J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 14, Number 2 (May 2007), Pages 127–137

BOHR'S INEQUALITIES IN n-INNER PRODUCT SPACES

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ABSTRACT. The classical Bohr's inequality states that

$$|z+w|^2 \le p|z|^2 + q|w|^2$$

for all $z, w \in \mathbb{C}$ and all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. In this paper, Bohr's inequality is generalized to the setting of n-inner product spaces for all positive conjugate exponents $p, q \in \mathbb{R}$. In particular, the parallelogram law is recovered and an interesting operator inequality is obtained.

1. INTRODUCTION

The classical Bohr's inequality [1, 6] states that, for any $z, w \in \mathbb{C}$ and conjugate exponents $p, q > 1$,

$$|z+w|^2 \le p|z|^2 + q|w|^2\,, \tag{1.1}$$

with the equality if and only if $w = (p-1)z$. In the past few decades, various generalizations of (1.1) have been obtained (see, e.g., [3, 5, 6, 8, 9, 10, 11]). A special direction of these is in the setting of normed vector spaces. In [7], Pečarić and Dragomir showed that, if $(X, \|\cdot\|)$ is a normed vector space and $p, q > 1$ are any conjugate exponents, then

$$\|v+w\|^2 \le p\|v\|^2 + q\|w\|^2 \tag{1.2}$$

for all $v, w \in X$. In [4], Hirzallah further generalized the inequality (1.2) to the context of operator algebras. It was shown that, if $\mathbb{H}$ is a complex separable Hilbert space and $B(\mathbb{H})$ is the algebra of all bounded linear operators on $\mathbb{H}$, then, for any $A, B \in B(\mathbb{H})$ and conjugate exponents p, q with $q \ge p > 1$,

$$|A-B|^2 + |(1-p)A - B|^2 \le p|A|^2 + q|B|^2,$$

Received by the editors March 6, 2007 and, in revised form April 28, 2007.
2000 *Mathematics Subject Classification.* 47A30, 26D15, 30A10, 46B20.
Key words and phrases. Bohr's inequality, n-inner product space, n-normed linear space.
This research is supported in part by the Research Grants Council of the Hong Kong SAR, China (Project No. HKU7017/05P).

where $|X| := (X^*X)^{1/2}$. It is worthwhile noting that, in [4], only the situation where $q \geq p > 1$, or equivalently, only the situation where $q \geq 2$ and $1 < p \leq 2$ was considered, while the other situations were left unattended. Very recently, in [2], Cheung and Pečarić continued working in the setting of [4], but without restriction to the conjugate components p, q.

In this paper, the results in [2] are further generalized to the context of n-inner product spaces. As an application of these Bohr-type inequalities, an interesting inequality on operators in an n-inner product space is given.

2. Basic Terminologies and Fundamental Results

Let $n \geq 2$ and X be a linear space of dimension greater than or equal to n over the complex numbers field $\mathbb{C}$. A complex-valued function $(\cdot,\cdot|\cdot,\cdots,\cdot) : X^{n+1} \to \mathbb{C}$ satisfying the following properties:

(I_1) $(x,x|a_2,\cdots,a_n) \geq 0$ and $(x,x|a_2,\cdots,a_n) = 0$ if and only if the vectors $x, a_2, \cdots, a_n$ are linearly dependent;

(I_2) $(x,x|a_2,\cdots,a_n) = (a_2,a_2|x,\cdots,a_n)$;

(I_3) $(x,y|a_{i_2},\cdots,a_{i_n}) = (x,y|a_2,\cdots,a_n)$ for any permutation $(i_2,\cdots,i_n)$ of $(2,\cdots,n)$;

(I_4) $(y,x|a_2,\cdots,a_n) = \overline{(x,y|a_2,\cdots,a_n)}$;

(I_5) $(\alpha x,y|a_2,\cdots,a_n) = \alpha(x,y|a_2,\cdots,a_n)$ for any scalar $\alpha \in \mathbb{C}$;

(I_6) $(x_1+x_2,y|a_2,\cdots,a_n) = (x_1,y|a_2,\cdots,a_n) + (x_2,y|a_2,\cdots,a_n)$

is called an *n-inner product* on X and $(X,(\cdot,\cdot|\cdot,\cdots,\cdot))$ is called an *n-inner product space.*

