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OSCILLATION AND NONOSCILLATION OF HIGHER-ORDER DIFFERENCE EQUATIONS WITH NONLINEAR NEUTRAL TERMS†

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ABSTRACT. In this paper, the oscillation and existence of nonoscillatory solutions of odd order difference equations with nonlinear neutral terms are studied respectively. Some new criteria are established. Furthermore, some examples are given to illustrate the advance of our results.

1. INTRODUCTION

Consider the higher-order difference equations of the form

$$\Delta^m(x_n - g(x_{n-\tau})) + f(n, x_{n-\sigma}) = 0, \ \ n \geq n_0, \tag{1.1}$$

$$\Delta^m(x_n - p_n g(x_{\tau_n})) + q_n h(x_{\sigma_n}) = 0, \ \ n \geq n_0, \tag{1.2}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, and for $i \geq 1$, Δ^i is the i^{th} order forward difference operator $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$. The following conditions are always assumed to hold for Eq.(1.1) and (1.2):

(a) $m \geq 1$ is an odd integer;
(b) $g,\ h \in C(\mathbb{R}, \mathbb{R})$, $f \in C(N(n_0) \times \mathbb{R}, \mathbb{R})$;
(c) τ, σ are nonnegative integers, $\{\tau_n\}$ and $\{\sigma_n\}$ are integer sequences, and $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n = \infty$;
(d) $\{p_n\}$ and $\{q_n\}$ are real sequences.

The neutral delay difference equations arise in a number of important applications including problems in population dynamics when maturation and gestation are

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included, in cobweb models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers.

Oscillation theory of neutral difference equations has developed very rapidly in recent years. It has concerned itself largely with the oscillatory and nonoscillatory properties of solutions (see [1-15] and the references cited therein). Agarwal and Wong [2], Agarwal et al. [3], Agarwal and Grace [4], and Zhang and Yang [9] investigate the oscillatory behavior of solutions of nonlinear neutral difference equations of order $m(\geq 1)$ of the following form

$$\Delta^m(x_n + cx_{n-k}) + p_n f(x_{n-r}) = 0 \text{ for } n \geq n_0. \tag{1.3}$$

Liu, Wu and Zhang [15] has studied the oscillation of solutions of even order difference equations with nonlinear neutral term of the form

$$\Delta^{m-1}(a_n \Delta(x_n + \varphi(n, \tau_n))) + q_n f(x_{gn}) = 0 \text{ for } m \text{ is even.} \tag{1.4}$$

Whereas it seems that the odd order difference equations with nonlinear neutral terms received much less attention, the purpose of this paper is to establish some criteria for the oscillatory and nonoscillatory properties of Eq.(1.1) and Eq.(1.2). Some examples are given to illustrate our results.

Let $\gamma = \max\{\tau, \sigma\}$ ($\gamma = \min\limits_{n \geq n_0}\{\tau_n, \sigma_n\}$), and n_0 be a fixed nonnegative integer. By a solution of Eq.(1.1) (Eq.(1.2)) we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0 \geq \gamma$ ($n \geq \gamma$), and satisfies Eq.(1.1) (Eq.(1.2)) for $n \geq n_0$. A solution $\{x_n\}$ of Eq.(1.1) (Eq.(1.2)) is said to be nonoscillatory if all terms x_n are eventually of one sign. Otherwise the solution $\{x_n\}$ is called oscillatory. In this paper, we shall be concerned only with the nontrivial solution of Eq.(1.1) (Eq.(1.2)).

The plan of the paper is as follows. In Section 2, we present preliminary lemmas that are needed to prove our main results, and in Section 3, we obtain some sufficient conditions for the oscillation of Eq.(1.1) and (1.2). Finally, in Section 4, we establish the existence of nonoscillatory solutions of Eq.(1.1) and (1.2). Examples are inserted to illustrate the results.

2. RELATED LEMMAS

To obtain our main results, we need the following lemmas.

Lemma 2.1 ([6]). *Let y_n be a real function defined on $N(n_0) = \{n_0, n_0 + 1, \cdots\}$, if $y_n > 0$ with $\Delta^m y_n \leq 0$, then there exists an integer k, $0 \leq k \leq m$ with $(m + k)$*

odd and $n_1 \geq n_0$ *sufficiently large, such that*

$$\Delta^j y_n > 0 \quad \textit{for all } n \geq n_1,\ j = 0,\ 1,\ \cdots,\ k$$

$$(-1)^{m+j+1}\Delta^j y_n > 0 \ \textit{for all } n \geq n_1,\ j = k+1,\ \cdots,\ m. \tag{2.1}$$

Lemma 2.2. *Let* y_n *be a real function defined on* $N(n_0) = \{n_0,\ n_0+1,\ \cdots\}$, *if* $y_n > 0$ *with* $\Delta^m y_n \leq 0$, *and* $\{y_n\}$ *is bounded, then there exists an integer* $n_1 \geq n_0$ *sufficiently large, such that*

$$(-1)^{m+j+1}\Delta^j y_n > 0 \ \textit{for all } n \geq n_1,\ j = 1,\ \cdots,\ m$$

and

$$\lim_{n\to\infty} \Delta^j y_n = 0 \ \textit{for } j = 1,\ 2,\ \cdots,\ m-1. \tag{2.2}$$

Lemma 2.3. *Assume* $p \in (0,\ 1]$, $q \in (0,\ \infty)$, τ *is a positive integer and* σ *is a nonnegative integer,* m *is odd, and every solution of the equation*

$$\Delta^m(x_n - p x_{n-\tau}) + q x_{n-\sigma} = 0 \tag{2.3}$$

is oscillatory. Then there exists an $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0,\ \varepsilon_0]$, *every solution of the equation*

$$\Delta^m(y_n - (p-\varepsilon) y_{n-\tau}) + (q-\varepsilon) y_{n-\sigma} = 0 \tag{2.4}$$

is oscillatory.

