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DIRECT PRODUCTS OF L_δ-GROUPS

JON CORSON* AND DOHYOUNG RYANG**

ABSTRACT. Recently L_δ-groups were introduced in the study of geometric group theory. Three levels of L_δ-groups are difined and discussed. It is shown that each of these levels of L_δ-groups is closed under taking a direct product.

1. INTRODUCTION

Recently L_δ-metric spaces and L_δ-groups were introduced in the study of geometric group theory as a generalization of hyberbolic metric spaces and hyperbolic groups, respectively. Hyperbolic groups have been a central topic and have provided motivation for much of geometric group theory since Gromov's seminal paper [6]. The class of hyperbolic groups includes finite groups, free groups, surface groups, hyperbolic small cancellation groups, etc., and is closed under free products, free factors, and direct factors. But a direct product of hyperbolic groups may not be hyperbolic. For example, $\mathbb{Z} \times \mathbb{Z}$ is not hyperbolic whereas $\mathbb{Z}$ is hyperbolic.

Elder studied a strong form of L_δ-groups, namely finitely generated groups that have a Cayley graph with the L_δ-property. He showed that such groups are finitely presented and have a sub-cubic Dehn function [5]. Chatterji defined L_δ-groups in a perhaps more general sense than in Elder's work in her dissertation [2]. Nevertheless, we showed that Elder's result is also true for Chatterji's L_δ-groups [4]. In addition to these two notions of L_δ-groups we introduce what might be an intermediate level of L_δ-groups using a weighted Cayley graph. The current paper is a discussion of these different levels of L_δ-groups which we call VSL_δ-groups, SL_δ-groups, and L_δ-groups; see Section 2 for the precise definitions. The following chain of inclusions is an easy consequence of their definitions (see Section 3):

$$\text{Hyperbolic groups} \Longrightarrow VSL_\delta\text{-groups} \Longrightarrow SL_\delta\text{-groups} \Longrightarrow L_\delta\text{-groups}.$$

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It is unknown whether or not any of these three levels of L_δ-groups are actually the same. Consequently, it is of interest to investigate differences and similarities among the three levels of L_δ-groups. As noted above, as a similarity, all three levels of L_δ-groups are finitely presented and have a sub-cubic Dehn function. This paper is an investigation of another similarity among the three classes. It was remarked in [3] that L_δ-groups are closed under taking a direct product. We show that indeed all three different levels of L_δ-groups are closed under taking a direct product.

All groups throughout the paper, unless stated otherwise, are assumed to be finitely generated. And all generating sets are assumed to be *symmetric*, meaning that they contain the inverse of each of their members.

2. Background and Definitions

Let G be a group and (X, d) be a metric space. A (left) action of G on X is is said to be *cocompact* if there exists a compact subset $K \subset X$ such that $X = GK$, i.e., the orbit space X/G is compact. We say that G is acting *by isometries* on X if for any $x, y \in X$ and $g \in G$, $d(x,y) = d(g \cdot x, g \cdot y)$. An action is said to be *proper* if for each $x \in X$ there exists a number $r > 0$ such that the set $\{g \in G \mid gB(x,r) \cap B(x,r) \neq \emptyset\}$ is finite. Or alternatively, G acts *properly* on X if for every compact subset $K \subseteq X$ there are only finitely many $g \in G$ such that $gK \cap K \neq \emptyset$. We will be interested in groups that are simultaneously acting properly, compactly, and by isometries on a metric space. These special actions are often called *geometric actions.*

Suppose that $\mathcal{P}$ is a property of metric spaces. We call a finitely generated group G a *(geometric) $\mathcal{P}$-group* if G acts geometrically on a geodesic space with property $\mathcal{P}$. For example, a hyperbolic group can be defined in this way. First let us recall the definition of a hyperbolic space.

Definition 2.1 (Gromov's hyperbolicity)**.** Let X be a geodesic metric space and $\delta \geq 0$ be a constant. A geodesic triangle in X is said to be *δ-thin* if each of its sides is contained in the δ-neighborhood of the other two sides. X is called *δ-hyperbolic* if every geodesic triangle in X is δ-thin.