In an inner product space $(X,(\cdot,\cdot|\cdot,\cdots,\cdot))$, the following extension of Cauchy-Schwarz-Buniakowsky inequality

$$|(x,y|a_2,\cdots,a_n)| \leq \sqrt{(x,x|a_2,\cdots,a_n)} \cdot \sqrt{(y,y|a_2,\cdots,a_n)}$$

for any $x, y, a_2, \cdots, a_n \in X$ is valid and it is easy to verify that the real valued function $\|\cdot,\cdot,\cdots,\cdot\| : X^n \to \mathbb{R}$ defined by

$$\|a_1,a_2,\cdots,a_n\| = \sqrt{(a_1,a_1|a_2,\cdots,a_n)}$$

satisfies the following conditions:

(N_1) $\|a_1,a_2,\cdots,a_n\| \geq 0$ and $\|a_1,a_2,\cdots,a_n\| = 0$ if and only if $a_1, a_2, \cdots, a_n$ are linearly dependent;

(N_2) $\|a_{i_1}, a_{i_2}, \cdots, a_{i_n}\| = \|a_1, a_2, \cdots, a_n\|$ for any permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$;

(N_3) $\|\alpha a_1, a_2, \cdots, a_n\| = |\alpha| \|a_1, a_2, \cdots, a_n\|$ for any scalar $\alpha \in \mathbb{C}$;

(N_4) $\|x + y, a_2, \cdots, a_n\| \leq \|x, a_2, \cdots, a_n\| + \|y, a_2, \cdots, a_n\|$.

Any real valued function $\|\cdot, \cdots, \cdot\|$ defined on X^n satisfying conditions $(N_1)\sim(N_4)$ is called an *n-norm* and $(X, \|\cdot, \cdots, \cdot\|)$ is called an *n-normed linear space.*

Throughout this paper, X will denote an n-inner product space equipped with the n-norm

$$\|a_1, a_2, \cdots, a_n\| := \sqrt{(a_1, a_1 | a_2, \cdots, a_n)}.$$

3. Bohr's Inequality and the Parallelogram Law

Theorem 3.1. For any $x, y, a_2, \cdots, a_n \in X$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $1 < p \leq 2$, then we have the following:

$$(1) \qquad \begin{aligned} &\|x - y, a_2, \cdots, a_n\|^2 + \|(1-p)x - y, a_2, \cdots, a_n\|^2 \\ &\leq p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2, \end{aligned}$$

$$(2) \qquad \begin{aligned} &\|x - y, a_2, \cdots, a_n\|^2 + \|x - (1-q)y, a_2, \cdots, a_n\|^2 \\ &\geq p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2. \end{aligned}$$

Furthermore, in both (1) and (2), the equality holds if and only if $p = q = 2$ or $px + qy, a_2, \cdots, a_n$ are linearly dependent.

Proof. (1) We have

$$\begin{aligned} &\|x - y, a_2, \cdots, a_n\|^2 \\ &= \|x, a_2, \cdots, a_n\|^2 + \|y, a_2, \cdots, a_n\|^2 - 2Re(x, y | a_2, \cdots, a_n) \end{aligned}$$

and

$$\begin{aligned} &\|(1-p)x - y, a_2, \cdots, a_n\|^2 \\ &= (1-p)^2\|x, a_2, \cdots, a_n\|^2 + \|y, a_2, \cdots, a_n\|^2 - 2(1-p)Re(x, y | a_2, \cdots, a_n). \end{aligned}$$