Proof. If not, for any $\varepsilon_0 > 0$ there exists an $\varepsilon \in (0,\ \varepsilon_0]$ such that (2.4) has an eventually positive solution. Then the characteristic equation of (2.4)

$$f(\lambda) = (\lambda - 1)^m[1 - (p-\varepsilon)\lambda^{-\tau}] + (q-\varepsilon)\lambda^{-\sigma} = 0 \tag{2.5}$$

has a real root $\lambda_0 \in (0,\ 1]$. On the other hand, by the assumption, the characteristic equation of (2.3)

$$F(\lambda) = (\lambda - 1)^m(1 - p\lambda^{-\tau}) + q\lambda^{-\sigma} = 0 \tag{2.6}$$

has no root on $(0, 1]$. Since $\lim_{\lambda\to 0} F(\lambda) = +\infty$, $F(1) = q > 0$, $F(\lambda)$ has a positive lower bound on $(0, 1]$, i.e. there exists $l > 0$ such that $F(\lambda) \geq l > 0$ for any $\lambda \in (0,\ 1]$. Set

$$G(\lambda) = (\lambda - 1)^m\Big(1 - \frac{p}{2}\lambda^{-\tau}\Big) + \frac{q}{2}\lambda^{-\sigma} \quad \text{for } \lambda \in (0,\ 1).$$

Since $\lim_{\lambda\to 0} G(\lambda) = +\infty$, there exists a $\lambda_1 \in (0,\, 1)$, such that $G(\lambda) > 0$ for $\lambda \in (0,\, \lambda_1]$. Choose $\varepsilon_0 > 0$ such that $p - \varepsilon_0 \geq \frac{p}{2}$, $q - \varepsilon_0 \geq \frac{q}{2}$, and

$$\varepsilon_0 \cdot \sup_{\lambda\in(\lambda_1,\, 1]} [(1-\lambda)^m \lambda^{-\tau} + \lambda^{-\sigma}] \leq \frac{l}{2}.$$

Thus for $\varepsilon \in (0,\, \varepsilon_0]$, if $\lambda \in (0,\, \lambda_1]$, then

$$\begin{aligned} f(\lambda) &= (\lambda-1)^m[1-(p-\varepsilon)\lambda^{-\tau}] + (q-\varepsilon)\lambda^{-\sigma} \\ &\geq (\lambda-1)^m\Big(1-\frac{p}{2}\lambda^{-\tau}\Big) + \frac{q}{2}\lambda^{-\sigma} = G(\lambda) > 0 \end{aligned}$$

if $\lambda \in (\lambda_1,\, 1]$, then

$$\begin{aligned} f(\lambda) &= (\lambda-1)^m[1-(p-\varepsilon)\lambda^{-\tau}] + (q-\varepsilon)\lambda^{-\sigma} \\ &\geq F(\lambda) - \varepsilon_0 \cdot \sup_{\lambda\in(\lambda_1,\, 1]} [(1-\lambda)^m\lambda^{-\tau} + \lambda^{-\sigma}] \\ &\geq l - \frac{l}{2} = \frac{l}{2} > 0 \end{aligned}$$

That is, there exists an $\varepsilon_0 > 0$ such that (2.5) has no real roots on $\lambda \in (0,\, 1]$ for every $\varepsilon \in (0,\, \varepsilon_0]$, we reach a contradiction. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Suppose that m is odd, $f \in C(N(n_0) \times \mathbb{R},\, \mathbb{R})$ is nondecreasing, $xf(n,x) \geq 0$ for $x \in \mathbb{R}$, and σ is a nonnegative integer. Then every solution of equation*

$$\Delta^m y_n + f(n, x_{n-\sigma}) = 0 \tag{2.7}$$

is oscillatory if and only if

$$\Delta^m y_n + f(n, x_{n-\sigma}) \leq 0 \tag{2.8}$$

has no eventually positive solutions, and

$$\Delta^m y_n + f(n, x_{n-\sigma}) \geq 0 \tag{2.9}$$

has no eventually negative solutions.

Proof. The sufficiency is obvious. To prove the necessary, without loss of generality, we assume that (2.8) has an eventually positive solution y_n. Thus $\Delta^m y_n \leq -f(n, y_{n-\sigma}) < 0$. According to Lemma 2.1, there exists an even integer k such that $0 \leq k \leq m-1$, and (2.1) holds.