There are some equivalent ways to formulate hyperbolicity. One way is as follows: Consider a geodesic triangle Δ with vertices x, y, and z in a geodesic space X. Then there exists a map α which maps Δ to the tripod T with four points x', y', z' and t' in $\mathbb{E}^2$, where $\alpha(x) = x', \alpha(y) = y', \alpha(z) = z'$ and t' is the common point of three edges. The map α is an isometry on each side. Then X is hyperbolic if and only if there is a constant k such that for each geodesic triangle Δ in X, the diameter of $\alpha^{-1}(p)$ is less than k for all $p \in T$ [6].

It is an important fact that hyperbolicity of geodesic metric spaces is a quasi-isometry invariant. Recall that quasi-isometry is a relation that equates spaces which look the same on the large scale:

Definition 2.2 (quasi-isometry). Let $\lambda \geq 1$ and $\varepsilon \geq 0$ be constants. A map $f : (X, d) \to (X', d')$ is a (λ, ε)-*quasi-isometric embedding* if

$$\frac{1}{\lambda}d(x,y) - \varepsilon \leq d'(f(x), f(y)) \leq \lambda d(x,y) + \varepsilon.$$

for all $x, y \in X$. In addition, f is called a (λ, ε)-*quasi-isometry* if there exists a constant $k \geq 0$ such that every point in X' lies in the k-neighborhood of the image of f. We say that (X, d) and (X', d') are *quasi-isometric* when such a map exists.

A finitely generated group G is a *hyperbolic group* if it acts properly, cocompactly, and by isometries on a hyperbolic space. Furthermore, it follows from the Švarc-Minor Theorem (see [1, 8]) that any two geodesic spaces on which G acts geometrically are quasi-isometric. Thus if a finitely generated group G is hyperbolic, then every geodesic metric space that G acts geometrically on is a hyperbolic space. The Cayley graph is a good example of geodesic space on which G acts geometrically. Hence the following definition is equivalent to the one given above.

Definition 2.3 (Hyperbolic group). A finitely generated group is *hyperbolic* if it has a Cayley graph that is a hyperbolic space.

Recall the L_δ-property of a geodesic space. Let (X, d) be a geodesic space. A finite sequence of points $(x_1, \ldots, x_n)$ in X is called a δ-*path* if there exists a non-negative constant δ such that

$$d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) \leq d(x_1, x_n) + \delta.$$

Let x, y, z in X. A point $t \in X$ is called a δ-*center* for the triple x, y, z if (x, t, y), (y, t, z), and (z, t, x) are all δ-paths. Note that if a triple of points has a δ-center, then it has a δ'-center for every $\delta' \geq \delta$. So we may assume that δ is an integer. We say that a geodesic space (X, d) has the L_δ-*property* and call it an L_δ-*space* if any triple x, y, z in X has a δ-center. Of course the L_δ-property makes sense for metric spaces in general, but here we are only interested in geodesic spaces.

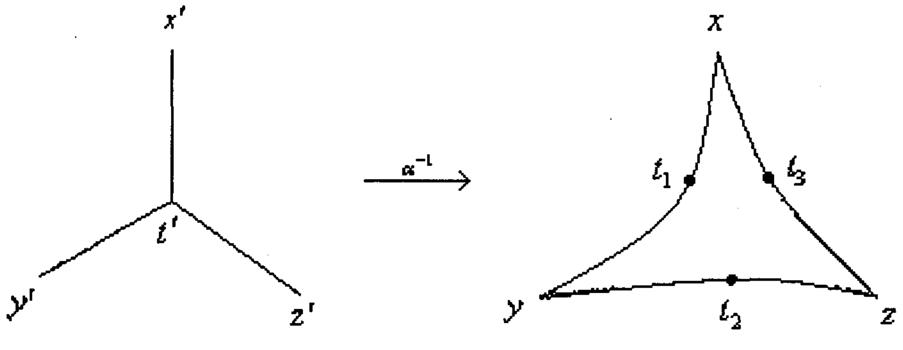

FIGURE 1. α^{-1} from T in $\mathbb{E}^2$ to a hyperbolic space X.