Thus

$$\begin{aligned} &|x - y, a_2, \cdots, a_n\|^2 + \|(1-p)x - y, a_2, \cdots, a_n\|^2 \\ &-p\|x, a_2, \cdots, a_n\|^2 - q\|y, a_2, \cdots, a_n\|^2 \\ &= (p^2 - 3p + 2)\|x, a_2, \cdots, a_n\|^2 \\ &+2(p-2)Re(x, y | a_2, \cdots, a_n) + (2-q)\|y, a_2, \cdots, a_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&= (p-2)(p-1)\|x, a_2, \cdots, a_n\|^2 \\
&\quad +2(p-2)Re(x, y|a_2, \cdots, a_n) + \frac{p-2}{p-1}\|y, a_2, \cdots, a_n\|^2 \\
&= (p-2)\Big[(p-1)\|x, a_2, \cdots, a_n\|^2 + 2Re(x, y|a_2, \cdots, a_n) \\
&\quad +\frac{1}{p-1}\|y, a_2, \cdots, a_n\|^2\Big] \\
&= (p-2)\left\|\left(\sqrt{p-1}\, x + \frac{1}{\sqrt{p-1}}\, y\right), a_2, \cdots, a_n\right\|^2 \le 0
\end{aligned}
\tag{3.1}
$$

and hence

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|(1-p)x - y, a_2, \cdots, a_n\|^2 \\
&\le p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2,
\end{aligned}
$$

with the equality if and only if $p = 2$ or $\left(\sqrt{p-1}\, x + \frac{1}{\sqrt{p-1}}\, y\right), a_2, \cdots, a_n$ are linearly dependent, that is, $p = q = 2$ or $px + qy, a_2, \cdots, a_n$ are linearly dependent.

(2) The proof is similar to (1). □

Remark 3.2. By combining (1) and (2) in Theorem 3.1, for any $1 < p \le 2$, we have

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|(1-p)x - y, a_2, \cdots, a_n\|^2 \\
&\le p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2 \\
&\le \|x-y, a_2, \cdots, a_n\|^2 + \|x - (1-q)y, a_2, \cdots, a_n\|^2.
\end{aligned}
$$

In particularly, if $p = q = 2$, then we have

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|x+y, a_2, \cdots, a_n\|^2 \\
&\le 2\|x, a_2, \cdots, a_n\|^2 + 2\|y, a_2, \cdots, a_n\|^2 \\
&\le \|x-y, a_2, \cdots, a_n\|^2 + \|x+y, a_2, \cdots, a_n\|^2
\end{aligned}
$$

and so we get the Parallelogram Law:

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|x+y, a_2, \cdots, a_n\|^2 \\
&= 2\|x, a_2, \cdots, a_n\|^2 + 2\|y, a_2, \cdots, a_n\|^2.
\end{aligned}
\tag{3.2}
$$

Equivalently, this is also obtained by directly writing out the equality in (1) or (2) for the case $p = 2$. This completes the proof.

Corollary 3.3. *For any $x, y, a_2, \cdots, a_n \in X$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $p > 2$, then we have the following:*

$$\begin{aligned}(1)\qquad &\|x-y,a_2,\cdots,a_n\|^2+\|(1-p)x-y,a_2,\cdots,a_n\|^2\\ &\geq p\|x,a_2,\cdots,a_n\|^2+q\|y,a_2,\cdots,a_n\|^2,\\ (2)\qquad &\|x-y,a_2,\cdots,a_n\|^2+\|x-(1-q)y,a_2,\cdots,a_n\|^2\\ &\leq p\|x,a_2,\cdots,a_n\|^2+q\|y,a_2,\cdots,a_n\|^2.\end{aligned}$$

Furthermore, in both (1) *and* (2), *the equality holds if and only if* $px+qy,a_2,\cdots,a_n$ *are linearly dependent.*

Proof. This follows from Theorem 3.1 by interchanging x and y, p and q. □

Combining Theorem 3.1 and Corollary 3.3, we have the following result:

Corollary 3.4. For any $x,y,a_2,\cdots,a_n\in X$ and $p,q\in\mathbb{R}$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\|x+y,a_2,\cdots,a_n\|^2\leq p\|x,a_2,\cdots,a_n\|^2+q\|y,a_2,\cdots,a_n\|^2,$$

with the equality holds if and only if $px-qy,a_2,\cdots,a_n$ are linearly dependent.