(I). If $k = 0$, choose N sufficiently large, for any $n \geq N$, summing (2.8) m times from n to ∞, we have

$$y_n \geq \frac{1}{(m-1)!} \sum_{s=n}^{\infty} (s-n+m-1)^{(m-1)} f(s, y_{s-\sigma}). \tag{2.10}$$

Let BC be a partially ordered Banach Space of all real sequences defined on $N(n_0)$. Set

$$\Omega = \Big\{ x = \{x_n\} \in BC : 0 \leq x_n \leq y_n,\ n \in N(n_0) \Big\}.$$

Now we define operator Γ on Ω as follows

$$\Gamma x_n = \begin{cases} \frac{1}{(m-1)!} \sum\limits_{s=n}^{\infty} (s-n+m-1)^{(m-1)} f(s, x_{s-\sigma}), & n \geq N \\ \Gamma x_N, & n_0 \leq n \leq N. \end{cases}$$

It is easy to see from (2.10) that $\Gamma\Omega \subset \Omega$. Since f is nondecreasing, Γ is increasing in x, i.e. if $x_n \leq y_n$, then $\Gamma x_n \leq \Gamma y_n$. Therefore, by Knaster's fixed point theorem, there exists $x \in \Omega$, such that

$$x_n = \frac{1}{(m-1)!} \sum_{s=n}^{\infty} (s-n+m-1)^{(m-1)} f(s, x_{s-\sigma}) \text{ for } n \geq N.$$

We see that x_n is an eventually positive solution of (2.7), this is a contradiction.
(II). If $2 \leq k \leq m-1$, summing (2.8) $m-k$ times from n to ∞, we have

$$\Delta^k y_n \geq \frac{1}{(m-k-1)!} \sum_{s=n}^{\infty} (s-n+m-k-1)^{(m-k-1)} f(s, y_{s-\sigma}). \tag{2.11}$$

Summing (2.11) k times from N to $n-1$, we have

$$\begin{aligned} y_n \geq y_N &+ \frac{1}{(k-1)!(m-k-1)!} \sum_{s=N}^{n-1} (n-s+k-2)^{(k-1)} \\ &\times \sum_{u=s}^{\infty} (u-s+m-k-1)^{(m-k-1)} f(u, y_{u-\sigma}), \end{aligned} \tag{2.12}$$

where $N \geq n_0$ sufficiently large such that $y_N > 0$. It is similar to the proof in (I), we can obtain that the corresponding equation of inequality (2.12)

$$\begin{aligned} x_n = y_N &+ \frac{1}{(k-1)!(m-k-1)!} \sum_{s=N}^{n-1} (n-s+k-2)^{(k-1)} \\ &\times \sum_{u=s}^{\infty} (u-s+m-k-1)^{(m-k-1)} f(u, x_{u-\sigma}) \end{aligned}$$

has an eventually positive solution x_n. Clearly, x_n is an eventually positive solution of (2.7), which is a contradiction. This completes the proof of Lemma 2.4. □

3. Oscillation

Theorem 3.1. *Assume that*

(i) *g is nondecreasing, $xg(x) \geq 0$ for $x \in \mathbb{R}$, and there exists an $\alpha \in (0, 1]$ such that* $\lim_{|x|\to\infty} \frac{|g(x)|}{|x|^\alpha} = \begin{cases} M \in [0, \infty) & \text{for } \alpha \in (0, 1) \\ c \in (0, 1) & \text{for } \alpha = 1; \end{cases}$

(ii) *f is nondecreasing in x, $xf(n, x) \geq 0$ for $(n, x) \in N(n_0) \times R$, and for any nonzero constant β*

$$\sum_{s=n_0}^{\infty} s^{(m-1)} f(s, \beta) = \infty \cdot sign\beta;$$

(iii) *there exists a positive integer M, such that every solution of equation*

$$\Delta^m y_n + f(n, y_{n-\sigma} + g(y_{n-\sigma-\tau} + g(y_{n-\sigma-2\tau} + \cdots + g(y_{n-\sigma-M\tau}) \ldots))) = 0 \tag{3.1}$$

is oscillatory.

Then every solution of Eq.(1.1) is oscillatory.

Proof. We prove it for the case that $\alpha \in (0, 1)$ in condition (i). The case that $\alpha = 1$ can be proved similarly. Assume the contrary, let x_n be an eventually positive solution of Eq.(1.1). Set

$$y_n = x_n - g(x_{n-\tau}). \tag{3.2}$$

Then eventually

$$\Delta^m y_n = -f(n, x_{n-\sigma}) \leq 0. \tag{3.3}$$

Therefore, y_n is constant sign eventually. Now let us consider the following two cases: $y_n < 0$ and $y_n > 0$ eventually.