Definition 2.4 (L_δ-group). A finitely generated group G is said to be an *L_δ-group* if it acts properly, cocompactly, and by isometries on an L_δ-space for some $\delta \geq 0$.

Easy examples show that the L_δ-property is not a quasi-isometry invariant. So we cannot define L_δ-groups in terms of their Cayley graphs as we did for hyperbolic groups. However, we use the Cayley graph and weighted Cayley graph to consider some stronger cases of L_δ-groups. A weighted Cayley graph is a Cayley graph with weights attached to its edges. These are discussed in the next section. An ordinary (unweighted) Cayley graph is a special case in which each edge has unit weight.

Definition 2.5 (VSL_δ-group, SL_δ-group). A group G is called a *strong L_δ-group*, and denoted by *SL_δ-group*, if a weighted Cayley graph of G has the L_δ-property. G is called a *very strong L_δ-group*, and denoted by *VSL_δ-group*, if an (unweighted) Cayley graph of G has the L_δ-property.

We now show that hyperbolic groups are L_δ-groups by proving that hyperbolic spaces are L_δ-spaces. It then follows by an observation made in the next section that hyperbolic groups are also SL_δ-groups and VSL_δ-groups.

Proposition 2.6. *A hyperbolic space is an L_δ-space.*

Proof. Let X be a k-hyperbolic space and x, y, z be in X. Let Δ be a geodesic triangle with vertices x, y, z. Then there is a map α from Δ to a tripod T with four points x', y', z' and $t' \in \mathbb{E}^2$, where $\alpha(x) = x', \alpha(y) = y', \alpha(z) = z'$ and t' is the unique triple point with a point from each side of Δ in its preimage. Let $\alpha^{-1}(t') = t_1, t_2, t_3$ where t_1 is the point on $[x, y]$, t_2 is the point on $[y, z]$, and t_3 is the point on $[z, x]$. See Figure 1.

Then $d(x,t_1) = d(x,t_3)$, $d(y,t_1) = d(y,t_2)$, and $d(z,t_2) = d(z,t_3)$. Since Δ is k-thin, it is easy to see that t_1 is within a distance $2k$ of t_2 or t_3, and likewise for t_2 and t_3. Thus one of the points t_1, t_2, or t_3 is within a distance $2k$ of the other two points. We check that this point, assume that it is t_1, is a $2k$-center for the triple x, y, z.

Since $t \in [x,y]$, $d(x,t) + d(t,y) = d(x,y)$. On the other hand, by the triangle inequality,

$$\begin{aligned} d(y,t_1) + d(t_1,z) &\le d(y,t_2) + d(t_1,t_2) + d(t_2,z) \\ &\le d(y,t_2) + 2k + d(t_2,z) \\ &= d(y,z) + 2k, \end{aligned}$$

and similarly $d(z,t_1)+d(t_1,x) \le d(z,x)+2k$. So t is a $2k$-center for the triple x, y, z, and hence X has the L_δ-property for $\delta = 2k$. □

3. Weighted Cayley Graph

In this section we review weighted Cayley graphs and some of their basic properties. Let G be a finitely generated group and let A be a weighted finite generating set. That is, A is a finite symmetric ($A^{-1} = A$) generating set for G with a weight function $\omega_A : A \to \mathbb{Z}^+$ satisfying $\omega_A(a^{-1}) = \omega_A(a)$ for all $a \in A$. Let A^* denote the set of all words on the set A, and denote the natural epimorphism $A^* \to G$ by $w \mapsto \overline{w}$. For a word $w = a_1 a_2 \cdots a_n$ in A^*, the weight of w is

$$\omega_A(w) = \omega_A(a_1 a_2 \cdots a_n) = \omega_A(a_1) + \omega_A(a_2) + \cdots + \omega_A(a_n).$$

Define a distance function $d_A : G \times G \to \mathbb{R}$ by

$$d_A(x,y) = \min\left\{\omega_A(w) \mid w \in A^* \text{ and } \overline{w} = x^{-1}y\right\},$$

for all x, y in G. It is easy to check that d_A is a metric for G. Construct the *weighted Cayley graph* $\Gamma(G,A)$ as follows. The vertex set of $\Gamma(G,A)$ is G and the set of directed edges is $G \times A$. For each $g \in G$ and $a \in A$, the directed edge (g,a) joins the vertex g to the vertex ga and is given the *label* a. The inverse of the edge (g,a) is (ga, a^{-1}). We extend d_A to a metric for $\Gamma(G,A)$ by making each edge with label $a \in A$ isometric to the interval $[0, \omega_A(a)] \subset \mathbb{R}$ and taking d_A to be the path metric. Note that we get an (unweighted) Cayley graph by putting the weight 1 on each $a \in A$.