Proof. This is immediate from (1) of Theorem 3.1 for the case $1<p\leq 2$ and (2) of Corollary 3.3 for the case $p>2$. □

Theorem 3.5. *For any* $x,y,a_2,\cdots,a_n\in X$ *and* $p,q\in\mathbb{R}$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *if* $p<1$, *then we have the following:*

$$\begin{aligned}(1)\qquad &\|x-y,a_2,\cdots,a_n\|^2+\|(1-p)x-y,a_2,\cdots,a_n\|^2\\ &\geq p\|x,a_2,\cdots,a_n\|^2+q\|y,a_2,\cdots,a_n\|^2,\\ (2)\qquad &\|x-y,a_2,\cdots,a_n\|^2+\|x-(1-q)y,a_2,\cdots,a_n\|^2\\ &\geq p\|x,a_2,\cdots,a_n\|^2+q\|y,a_2,\cdots,a_n\|^2.\end{aligned}$$

Furthermore, in both (1) *and* (2), *the equality holds if and only if* $px+qy,a_2,\cdots,a_n$ *are linearly dependent.*

Proof. (1) Since $p<1$, by (3.1)

$$\begin{aligned}&\|x-y,a_2,\cdots,a_n\|^2+\|(1-p)x-y,a_2,\cdots,a_n\|^2\\ &-p\|x,a_2,\cdots,a_n\|^2-q\|y,a_2,\cdots,a_n\|^2\\ =\ &(p^2-3p+2)\|x,a_2,\cdots,a_n\|^2\\ &+2(p-2)Re(x,y|a_2,\cdots,a_n)+(2-q)\|y,a_2,\cdots,a_n\|^2\\ =\ &(2-p)(1-p)\|x,a_2,\cdots,a_n\|^2\\ &-2(2-p)Re(x,y|a_2,\cdots,a_n)+\frac{2-p}{1-p}\|y,a_2,\cdots,a_n\|^2\end{aligned}$$

$$
\begin{aligned}
&= (2-p)\Big[(1-p)\|x, a_2, \cdots, a_n\|^2 \\
&\quad -2Re(x, y|a_2, \cdots, a_n) + \frac{1}{1-p}\|y, a_2, \cdots, a_n\|^2\Big] \\
&= (2-p)\left\|\left(\sqrt{1-p}\, x - \frac{1}{\sqrt{1-p}}\, y\right), a_2, \cdots, a_n\right\|^2 \\
&\geq 0
\end{aligned} \tag{3.3}
$$

and hence

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|(1-p)x - y, a_2, \cdots, a_n\|^2 \\
&\geq p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2,
\end{aligned}
$$

with the equality if and only if $px+qy, a_2, \cdots, a_n$ are linearly dependent.

(2) Since $p < 1$, we have $q < 1$ and

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|x-(1-q)y, a_2, \cdots, a_n\|^2 \\
&-p\|x, a_2, \cdots, a_n\|^2 - q\|y, a_2, \cdots, a_n\|^2 \\
&= (2-p)\|x, a_2, \cdots, a_n\|^2 \\
&\quad +2(q-2)Re(x, y|a_2, \cdots, a_n) + (q^2-3q+2)\|y, a_2, \cdots, a_n\|^2 \\
&= \frac{2-q}{1-q}\|x, a_2, \cdots, a_n\|^2 \\
&\quad -2(2-q)Re(x, y|a_2, \cdots, a_n) + (2-q)(1-q)\|y, a_2, \cdots, a_n\|^2 \\
&= (2-q)\Big[\frac{1}{1-q}\|x, a_2, \cdots, a_n\|^2 - 2Re(x, y|a_2, \cdots, a_n) \\
&\quad +(1-q)\|y, a_2, \cdots, a_n\|^2\Big] \\
&= (2-q)\left\|\left(\frac{1}{\sqrt{1-q}}\, x - \sqrt{1-q}\, y\right), a_2, \cdots, a_n\right\|^2 \geq 0
\end{aligned} \tag{3.4}
$$

and hence

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \|x-(1-q)y, a_2, \cdots, a_n\|^2 \\
&\geq p\|x, a_2, \cdots, a_n\|^2 + q\|y, a_2, \cdots, a_n\|^2,
\end{aligned}
$$

with equality if and only if $px+qy, a_2, \cdots, a_n$ are linearly dependent. This completes the proof. □