(I). Assume $y_n < 0$ eventually. Since m is odd, it follows from (3.3) that $\Delta y_n < 0$. In fact, if $\Delta^i y_n > 0$ and $\Delta^{i+1} y_n > 0$ for some $i = 1, 2, \cdots, m-2$, then $y_n > 0$ eventually, that is a contradiction. If $\Delta^i y_n < 0$ and $\Delta^{i+1} y_n < 0$ for some $i = 2, 3, \cdots, m-1$, then $\Delta y_n < 0$ eventually. If $\Delta^i y_n \cdot \Delta^{i+1} y_n < 0$ for $i = 1, 2, \cdots, m-1$, since m is odd, $\Delta y_n < 0$. Then either

$$\lim_{n\to\infty} y_n = -\infty \tag{3.4}$$

or

$$\lim_{n\to\infty} y_n = -r \in (-\infty, 0). \tag{3.5}$$

If (3.4) holds, from (3.2) we have $\lim_{n\to\infty} g(x_{n-\tau}) = \infty$, and so $x_n \to \infty$ as $n \to \infty$ from condition (i). This implies that there exists an infinite sequence $\{n_k\}$ such that $\lim_{k\to\infty} x_{n_k} = \infty$, where $x_{n_k} = \max\{x_n : n_0 \le n \le n_k\}$. Hence from condition (i) and (3.2) we have

$$\begin{aligned} y_{n_k} &= x_{n_k} - g(x_{n_k-\tau}) = x_{n_k}^{\alpha}\left[x_{n_k}^{1-\alpha} - \frac{g(x_{n_k-\tau})}{x_{n_k}^{\alpha}}\right] \\ &\ge x_{n_k}^{\alpha}\left[x_{n_k}^{1-\alpha} - \frac{g(x_{n_k})}{x_{n_k}^{\alpha}}\right] \to \infty \text{ as } k \to \infty, \end{aligned}$$

which contradicts (3.4) and so (3.5) holds, i.e. $\{y_n\}$ is bounded. Thus we have

$$(3.6) \qquad (-1)^i \Delta^i y_n > 0 \quad \text{for } i = 1, \cdots, m.$$

From (3.5) there exists a $n_1 \ge n_0$ sufficiently large, such that $y_n = x_n - g(x_{n-\tau}) < -\frac{r}{2}$ for $n \ge n_1$. So $g(x_{n-\tau}) \ge \frac{r}{2} > 0$ for $n \ge n_1$. Since g is nondecreasing in x, there exists a $\beta > 0$, such that $x_n \ge \beta$ for $n \ge n_1$. Since f is nondecreasing in x, we have

$$(3.7) \qquad \Delta^m y_n = -f(n, x_{n-\sigma}) \le -f(n, \beta) \quad \text{for } n \ge n_1 + \sigma.$$

Multiplying (3.7) by $n^{(m-1)}$, and summing it from $n_1 + \sigma$ to $n - 1$, we obtain that

$$(3.8) \qquad F(n) - F(n_1 + \sigma) \le - \sum_{s=n_1+\sigma}^{n-1} s^{(m-1)} f(s, \beta)$$

where

$$F(n) = \sum_{i=0}^{m-1} (-1)^i (\Delta^i n^{(m-1)}) \Delta^{m-i-1} y_{n+i}.$$

In view of (3.6), we have $F(n) > 0$, Thus

$$-F(n_1 + \sigma) \le - \sum_{s=n_1+\sigma}^{n-1} s^{(m-1)} f(s, \beta).$$

By condition (ii), we have

$$-F(n_1 + \sigma) \to -\infty \text{ as } n \to \infty.$$

This is a contradiction.

(II). Assume $y_n > 0$ eventually. Then from (3.2) we have

$$\begin{aligned} x_n &= y_n + g(x_{n-\tau}) = y_n + g(y_{n-\tau} + g(x_{n-2\tau})) \\ &\ge y_n + g(y_{n-\tau} + g(y_{n-2\tau} + \cdots + g(y_{n-M\tau}) \cdots)). \end{aligned}$$

Substituting this into Eq.(1.1), we find that

$$\Delta^m y_n + f(n, y_{n-\sigma} + g(y_{n-\sigma-\tau} + g(y_{n-\sigma-2\tau} + \cdots + g(y_{n-\sigma-M\tau})\ldots))) \le 0,$$

which contradicts condition (iii) by Lemma 2.4. By the same method we can prove Eq.(1.1) has no eventually negative solutions. □

Example 3.1. Consider

$$\Delta^m(x_n - \frac{1}{2}x_{n-3}^{\frac{1}{3}}) + 3 \cdot 2^{m-1}e^{(-1)^{n-1}}x_{n-2}e^{x_{n-2}} = 0 \tag{3.9}$$

Here, $m \ge 1$ is odd, $\tau = 3$, $\sigma = 2$, and $g(x) = \frac{1}{2}x^{\frac{1}{3}}$, $f(n, x) = 3 \cdot 2^{m-1}e^{(-1)^{n-1}}xe^x$. Choose $\alpha = \frac{1}{3}$, we can show that the conditions of Theorem 3.1 are all satisfied. Therefore, every solution of *Eq.*(3.9) is oscillatory. In particular, $x_n = (-1)^n$ is an oscillatory solution of the equation.

