\cdot,a) \in C(\bar{\Omega} - \{a\})$, $G_\Omega(\cdot,a) = 0$ on $b\Omega$, and $\Delta(G_\Omega(\cdot,a) + \log|\cdot - a|) = 0$ on Ω. Hence $G_\Omega(z,a) = -\log|z-a| + h(z)$ where $\Delta h = 0$ in Ω and $h|_{b\Omega} = \log|\cdot - a| \in C^2(b\Omega)$. By standard elliptic theory (see [4;p.102]), $h \in C^{2-\epsilon}(\bar{\Omega} - \{a\})$ for any $0 < \epsilon < 1$.

Since $G_\Omega(z,a) = G_U(f(z), f(a))$ where $G_U(\cdot,\cdot)$ is the Green's function associated to U,

$$-\frac{\partial}{\partial z}G_\Omega(z,a) = \frac{\partial}{\partial z}(\log|f(z)|) = \frac{\partial}{\partial z}\left(\frac{1}{2}\log|f(z)|^2\right) = \frac{1}{2}\frac{f'(z)}{f(z)}$$

and hence

$$f'(z) = -2f(z)\frac{\partial}{\partial z}G_\Omega(z,a). \tag{1}$$

By the facts that $f \in C(\bar{\Omega})$ by Carathéodory's theorem and $G_\Omega(\cdot,a)$ extends $C^{2-\epsilon}$ smoothly up to boundary of Ω, we get that $f' \in C(\bar{\Omega})$ and therefore $f \in C^1(\bar{\Omega})$. Again by using (1), we get $f' \in C^{1-\epsilon}(\bar{\Omega})$ and hence $f \in C^{2-\epsilon}(\bar{\Omega})$. □

The following theorem is the main result which tells continuous extendibility of the Szegő kernel when given domain has C^2 smooth boundary.

Theorem 4. *Let Ω be a bounded domain in $\mathbb{C}$ with C^2 smooth boundary. Then* $S(z,w) \in C(\bar{\Omega} \times \bar{\Omega} - \{(z,z) : z \in b\Omega\})$.

Proof. Note that in Lemma 1 and Lemma 2, we can relax the smoothness of $b\Omega_1$ from C^∞ to C^2 without difficulties, and therefore Ω is biholomorphically equivalent to a bounded domain D in $\mathbb{C}$ with real analytic boundary and same connectivity. Let $f : \Omega \to D$ be the biholomorphic mapping given in Lemma 1.

By using Lemma 3 and the fact that $1/(z-w) \in C^2(\bar{\Omega})$ with respect to z when $w \notin \Omega$, we find that $f \in C^{2-\epsilon}(\bar{\Omega})$ for any $0 < \epsilon < 1$. By the transformation formula for the Szegő kernel in Lemma 2,

$$S_\Omega(z,w) = \sqrt{f'(z)} S_D(f(z), f(w)) \overline{\sqrt{f'(w)}}$$

for $z, w \in \Omega$. Since $S_D(\cdot,\cdot) \in C(\bar{D} \times \bar{D} - \{(z,z) : z \in bD\})$ (see [1,p.103]) and $f \in C^{2-\epsilon}(\bar{\Omega})$ for any $0 < \epsilon < 1$,

$$S_D(f(\cdot), f(\cdot)) \in C(\bar{\Omega} \times \bar{\Omega} - \{(z,z) : z \in b\Omega\})$$

and $\sqrt{f'(\cdot)} \in C(\bar{\Omega})$. Therefore we see that $S_\Omega(z,w) \in C(\bar{\Omega} \times \bar{\Omega} - \{(z,z) : z \in b\Omega\})$. □

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