Theorem 3.6. *For any $x, y, a_2, \cdots, a_n \in X$ and $\alpha, \beta \in \mathbb{R}$ be nonzero constants.*

(*A*) *If $\alpha\beta > 0$ with $|\alpha| \geq |\beta| > 0$, then*

$$
\begin{aligned}
&\|x-y, a_2, \cdots, a_n\|^2 + \frac{1}{\alpha^2}\|\beta x + \alpha y, a_2, \cdots, a_n\|^2 \\
&\leq \frac{\alpha+\beta}{\alpha}\|x, a_2, \cdots, a_n\|^2 + \frac{\alpha+\beta}{\beta}\|y, a_2, \cdots, a_n\|^2,
\end{aligned}
$$

with the equality if and only if $\alpha = \beta$ *or* $\beta x + \alpha y, a_2, \cdots, a_n$ *are linearly dependent.*

(B) *Let* $\alpha\beta < 0$ *with* $|\alpha| > 0 > |\beta|$.

(1) *If* $\alpha > 0 > \beta \geq -\alpha$, *then*

$$\begin{aligned}&\|x - y, a_2, \cdots, a_n\|^2 + \frac{1}{\alpha^2}\|\beta x - \alpha y, a_2, \cdots, a_n\|^2\\ &\leq \frac{\alpha - \beta}{\alpha}\|x, a_2, \cdots, a_n\|^2 - \frac{\alpha - \beta}{\beta}\|y, a_2, \cdots, a_n\|^2,\end{aligned}$$

with the equality if and only if $\alpha = -\beta$ *or* $\beta x - \alpha y, a_2, \cdots, a_n$ *are linearly dependent.*

(2) *If* $\alpha > 0 > -\alpha \geq \beta$, *then*

$$\begin{aligned}&\|x - y, a_2, \cdots, a_n\|^2 + \frac{1}{\beta^2}\|\alpha x - \beta y, a_2, \cdots, a_n\|^2\\ &\leq -\frac{\alpha - \beta}{\beta}\|x, a_2, \cdots, a_n\|^2 + \frac{\alpha - \beta}{\alpha}\|y, a_2, \cdots, a_n\|^2,\end{aligned}$$

with the equality if and only if $\alpha = -\beta$ *or* $\alpha x - \beta y, a_2, \cdots, a_n$ *are linearly dependent.*

Proof. (A) If $\alpha \geq \beta > 0$, we write

$$p = \frac{\alpha + \beta}{\alpha}, \quad q = \frac{\alpha + \beta}{\beta}.$$

Then

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 < p \leq 2 \leq q.$$

Hence, by Theorem 3.1, we have

$$\begin{aligned}&\|x - y, a_2, \cdots, a_n\|^2 + \| - \frac{\beta}{\alpha}x - y, a_2, \cdots, a_n\|^2\\ &\leq \frac{\alpha + \beta}{\alpha}\|x, a_2, \cdots, a_n\|^2 + \frac{\alpha + \beta}{\beta}\|y, a_2, \cdots, a_n\|^2\end{aligned}$$

or

$$\begin{aligned}&\|x - y, a_2, \cdots, a_n\|^2 + \frac{1}{\alpha^2}\|\beta x + \alpha y, a_2, \cdots, a_n\|^2\\ &\leq \frac{\alpha + \beta}{\alpha}\|x, a_2, \cdots, a_n\|^2 + \frac{\alpha + \beta}{\beta}\|y, a_2, \cdots, a_n\|^2,\end{aligned}$$

with the equality if and only if

$$\frac{\alpha + \beta}{\alpha} = 2$$

or

$$\left(\frac{\alpha + \beta}{\alpha}\right)x + \left(\frac{\alpha + \beta}{\beta}\right)y, a_2, \cdots, a_n \text{ are linearly dependent,}$$

that is, $\alpha = \beta$ or $\beta x + \alpha y, a_2, \cdots, a_n$ are linearly dependent.