Theorem 3.2. *Assume that*

(i) $\tau_n = n - \tau,\ \sigma_n = n - \sigma$;

(ii) $0 < \lim_{n\to\infty} p_n = p < 1,\ \liminf_{n\to\infty} q_n = q > 0,\ \lim_{n\to\infty} \frac{q_n}{q_{n-\tau}} = l > 0$, *and* $p\,l < 1$;

(iii) $xg(x) > 0$ *for* $x \ne 0$, $|g(x)| \le |x|$ *as* $|x|$ *sufficiently large, and*

$$\lim_{x\to 0} \frac{g(x)}{x} = 1;$$

(iv) $xh(x) > 0$ *for* $x \ne 0$, $|h(x)| \ge h > 0$ *as* $|x|$ *sufficiently large, and*

$$\lim_{x\to 0} \frac{h(x)}{x} = 1;$$

(v) *every solution of the linear difference equation*

$$\Delta^m(y_n - p\,l y_{n-\tau}) + q y_{n-\sigma} = 0 \tag{3.10}$$

is oscillatory.

Then every solution of Eq.(1.2) is oscillatory.

Proof. Assume the contrary, let x_n be an eventually positive solution of Eq.(1.2). Set

$$y_n = x_n - p_n g(x_{n-\tau}). \tag{3.11}$$

Then eventually $\Delta^m y_n = -q_n h(x_{n-\sigma}) \le 0$, therefore $\lim_{n\to\infty} \Delta^{m-1} y_n = l < \infty$. If $l = -\infty$, then $\lim_{n\to\infty} y_n = -\infty$, thus from (3.11), $\{x_n\}$ is unbounded, i.e. $\limsup_{n\to\infty} x_n = \infty$, and there exists an infinite sequence $\{n_k\}$ such that $\lim_{k\to\infty} x_{n_k} = \infty$, where $x_{n_k} = \max\{x_n : n_0 \le n \le n_k\}$. Hence from (3.11) and conditions (ii) (iii), we have

$$\begin{aligned} y_{n_k} &= x_{n_k} - p_{n_k} g(x_{n_k-\tau}) = x_{n_k} - p_{n_k} \cdot \frac{g(x_{n_k-\tau})}{x_{n_k-\tau}} x_{n_k-\tau} \\ &\ge x_{n_k}[1-(p+\varepsilon_0)] \to \infty \text{ as } k \to \infty, \end{aligned}$$

where $\varepsilon_0 > 0$ and $0 < p + \varepsilon_0 < 1$, which contradicts $l = -\infty$. Therefore, $l \in \mathbb{R}$. Then summing Eq.(1.2) from n_1 to ∞ we find that

$$\sum_{s=n_1}^{\infty} q_s h(x_{s-\sigma}) < \infty, \tag{3.12}$$

which implies that $\liminf_{n\to\infty} x_n = 0$. In fact, if not, then $q_n h(x_{n-\sigma})$ has a positive lower bound, which contradicts (3.12). Thus there exists an infinite sequence $\{n_k\}$ such that $\lim_{k\to\infty} x_{n_k} = 0$, then $\lim_{k\to\infty} y_{n_k} \le 0$. On the other hand, from (3.11) and condition (iii), we have

$$y_{n_k+\tau} \ge -p_{n_k+\tau} g(x_{n_k}) = -p_{n_k+\tau} \frac{g(x_{n_k})}{x_{n_k}} x_{n_k} \to 0 \text{ as } k \to \infty.$$

Since y_n is monotonic, we have $\lim_{n\to\infty} y_n = 0$. Then from Lemma 2.2, it is easy to see that

$$\lim_{n\to\infty} \Delta^i y_n = 0 \text{ for } i = 0, 1, \cdots, m-1, \tag{3.13}$$

and $\lim_{n\to\infty} x_n = 0$. We rewrite Eq.(1.2) in the form

$$\Delta^m (x_n - P_n x_{n-\tau}) + Q_n x_{n-\sigma} = 0, \tag{3.14}$$

where $P_n = p_n \frac{g(x_{n-\tau})}{x_{n-\tau}}$, $Q_n = q_n \frac{h(x_{n-\sigma})}{x_{n-\sigma}}$. Hence $y_n = x_n - P_n x_{n-\tau}$, and (3.14) becomes

$$\Delta^m y_n - P_{n-\sigma} \frac{Q_n}{Q_{n-\tau}} \Delta^m y_{n-\tau} + Q_n y_{n-\sigma} = 0. \tag{3.15}$$

From condition (iii) (iv), for any $\varepsilon > 0$, $n_2 \ge n_1$ sufficiently large,

$$Q_n \ge q - \varepsilon \text{ and } P_{n-\sigma} \cdot \frac{Q_n}{Q_{n-\tau}} \ge p\,l - \varepsilon \text{ for } n \ge n_2.$$

Then from (3.15) we have

$$\Delta^m y_n - (p\,l - \varepsilon)\Delta^m y_{n-\tau} + (q-\varepsilon) y_{n-\sigma} \le 0. \tag{3.16}$$

Summing (3.16) m times from n to ∞, use (3.13), we have

$$y_n \ge (p\,l-\varepsilon) y_{n-\tau} + \frac{q-\varepsilon}{(m-1)!} \sum_{s=n}^{\infty} (s-n+m-1)^{(m-1)} y_{s-\sigma} \text{ for } n \ge n_2.$$