Remark. Let $\omega : A \to \mathbb{Q}^+$ be a rational weight function. For all $a_i \in A$, the weight $\omega(a_i) = \frac{p_i}{q_i}$, where $p_i,\, q_i \in \mathbb{Z}$. Since A is finite, there exists a least common multiple m of all q_i. Then there exist $m_i \in \mathbb{Z}$ such that $m\frac{p_i}{q_i} = m_i p_i \in \mathbb{Z}$. We reassign the weight $\omega(a_i) = m_i p_i$ for all $a_i \in A$. So, by scaling we obtain integer weights, and hence nothing is gained by allowing rational, non-integer weights.

Lemma 3.1. *Let G be a finitely generated group and A be a finite weighted generating set for G. Then $(\Gamma(G,A), d_A)$ is a geodesic metric space on which G acts properly, cocompactly, and by isometries.*

Proof. We first need to show that $(\Gamma(G,A), d_A)$ is a geodesic space. Since A generates G, $\Gamma(G,A)$ is path connected. Choose $x, y \in G$ and let $\{w_j\}$ be the set of all words in A^* such that $\overline{w}_j = x^{-1}y$. Since all $\omega_A(w_j)$ are positive integers, there exists a word $w \in \{w_j\}$ such that $d_A(x,y) = \inf_j \omega_A(w_j) = \omega_A(w)$. Thus, the path in $\Gamma(G,A)$ from x to y with label w is a geodesic path from x to y.

Choose $x, y \in \Gamma(G,A) \setminus G$. Let e_x and e_y be the edges containing x and y, respectively. Then there are endpoints x' of e_x and y' of e_y such that $d_A(x,y) = d_A(x,x') + d_A(x',y') + d_A(y',y)$. Thus there exist geodesics from x to x', x' to y', and y' to y, that together form a geodesic from x to y. Hence $(\Gamma(G,A), d_A)$ is a geodesic space.

We next check that G acts properly, cocompactly, by isometries on $\Gamma(G,A)$. Since the orbit space $\Gamma(G,A)/G$ is a finite graph with one vertex and with $|A|$ edges, G acts cocompactly on $(\Gamma(G,A), d_A)$. For any $x, y \in \Gamma(G,A)$, let γ be a geodesic from x to y. Then $g\gamma$ is a path in $\Gamma(G,A)$ between gx and gy with the same weight as γ for all $g \in G$. Thus

$$d_A(gx, gy) \le \omega_A(g\gamma) = \omega_A(\gamma) = d_A(x,y).$$

Then we also have that

$$d_A(x,y) = d_A(g^{-1}gx, g^{-1}gy) \le d_A(gx, gy).$$

So, $d_A(gx,gy) = d_A(x,y)$ and hence G acts on $(\Gamma(G,A), d_A)$ by isometries. Since G acts freely on $\Gamma(G,A)$, it follows easily that the action is proper. □

Now suppose that G is a hyperbolic group. By Lemma 3.1 , G acts geometrically on its (unweighted) Cayley graph $\Gamma(G,A)$ and thus $\Gamma(G,A)$ is a hyperbolic space. So by Proposition 2.6, $\Gamma(G,A)$ is an L_δ-space and hence G is a VSL_δ-group. Furthermore, it is obvious that a VSL_δ-group is a SL_δ-group, and by Lemma 3.1 we see that a SL_δ-group is an L_δ-group. That is,

$$\text{Hyperbolic groups} \subseteq VSL_\delta\text{-groups} \subseteq SL_\delta\text{-groups} \subseteq L_\delta\text{-groups}.$$

It also follows from Lemma 3.1 and the Švarc-Minor Theorem (Proposition 8.19 [1]) that weighted Cayley graphs of G for different weighted finite generating sets are quasi-isometric. This also follows from our next observation.