If $0 > \beta \geq \alpha$, then $-\alpha \geq -\beta > 0$ and so, from above,

$$\begin{aligned}&\|x-y, a_2, \cdots, a_n\|^2 + \frac{1}{(-\alpha)^2}\|-\beta x - \alpha y, a_2, \cdots, a_n\|^2\\ &\leq \frac{-\alpha-\beta}{-\alpha}\|x, a_2, \cdots, a_n\|^2 + \frac{-\alpha-\beta}{-\beta}\|y, a_2, \cdots, a_n\|^2\end{aligned}$$

or

$$\begin{aligned}&\|x-y, a_2, \cdots, a_n\|^2 + \frac{1}{\alpha^2}\|\beta x + \alpha y, a_2, \cdots, a_n\|^2\\ &\leq \frac{\alpha+\beta}{\alpha}\|x, a_2, \cdots, a_n\|^2 + \frac{\alpha+\beta}{\beta}\|y, a_2, \cdots, a_n\|^2,\end{aligned}$$

with the equality if and only if $\alpha = \beta$ or $\beta x + \alpha y, a_2, \cdots, a_n$ are linearly dependent.

(B) (1) If $\alpha > 0 > \beta \geq -\alpha$, the assertion follows immediately by applying (A) to $\alpha \geq -\beta > 0$.

(2) If $\alpha > 0 > -\alpha \geq \beta$, the assertion follows immediately by applying (A) to $-\beta \geq \alpha > 0$. This completes the proof. □

4. Applications

Interesting inequalities on operators in the n-inner product space X can easily be derived from the Bohr-type inequalities obtained in Section 3 above. For this, we first observe the following generalization of Adamović's result to X:

Lemma 4.1. *For any $x_i \in X$, $i = 1, 2, \cdots, n$,*

$$\begin{aligned}&\left\|\sum_{i=1}^{n} x_i, a_2, \cdots, a_n\right\|^2 - \left(\sum_{i=1}^{n}\|x_i, a_2, \cdots, a_n\|\right)^2\\ &= \sum_{1\leq i<j\leq n}\left[\|x_i + x_j, a_2, \cdots, a_n\|^2 - (\|x_i, a_2, \cdots, a_n\| + \|x_j, a_2, \cdots, a_n\|)^2\right].\end{aligned}$$

Proof. We will use induction on n. Clearly, the result holds for $n = 2$. Suppose now that the result holds for $n = k$. Then we have

$$\begin{aligned}&\left\|\sum_{i=1}^{k+1} x_i, a_2, \cdots, a_n\right\|^2 - \left(\sum_{i=1}^{k+1}\|x_i, a_2, \cdots, a_n\|\right)^2\\ &= \left\|\sum_{i=1}^{k} x_i + x_{k+1}, a_2, \cdots, a_n\right\|^2 - \left(\sum_{i=1}^{k}\|x_i, a_2, \cdots, a_n\| + \|x_{k+1}, a_2, \cdots, a_n\|\right)^2\\ &= \left\|\sum_{i=1}^{k} x_i, a_2, \cdots, a_n\right\|^2 - \left(\sum_{i=1}^{k}\|x_i, a_2, \cdots, a_n\|\right)^2\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=1}^{k} Re(x_i, x_{k+1}|a_2,\cdots,a_n) - 2\|x_{k+1},a_2,\cdots,a_n\|\sum_{i=1}^{k}\|x_i,a_2,\cdots,a_n\| \\
= &\sum_{1\le i<j\le k}\left[\|x_i+x_j,a_2,\cdots,a_n\|^2 - (\|x_i,a_2,\cdots,a_n\| + \|x_j,a_2,\cdots,a_n\|)^2\right] \\
&+2\sum_{i=1}^{k} Re(x_i, x_{k+1}|a_2,\cdots,a_n) - 2\|x_{k+1},a_2,\cdots,a_n\|\sum_{i=1}^{k}\|x_i,a_2,\cdots,a_n\| \\
= &\sum_{1\le i<j\le k}\left[2Re(x_i,x_j|a_2,\cdots,a_n) - 2\|x_i,a_2,\cdots,a_n\|\|x_j,a_2,\cdots,a_n\|\right] \\
&+2\sum_{i=1}^{k} Re(x_i, x_{k+1}|a_2,\cdots,a_n) - 2\|x_{k+1},a_2,\cdots,a_n\|\sum_{i=1}^{k}\|x_i,a_2,\cdots,a_n\| \\
= &\sum_{1\le i<j\le k+1}\left[2Re(x_i,x_j|a_2,\cdots,a_n) - 2\|x_i,a_2,\cdots,a_n\|\|x_j,a_2,\cdots,a_n\|\right] \\
= &\sum_{1\le i<j\le k+1}\left[\|x_i+x_j,a_2,\cdots,a_n\|^2 - (\|x_i,a_2,\cdots,a_n\| + \|x_j,a_2,\cdots,a_n\|)^2\right].
\end{aligned}
$$