So the equation

$$u_n = (p\,l - \varepsilon)u_{n-\tau} + \frac{q-\varepsilon}{(m-1)!}\sum_{s=n}^{\infty}(s-n+m-1)^{(m-1)}u_{s-\sigma}$$

has an eventually positive solution u_n. Hence the equation

$$\Delta^m(u_n - (p\,l-\varepsilon)u_{n-\tau}) + (q-\varepsilon)u_{n-\sigma} = 0$$

has an eventually positive solution. This contradicts (v) by Lemma 2.3. □

Example 3.2. Consider

$$\Delta^m\Big(x_n - \frac{1}{2}x_{n-2}e^{-x_{n-2}}\Big) + q_n x_{n-2}e^{x_{n-2}} = 0 \tag{3.17}$$

Here, $m \geq 1$ is odd, $p_n = \frac{1}{2}$, $\tau = 2$, $\sigma = 2$, and $g(x) = xe^{-x}$, $h(x) = xe^{x}$, $q_n = 2^m e^{(-1)^{n+1}}$. Clearly, $\liminf_{n\to\infty} q_n = \frac{2^m}{e} > 0$, and $\lim_{n\to\infty}\frac{q_n}{q_{n-\tau}} = 1$, We can show that the conditions of Theorem 3.2 are all satisfied. Therefore, every solution of Eq.(3.17) is oscillatory. For example, $x_n = (-1)^n$ is an oscillatory solution of the equation.

4. NONOSCILLATION

Theorem 4.1. *Assume that*

(i) *g is nondecreasing, $xg(x) \geq 0$ for $x \in R$, and there exists a $d > 0$ such that $g(d) < d$;*

(ii) *f is nondecreasing in x, $xf(n, x) \geq 0$ for $(n, x) \in N(n_0) \times R$, and*

$$\sum_{n=n_0}^{\infty} n^{(m-1)}|f(n, c)| < \infty \text{ for constant } c \neq 0. \tag{4.1}$$

Then Eq.(1.1) has an eventually positive solution.

Proof. Choose $\beta > 0$ such that $\beta + g(d) < d$, then by (4.1) there exists a $N \geq n_0$ such that

$$\frac{1}{(m-1)!}\sum_{s=n}^{\infty} s^{(m-1)}f(s, d) < d - g(d) - \beta \text{ for } n \geq N \tag{4.2}$$

Let BC be a partially ordered Banach Space of all bounded sequences defined on $N(n_0)$. Set

$$\Omega = \Big\{x = \{x_n\} \in BC : \beta \leq x_n \leq d,\ n \in N(n_0)\Big\}.$$

Now we define operator Γ on Ω as follows

$$\Gamma x_n = \begin{cases} \beta + g(x_{n-\tau}) + \frac{1}{(m-1)!}\sum_{s=n}^{\infty}(s-n+m-1)^{(m-1)}f(s,\, x_{s-\sigma}) & n \geq N \\ \Gamma x_N & n_0 \leq n \leq N. \end{cases}$$

From condition (i) and (4.2), it is easy to see that $\beta \leq \Gamma x_n \leq \beta + g(d) + (d - g(d) - \beta) = d$, thus $\Gamma\Omega \subset \Omega$. Conditions (i) and (ii) imply that Γ is increasing in x, i.e. if $x_n \leq y_n$, then $\Gamma x_n \leq \Gamma y_n$. Therefore, by Knaster's fixed point theorem, there exists $x \in \Omega$, such that

$$x_n = \beta + g(x_{n-\tau}) + \frac{1}{(m-1)!}\sum_{s=n}^{\infty}(s-n+m-1)^{(m-1)}f(s,\, x_{s-\sigma}) \text{ for } n \geq N.$$

We see that x_n is an eventually positive solution of Eq.(1.1). This completes the proof of Theorem 4.1. □

Example 4.1. Consider

$$\Delta^3(x_n - x_{n-2}^3) + f(n, x_{n-1}) = 0 \tag{4.3}$$

Here, $m = 3$, $\tau = 2$, $\sigma = 1$, and $g(x) = x^3$, $f(n,x) = [e^{-\frac{8}{3}n-3}(1 - 3e^3 + 3e^6 - e^9) - e^{-\frac{2}{3}n-3}(1-3e+3e^2-e^3)]x^{\frac{1}{3}}$. It can be easily checked that the conditions of Theorem 4.1 are all satisfied. Therefore, *Eq.*(4.3) has an eventually positive solution. In fact, $x_n = e^{-n}$ is such a solution.

For Eq.(1.1), combining Theorem 3.1 and 4.1, we obtain the following corollary.