Proposition 3.2. *Let G be a finitely generated group and let A and B be weighted finite generating sets for G. Then the inclusion map $\iota : (G, d_A) \to (\Gamma(G, B), d_B)$ is a quasi-isometry.*

Proof. Let $\alpha = \max\{\omega_B(a) \mid a \in A\}$ and $\beta = \max\{\omega_A(b) \mid b \in B\}$, and let $\lambda = \max\{\alpha, \beta\}$. Choose $x, y \in G$. Then there exist $a_1, \ldots, a_n \in A$ such that $x^{-1}y = a_1 \cdots a_n$ and $d_A(x, y) = \omega_A(a_1 \cdots a_n)$. Thus,

$$d_B(x, y) \leq \omega_B(a_1) + \cdots + \omega_B(a_n) \leq \alpha \cdot n \leq \lambda \cdot d_A(x, y).$$

Similarly, $d_A(x, y) \leq \beta \cdot d_B(x, y) \leq \lambda \cdot d_B(x, y)$. So

$$\frac{1}{\lambda} d_A(x, y) \;\leq\; d_B(x, y) \;\leq\; \lambda\, d_A(x, y),$$

and hence $\iota : (G, d_A) \to (\Gamma(G, B), d_B)$ is a quasi-isometric embedding.

Note that every point in $\Gamma(G, B)$ is no more than one edge from a vertex, and each edge is weighted by at most $k = \max\{\omega_B(b) \mid b \in B\}$. Therefore every point in $\Gamma(G, B)$ is in the k-neighborhood of the image of ι, and thus the inclusion map ι is a quasi-isometry. □

4. Main Results

In this section we investigate direct products of the three levels of L_δ-groups. We begin with the case of SL_δ-groups.

Theorem 4.1. *Let G be a finitely generated group and $G = M \times N$. If M and N are SL_δ-groups, then G is a SL_δ-group.*

Proof. Let G be a finitely generated group and M, N be normal subgroups of G so that $G = MN$, $M \cap N = 1$, M and N are SL_δ-groups. There exists a weighted finite generating sets A and B for M and N, respectively, such that $(\Gamma(M, A), d_A)$ has the L_{δ_1}-property and $(\Gamma(N, B), d_B)$ have the L_{δ_2}-property for some $\delta_i \geq 0$. Let

$E = A \cup B$. Then E is a weighted finite generating set for G, where the weight function $\omega_E : E \to \mathbb{Z}^+$ is determined by $\omega_A : A \to \mathbb{Z}^+$ and $\omega_B : B \to \mathbb{Z}^+$.

It suffices to show that $(\Gamma(G,E), d_E)$ has the L_δ-property for some $\delta \geq 0$. We make use of the following observation about the metrics d_E, d_A, and d_B. Let g_1 and g_2 be in G. Then there exist unique $m_1, m_2 \in M$ and $n_1, n_2 \in N$ such that $g_1 = m_1 n_1$ and $g_2 = m_2 n_2$. We claim that $d_E(g_1, g_2) = d_A(m_1, m_2) + d_B(n_1, n_2)$.

First note that

$$\begin{aligned} d_E(g_1, g_2) &= d_E(m_1 n_1, m_2 n_2) \\ &\leq d_E(m_1 n_1, m_2 n_1) + d_E(m_2 n_1, m_2 n_2) \\ &\leq d_E(n_1 m_1, n_1 m_2) + d_E(m_2 n_1, m_2 n_2) \\ &\leq d_E(m_1, m_2) + d_E(n_1, n_2) \\ &\leq d_A(m_1, m_2) + d_B(n_1, n_2). \end{aligned}$$

On the other hand, choose $u \in A^*$ and $v \in B^*$ such that $m_1^{-1} m_2 = \overline{u}$, $n_1^{-1} n_2 = \overline{v}$, $d_A(m_1, m_2) = \omega_A(u)$, and $d_B(n_1, n_2) = \omega_B(v)$. Then $uv \in E^*$ and $g_1^{-1} g_2 = n_1^{-1} m_1^{-1} m_2 n_2 = m_1^{-1} m_2 n_1^{-1} n_2 = \overline{uv}$. So

$$d_E(g_1, g_2) \leq \omega_E(uv) = \omega_A(u) + \omega_B(v) = d_A(m_1, m_2) + d_B(n_1, n_2),$$

and hence the claim follows.