This completes the proof. □

Theorem 4.2. *For any $x_i \in X$, $i = 1, 2, \cdots, n$, and $p_{ij} > 1, q_{ij} \in \mathbb{R}$ with $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $1 \le i < j \le n$, we have*

$$
\begin{aligned}
&\left\|\sum_{i=1}^{n} x_i, a_2,\cdots,a_n\right\|^2 \\
&\le \sum_{k=1}^{n}\left(1 + \sum_{j=k+1}^{n}(p_{ij}-1) + \sum_{j=1}^{k-1}(q_{ij}-1)\right)\|x_k,a_2,\cdots,a_n\|^2,
\end{aligned}
$$

the equality holds if and only if $(p_{ij}x_i - q_{ij}x_j), a_2, \cdots, a_n$ are linearly dependent for all i, j with $1 \le i < j \le n$.

Proof. By Lemma 4.1,

$$
\begin{aligned}
&\left\|\sum_{i=1}^{n} x_i, a_2,\cdots,a_n\right\|^2 - \left(\sum_{i=1}^{n}\|x_i,a_2,\cdots,a_n\|\right)^2 \\
= &\sum_{1\le i<j\le n}\left[\|x_i+x_j,a_2,\cdots,a_n\|^2 - (\|x_i,a_2,\cdots,a_n\| + \|x_j,a_2,\cdots,a_n\|)^2\right].
\end{aligned}
$$

Equivalently, we have

$$\left\| \sum_{i=1}^{n} x_i, a_2, \cdots, a_n \right\|^2 - \sum_{i=1}^{n} \|x_i, a_2, \cdots, a_n\|^2$$
$$= \sum_{1\le i<j\le n} \left[\|x_i + x_j, a_2, \cdots, a_n\|^2 - (\|x_i, a_2, \cdots, a_n\|^2 + \|x_j, a_2, \cdots, a_n\|^2) \right].$$

Applying Corollary 3.4 to $\|x_i + x_j, a_2, \cdots, a_n\|$, we have

$$\left\| \sum_{i=1}^{n} x_i, a_2, \cdots, a_n \right\|^2 - \sum_{i=1}^{n} \|x_i, a_2, \cdots, a_n\|^2$$
$$\le \sum_{1\le i<j\le n} [(p_{ij}-1)\|x_i, a_2, \cdots, a_n\|^2 + (q_{ij}-1)\|x_j, a_2, \cdots, a_n\|^2],$$

with the equality if and only if $(p_{ij}x_i - q_{ij}x_j), a_2, \cdots, a_n$ are linearly dependent for all i, j with $1 \le i < j \le n$, that is,

$$\left\| \sum_{i=1}^{n} x_i, a_2, \cdots, a_n \right\|^2$$
$$\le \sum_{i=1}^{n} \|x_i, a_2, \cdots, a_n\|^2$$
$$+ \sum_{1\le i<j\le n} [(p_{ij}-1)\|x_i, a_2, \cdots, a_n\|^2 + (q_{ij}-1)\|x_j, a_2, \cdots, a_n\|^2]$$
$$\le \sum_{k=1}^{n} \left(1 + \sum_{j=k+1}^{n} (p_{ij}-1) + \sum_{j=1}^{k-1} (q_{ij}-1)\right) \|x_k, a_2, \cdots, a_n\|^2,$$

with the equality if and only if $(p_{ij}x_i - q_{ij}x_j), a_2, \cdots, a_n$ are linearly dependent for all i, j with $1 \le i < j \le n$. This completes the proof. □

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