Corollary 4.1. *Let m=1, assume that*

(i) *g is nondecreasing, $xg(x) \geq 0$ for $x \in \mathbb{R}$, and* $\limsup_{|x|\to\infty} \frac{|g(x)|}{|x|} = c \in (0,\, 1)$;

(ii) *f is nondecreasing in x, $xf(n,\, x) > 0$ for $x \neq 0$, and there exists $\beta \in (0,\, 1)$ such that $|f(n,\, x)|/|x|^\beta$ is nonincreasing in $|x|$.*

Then every solution of Eq.(1.1) is oscillatory if and only if

$$\sum_{s=n_0}^{\infty} |f(s,\, c)| = \infty \;\; \textit{for constant } c \neq 0. \tag{4.4}$$

Proof. The necessity follows from Theorem 4.1, and the sufficiency follows from Theorem 3.1. □

Example 4.2. Consider

$$\Delta\Big(x_n - \frac{1}{2}x_{n-3} - x_{n-3}^{\frac{1}{3}}\Big) + (3e^{\frac{2}{3}} + 2)x_{n-2}^{\frac{1}{3}} = 0 \tag{4.5}$$

Here, $m = 1$, $\tau = 3$, $\sigma = 2$, and $g(x) = \frac{1}{2}x + x^{\frac{1}{3}}$, $f(n,x) = (3e^{\frac{2}{3}} + 2)x^{\frac{1}{3}}$. Choose $\beta = \frac{1}{2}$, we can show that the conditions of Corollary 4.1 are all satisfied. Therefore, every solution of Eq.(4.5) is oscillatory. In particular, $x_n = (-1)^n e$ is an oscillatory solution of the equation.

Theorem 4.2. *Assume that*

(i) $xg(x) > 0$, $xh(x) > 0$ *for* $x \neq 0$, *and* $|g(x) - g(y)| \leq |x - y|$ *for* $x, y \in (0, 1]$, h *is nondecreasing;*

(ii) $0 \leq n - \tau_n \leq M$, *where* M *is a positive constant;*

(iii) *there exists* λ, $c \in (0, 1)$, *such that* $p_n\lambda^{-(n-\tau_n)} \leq c < 1$ *and*

$$p_n\lambda^{-(n-\tau_n)} + \frac{\lambda^{-n}}{(m-1)!}\sum_{s=n}^{\infty}(s-n+m-1)^{(m-1)}q_s h(\lambda^{\sigma_s}) \leq 1 \ for\, n \geq n_0.$$

Then Eq.(1.2) has an eventually positive solution x_n *which tends to zero exponentially as* $n \to \infty$.

Proof. Denote $N = \min\left\{\inf_{n\geq n_0} \tau_n, \inf_{n\geq n_0} \sigma_n\right\}$, $N_1 = N + M$.
Let BC be the Banach Space of all bounded sequences defined on $n \geq N$, with the norm $||x|| = \sup_{n\geq N} |x_n\lambda^n|$. Set

$$\Omega = \left\{x = \{x_n\} \in BC : \ 0 \leq x_n \leq 1, \, n \geq N\right\}.$$

Clearly, Ω is a nonempty, bounded, closed and convex subset of BC.
Now we define operators Γ_1 and Γ_2 on Ω as follows

$$\Gamma_1 x_n = \begin{cases} p_n\lambda^{-n}g(x_{\tau_n}\lambda^{\tau_n}), & n \geq N_1 \\ \frac{n}{N_1}\Gamma_1 x_{N_1} + (1 - \frac{n}{N_1}), & N \leq n \leq N_1 \end{cases}$$

$$\Gamma_2 y_n = \begin{cases} \frac{\lambda^{-n}}{(m-1)!}\sum\limits_{s=n}^{\infty}(s-n+m-1)^{(m-1)}q_s h(y_{\sigma_s}\lambda^{\sigma_s}), & n \geq N_1 \\ \frac{n}{N_1}\Gamma_2 y_{N_1}, & N \leq n \leq N_1 \, . \end{cases} \tag{4.6}$$

In view of conditions (i) and (iii), For any $x, y \in \Omega$, we have $0 \leq \Gamma_1 x_n + \Gamma_2 y_n \leq 1$, thus $\Gamma_1 x + \Gamma_2 y \in \Omega$, and from conditions (i), (ii) and (iii) we have

$$\begin{aligned} |(\Gamma_1 x_n - \Gamma_1 y_n)\lambda^n| &= |p_n| \cdot |g(x_{\tau_n}\lambda^{\tau_n}) - g(y_{\tau_n}\lambda^{\tau_n})| \\ &\leq p_n|x_{\tau_n}\lambda^{\tau_n} - y_{\tau_n}\lambda^{\tau_n}| \\ &= p_n\lambda^{\tau_n - n}\lambda^n|x_{\tau_n} - y_{\tau_n}| \\ &\leq p_n\lambda^{\tau_n - n}\lambda^{\tau_n}|x_{\tau_n} - y_{\tau_n}| \\ &\leq c \cdot |x_{\tau_n}\lambda^{\tau_n} - y_{\tau_n}\lambda^{\tau_n}| \end{aligned}$$

Hence $\|\Gamma_1 x - \Gamma_2 y\| \le c\|x - y\|$, thus Γ_1 is a contraction on Ω. From (iii) we have

$$\frac{1}{(m-1)!}\sum_{s=n}^{\infty}(s-n+m-1)^{(m-1)}q_s h(y_{\sigma_s}\lambda^{\sigma_s}) < \lambda^n, \tag{4.7}$$

thus for $\lambda \in (0, 1)$, the series in (4.7) is uniformly convergent in $y \in \Omega$. Since h is continuous in y, Γ_2 is continuous in y. Condition (iii) and (4.6) imply that the family of $\Gamma_2\Omega$ is uniformly bounded. We now shall show that $\Gamma_2\Omega$ is uniformly Cauchy. For any $y \in \Omega$, from (4.6) and (4.7), we have