We now check the L_δ-property of $(\Gamma(G,E), d_E)$. Let $g_1, g_2, g_3 \in G$, i.e., vertices of $\Gamma(G,E)$. Then there exist unique m_1, m_2, and m_3 in M and n_1, n_2, and n_3 in N such that $g_1 = m_1 n_1$, $g_2 = m_2 n_2$, and $g_3 = m_3 n_3$. Since $\Gamma(M, A)$ has the L_{δ_1}-property, there exist a δ_1-center $t \in \Gamma(M, A)$ for the triple m_1, m_2, m_3. Let $m \in M$ be a nearest vertex to t. Note that $d_A(t, m) \leq k/2$, where $k = \max\{\omega_E(e) \mid e \in E\}$. Thus for $i \neq j$,

$$\begin{aligned} d_A(m_i, m) + d_A(m, m_j) &\leq d_A(m_i, t) + d_A(t, m) + d_A(m, t) + d_A(t, m_j) \\ &\leq d_A(m_i, m_j) + \delta_1 + k. \end{aligned}$$

Likewise, there exists an $n \in N$ such that for $i \neq j$,

$$d_B(n_i, n) + d_B(n, n_j) \leq d_B(n_i, n_j) + \delta_2 + k.$$

Let $g = mn \in G$. Then g is a vertex of $\Gamma(G,E)$ satisfying: for $i \neq j$,

$$\begin{aligned} d_E(g_i,g) + d_E(g,g_j) &= d_A(m_i,m) + d_B(n_i,n) + d_A(m,m_j) + d_B(n,n_j) \\ &\leq d_A(m_i,m_j) + \delta_1 + k + d_B(n_i,n_j) + \delta_2 + k \\ &= d_E(g_i,g_j) + \delta_1 + \delta_2 + 2k. \end{aligned}$$

Hence the vertex g is a $(\delta_1 + \delta_2 + 2k)$-center in $\Gamma(G,E)$ for the triple of vertices g_1, g_2, g_3.

Now let x_1, x_2, and x_3 be any three points in $\Gamma(G,E)$. Pick three vertices g_1, g_2 and g_3 in G which are closest to the points x_1, x_2, and x_3, respectively. Note that $d_E(x_i,g_i) \leq k/2$, where as before $k = \max\{\omega_E(e) \mid e \in E\}$. As we saw above, there exists $g \in G$ which is a $(\delta_1 + \delta_2 + 2k)$-center for g_1, g_2, g_3 in $\Gamma(G,E)$. Thus, by the triangle inequality and the above inequalities, we obtain the following inequalities: for $i \neq j$,

$$\begin{aligned} d_E(x_i,g) + d_E(g,x_j) &\leq d_E(x_i,g_i) + d_E(g_i,g) + d_E(g,g_j) + d_E(g_j,x_j) \\ &\leq \frac{k}{2} + d_E(g_i,g_j) + \delta_1 + \delta_2 + 2k + \frac{k}{2} \\ &\leq d_E(g_i,x_i) + d_E(x_i,x_j) + d_E(x_j,g_j) + \delta_1 + \delta_2 + 3k \\ &\leq d_E(x_i,x_j) + \delta_1 + \delta_2 + 4k. \end{aligned}$$

It follows that g is a $(\delta_1+\delta_2+4k)$-center for the triple x_1, x_2, x_3. Hence $(\Gamma(G,E), d_E)$ has the L_δ-property for $\delta = \delta_1 + \delta_2 + 4k$, and therefore G is an SL_δ-group. □

By the same argument as in the proof of Theorem 4.1 we obtain the result for VSL_δ-groups:

Corollary 4.2. *Let G be a finitely generated group and $G = M \times N$. If M and N are VSL_δ-groups, then G is a VSL_δ-group.*

Proof. Since, by assumption, M and N are VSL_δ-groups, there exists finite generating sets A for M and B for N (with unit weights) such that $(\Gamma(M,A), d_A)$ has the L_{δ_1}-property and $(\Gamma(N,B), d_B)$ has the L_{δ_2}-property for some $\delta_i \geq 0$. Let $E = A \cup B$. Then E is a finite generating set for G, where ω_E is the unit weight function, i.e., $\omega_E(e) = 1$ for all $e \in E$. Furthermore, by the argument in the proof of Theorem 4.1, $(\Gamma(G,E), d_E)$ has the L_δ-property for $\delta = \delta_1 + \delta_2 + 4$ (as here $k = 1$). Hence G is a VSL_δ-group. □

Turning to the general case, we make the following convention. Let (X, d_X) and (Y, d_Y) be metric spaces. Then, unless stated otherwise, we give the product space $X \times Y$ the metric d, where for any two points (x_1, y_1) and (x_2, y_2) in $X \times Y$,

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Lemma 4.3. *Let (X, d_X) and (Y, d_Y) be L_δ-spaces and let d be the metric on $X \times Y$ defined above. Then $(X \times Y, d)$ is an $L_{\delta'}$-space for $\delta' = 2\delta$.*

Proof. We may assume that (X, d_X) and (Y, d_Y) are L_δ-spaces for the same $\delta \geq 0$. Choose three points $(x_i, y_i) \in X \times Y, i = 1, 2, 3$. Since X is an L_δ-space, there exists a δ-center $s \in X$ for the triple x_1, x_2, x_3 in X and there exists a δ-center $t \in Y$ for the triple y_1, y_2, y_3 in Y. We see that $((x_1, y_1), (s, t), (x_2, y_2))$ is δ'-path for $\delta' = 2\delta$, since

$$\begin{aligned}
&d((x_1, y_1), (s, t)) + d((s, t), (x_2, y_2)) \\
&= d_X(x_1, s) + d_Y(y_1, t) + d_X(s, x_2) + d_Y(t, y_2) \\
&\leq d_X(x_1, x_2) + \delta + d_Y(y_1, y_2) + \delta \\
&= d((x_1, y_1), (x_2, y_2)) + 2\delta.
\end{aligned}$$

Similarly $((x_2, y_2), (s, t), (x_3, y_3))$ and $((x_3, y_3), (s, t), (x_1, y_1))$ are δ'-paths, so $(s, t) \in X \times Y$ is a δ'-center for the triple (x_1, y_1), (x_2, y_2), (x_3, y_3) in $X \times Y$. Hence $(X \times Y, d)$ has the $L_{\delta'}$-property for $\delta' = 2\delta$. □

Lemma 4.4. *Let G and H be finitely generated groups acting properly, cocompactly, and by isometries on a metric space (X, d_X) and (Y, d_Y), respectively. Then $G \times H$ acts properly, cocompactly, and by isometries on the metric space $(X \times Y, d)$.*

Proof. Define the group action of $G \times H$ on the product space $X \times Y$ by

$$(g, h) \cdot (x, y) = (g \cdot x, h \cdot y)$$

for all $g \in G, h \in H$ and for all $x \in X, y \in Y$. We claim that this action is proper, cocompact, and by isometries.

Let K be a compact subset of $X \times Y$, and let K_X be the projection of K on X and K_Y be the projection of K on Y. Then $K \subset K_X \times K_Y$. Note that K_X and K_Y are compact since a continuous map preserves compactness. We observe that

$$\begin{aligned}
&\{(g, h) \in G \times H \mid (g, h)K \cap K \neq \emptyset\} \\
&\subset \{(g, h) \in G \times H \mid (g, h)(K_X \times K_Y) \cap (K_X \times K_Y) \neq \emptyset\}
\end{aligned}$$

$$= \{(g,h) \in G \times H \mid (gK_X \times hK_Y) \cap (K_X \times K_Y) \neq \emptyset\}$$
$$= \{(g,h) \in G \times H \mid (gK_X \cap K_X) \neq \emptyset \text{ and } (hK_Y \cap K_Y) \neq \emptyset\}$$
$$= \{g \in G \mid gK_X \cap K_X \neq \emptyset\} \times \{h \in H \mid hK_Y \cap K_Y \neq \emptyset\}.$$

We know that $\{g \in G \mid gK_X \cap K_X \neq \emptyset\}$ is finite since K_X is a compact subset of X and G acts properly on X. Similarly, $\{h \in H \mid hK_Y \cap K_Y \neq \emptyset\}$ is finite. So, $\{(g,h) \in G \times H \mid (g,h)K \cap K \neq \emptyset\}$ is finite and hence $G \times H$ acts properly on $X \times Y$.

Let K_X be a compact subset of X such that $GK_X = X$ and let K_Y be a compact subset of Y such that $HK_Y = Y$. Let $K = K_X \times K_Y$. Then

$$\begin{aligned}(G \times H)(K_X \times K_Y) &= \{(g,h) \cdot (x,y) \mid g \in G, h \in H, x \in K_X, y \in K_Y\}\\ &= \{(gx, hy) \mid gx \in GK_X, hy \in HK_Y\}\\ &= GK_X \times HK_Y\\ &= X \times Y.\end{aligned}$$

And note that $K_X \times K_Y$ is a compact subset of $X \times Y$ since the d-metric topology and the usual product topology for $X \times Y$ are the same. So, $G \times H$ acts cocompactly on $X \times Y$.

It is easy to check that $G \times H$ acts on $X \times Y$ by isometries:

$$\begin{aligned}d((g,h) \cdot (x_1,y_1), (g,h) \cdot (x_2,y_2)) &= d((gx_1, hy_1), (gx_2, hy_2))\\ &= d_X(gx_1, gx_2) + d_Y(hy_1, hy_2)\\ &= d_X(x_1, x_2) + d_Y(y_1, y_2)\\ &= d((x_1,y_1),(x_2,y_2)),\end{aligned}$$

for all $g \in G$, $h \in H$, $x_i \in X$, and $y_i \in Y$. □

The previous two lemmas are combined to prove the following main theorem.

Theorem 4.5. *If G and H are L_δ-groups, then $G \times H$ is an L_δ-group.*

Proof. Let X be an L_δ-space on which G acts properly, cocompactly, and by isometries and let Y be an L_δ-space on which H acts properly, cocompactly, and by isometries. Then, by Lemma 4.4, $G \times H$ acts properly, cocompactly, and by isometries on $X \times Y$. By Lemma 4.3, $X \times Y$ is an $L_{\delta'}$-space for $\delta' = 2\delta$. So the result follows. □

REFERENCES

1. M. Bridson & A. Haefliger: *Metric Spaces of Non-Positive Curvature.* Springer-Verlag, 1999.
2. I. Chatterji: *On property RD for certain discrete groups*, PhD thesis. ZTH Zürich, 2001. Available at http://www.math.ohio-state.edu/∼ indira.
3. I. Chatterji & K. Ruane: Some Geometric Groups with Rapid Decay. *GAFA* **15** (2005), 311 - 339.
4. J. Corson & D. Ryang: Isoperimetric functions of groups acting on L_δ-spaces. *Glasgow Mathematical Journal*, to appear.
5. M. Elder: L_δ groups are almost convex and have a sub-cubic Dehn function. *Agebraic and Geometric Topology* **4** (2004), 23-29.
6. M. Gromov: *Hyperbolic groups.* In Essays in Group Theory (ed. S. M. Gersten), MSRI Publ. **8**, 75-263. Springer-Verag, 1987.
7. P. de la Harpe: *Topics in Geometric Group Theory.* The University of Chicago Press, 2000.
8. M. Neumann & M. Shapiro: *A short course in geometric group theory.* Notes for the ANU Workshop, 1996.

*UNIVERSITY OF ALABAMA,TUSCALOOSA, ALABAMA 35487, USA
Email address: `jcorson@bama.ua.edu`

**TALLADEGA COLLEGE, TALLADEGA, ALABAMA 35160, USA
Email address: `dryang@talladega.edu`