$$\begin{aligned}|\Gamma_2 y_{n_1}\lambda^{n_1} - \Gamma_2 y_{n_2}\lambda^{n_2}| &= \Bigg|\frac{1}{(m-1)!}\sum_{s=n_1}^{\infty}(s-n_1+m-1)^{(m-1)}q_s h(y_{\sigma_s}\lambda^{\sigma_s}) \\ &\quad - \frac{1}{(m-1)!}\sum_{s=n_2}^{\infty}(s-n_2+m-1)^{(m-1)}q_s h(y_{\sigma_s}\lambda^{\sigma_s})\Bigg| \\ &< \lambda^{n_1} + \lambda^{n_2} \ \ for \ n_1, n_2 \ge N_1.\end{aligned}$$

Since the above series is uniform convergent, for any $\varepsilon > 0$, there exists a $N_2 \ge N_1$, for any $y \in \Omega$, we have

$$|\Gamma_2 y_{n_1}\lambda^{n_1} - \Gamma_2 y_{n_2}\lambda^{n_2}| < \varepsilon \text{ for } n_1, n_2 \ge N_2$$

Thus $\Gamma_2\Omega$ is uniformly Cauchy. Therefore, by Discrete Krasnostlskii's fixed point theorem, there exists $y \in \Omega$, such that $\Gamma_1 y + \Gamma_2 y = y$. Clearly, $0 < y_n \le 1$ for $n \ge N_2$. Let $x_n = y_n\lambda^n$, then x_n is an eventually positive solution of Eq.(1.2). Clearly, x_n tends to zero exponentially as $n \to \infty$. The proof is complete. □

Example 4.3. Consider

$$\Delta^3(x_n - 2^{-n}x^3_{n-2}) + q_n x^{\frac{1}{3}}_{2n+3} = 0 \text{ for } n \ge 4. \tag{4.8}$$

Here, $m = 3$, $p_n = 2^{-n}$, $\tau_n = n - 2$, $\sigma_n = 2n + 3$, and $g(x) = x^3$, $h(x) = x^{\frac{1}{3}}$, $q_n = 2^{-\frac{n}{3}-2} - 4335 \cdot 2^{-\frac{10n}{3}-5}$. Choose $\lambda = 2^{-\frac{1}{2}}$, $c = \frac{1}{8}$. It can be easily checked that the conditions of Theorem 4.2 are all satisfied. Therefore, Eq.(4.8) has an eventually positive solution which tends to zero exponentially as $n \to \infty$. In fact, $x_n = 2^{-n}$ is such a solution.

References

1. R. P. Agarwal: *Difference equations and inequalities.* Second Edition, Marcel Dekker, New York, (2000).
2. R. P. Agarwal and P. J. Y. Wong: *Advanced topics in difference equations.* Kluwer Academic, Dordrecht, (1997).

3. R. P. Agarwal, E. Thandapani & P. J. Y. Wong: Oscillations of higher-order neutral difference equations. *Appl. Math. Lett.* **10**(1) (1997), 71-78.
4. R. P. Agarwal & S. R. Grace: The oscillation of higher-order nonlinear difference equations of neutral type. *Appl. Math. Lett.* **12**(8) (1999), 77-83.
5. L. H. Erbe, Q. Kong & B. G. Zhang: *Oscillatory theory for functional differential equation.* Dekker, New York,1995.
6. R. P. Agarwal, S. R. Grace & D. ORegan: *Oscillation theory for difference and functional differential equations.* Kluwer Academic, (2000).
7. W. T. Li & S. S. Cheng: Asymptotic trichotomy for positive solutions of a class of odd order nonlinear neutral difference equations. *Com. Math. Appl.* **35** (1998), 101-108.
8. Yong Zhou: Existence of nonoscillatory solutions of higher-order neutral difference equations with general coefficients. *Appl. Math. Lett.* **15** (2002), 785-791.
9. B. G. Zhang & B. Yang: *Oscillation in higher order nonlinear difference equation. Chinese Ann. Math.* **20** (1999), 71-80.
10. Yong Zhou & Y. Q. Huang: Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations. *J. Math. Anal. Appl.* **280** (2003), 63-76.
11. Yong Zhou & B. G. Zhang: Existence of nonoscillatory solutions of higher-Order neutral delay difference equations with variable coefficients. *Com. Math. Appl.* **45** (2003), 991-1000.
12. N. Parhi & A. K. Tripathy: Oscillation of a class of nonlinear neutral difference equations of higher order. *J. Math. Anal. Appl.* **284** (2003), 756-774.
13. Y. Bolat & Ö. Akin: Oscillatory behavior of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients. *Appl. Math. Lett.* **17** (2004), 1073-1078.
14. Z. G. Zhang, J. F. Chen & C. S. Zhang: Oscillation of solutions for second-order nonlinear difference equations with nonlinear neutral term. *Com. Math. Appl.* **12** (2001), 1487-1494.
15. Z. S. Liu, S. H. Wu & Z. G. Zhang: Oscillation of solutions for even order difference equations with nonlinear neutral term. *Indian J. Pure. Appl. Math.* **34** (2003), 1585-1598.

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