J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 14, Number 2 (May 2007), Pages 71–84

ON C-STIELTJES INTEGRAL OF BANACH-VALUED FUNCTIONS

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ABSTRACT. In this paper, we define the C-Stieltjes integral of the functions mapping an interval [a,b] into a Banach space X with respect to g on [a,b], and the C-Stieltjes representable operators for the vector-valued functions which are the generalizations of the Henstock-Stieltjes representable operators. Some properties of the C-Stieltjes operators and the convergence theorems of the C-Stieltjes integral are given.

1. INTRODUCTION

In 1996 [6] B. Bongiorno introduced a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function. B. Bongiorno and L. Di Piazza [6, 7] discussed some properties of the C-integral of real-valued functions. In [3] ap-Henstock-Stieltjes integral in Banach space has been given, and the dominated convergence theorem also has been proved. J. Han Yoon, J. Sul Lim and G. Sik Eun defined the Henstock-Stieltjes integral and its representable and nearly representable operators for vector-valued function in [2].

In this paper, we define the C-Stieltjes integral and the C-Stieltjes representable operators for Banach-valued functions. The basic properties of C-Stieltjies integral will be discussed. Finally, we prove two convergence theorems of the C-Stieltjes integral.

2. DEFINITIONS AND BASIC PROPERTIES

Throughout this paper $[a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. A partition D is a finite collection of interval-point pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^n$, where $\{[u_i, v_i]\}_{i=1}^n$ are non-overlapping

Received by the editors December 14, 2006 and, in revised form April 20, 2007.
2000 *Mathematics Subject Classification*. 26A39, 28B05, 46G10.
Key words and phrases. C-integral, C-Stieltjes integral, representable operators.

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subintervals of $[a,b]$. $f:[a,b]\to X$, $\delta(\xi)$ is a positive function on $[a,b]$, i.e., $\delta(\xi):[a,b]\to\mathrm{R}^+$. We say that $D=\{[u_i,v_i]\}_{i=1}^n$ is

(1) a partial partition of $[a,b]$ if $\bigcup_{i=1}^n[u_i,v_i]\subset[a,b]$;

(2) a partition of $[a,b]$ if $\bigcup_{i=1}^n[u_i,v_i]=[a,b]$;

(3) a δ-fine McShane partition of $[a,b]$ if $[u_i,v_i]\subset B(\xi_i,\delta(\xi))=(\xi_i-\delta(\xi),\xi_i+\delta(\xi))$ and $\xi_i\in[a,b]$ for all $i=1,2,...,n$;

(4) a δ-fine C-partition of $[a,b]$ if for the given $\varepsilon>0$, it is a δ-fine McShane partition of $[a,b]$ and satisfying the condition

$$\sum_{i=1}^n dist(\xi_i,[u_i,v_i])<\frac{1}{\varepsilon},$$

here $dist(\xi_i,[u_i,v_i])=\inf\{|t_i-\xi_i|:t_i\in[u_i,v_i]\}$.

Given an δ-fine C-partition $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ we write

$$S(f,D)=\sum_{i=1}^n f(\xi_i)(v_i-u_i)$$

for the integral sums over D, whenever $f:[a,b]\to X$.

Definition 1. A function $f:[a,b]\to X$ is *C-integrable* if there exists a vector $A\in X$ such that for every $\varepsilon>0$ there is a positive function $\delta(\xi):[a,b]\to R^+$ such that

$$\|S(f,D)-A\|<\varepsilon$$

for each δ-fine C-partition $D=\{[u_i,v_i],\xi_i\}_{i=1}^n$ of $[a,b]$. A is called the*C-integral* of f on $[a,b]$ and we write $A=\int_a^b f$ or $A=(\mathrm{C})\int_a^b f$.

The function f is *C-integrable on the set* $E\subset[a,b]$ if the function $f\chi_E$ is C-integrable on $[a,b]$. We write $\int_E f=\int_a^b f\chi_E$.

Definition 2. Let $g:[a,b]\to R$ be an increasing function. A function $f:[a,b]\to X$ is *C-Stieltjes integrable* with respect to g on $[a,b]$ if there exists a vector $A\in X$ such that for every $\varepsilon>0$ there is a positive function $\delta(\xi):[a,b]\to R^+$ such that

$$\|S(f,g,D)-A\|<\varepsilon$$

for each δ-fine C-partition $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[a,b]$, whenever

$$S(f,g,D)=\sum_{i=1}^n f(\xi_i)[g(v_i)-g(u_i)]$$

for the integral sums over D. A is called the *C-Stieltjes integral* of f with respect to g on $[a,b]$, and we write $A = \int_a^b f dg$.

We can easily get the following basic properties of C-Stieltjes integral.

Theorem 3. *Let $g : [a,b] \to R$ be an increasing function.*

(1) *If f is C-Stieltjes integrable with respect to g on $[a,b]$, then f is C-Stieltjes integrable with respect to g on every subinterval $[c,d] \subseteq [a,b]$. In addition, if $c \in (a,b)$, then*

$$\int_a^c f dg + \int_c^b f dg = \int_a^b f dg.$$

(2) *If f_1 and f_2 are C-Stieltjes integrable with respect to g on $[a,b]$ and α, β are real numbers, then $\alpha f_1 + \beta f_2$ is C-Stieltjes integrable with respect to g on $[a,b]$ and*

$$\int_a^b (\alpha f_1 + \beta f_2) dg = \alpha \int_a^b f_1 dg + \beta \int_a^b f_2 dg.$$

(3) *Let g_1, g_2 be increasing real functions on $[a,b]$ and α, β be real numbers. If f is C-Stieltjes integrable with respect to both g_1 and g_2 on $[a,b]$, then the function f is C-Stieltjes integrable with respect to $\alpha g_1 + \beta g_2$ on $[a,b]$ and*

$$\int_a^b f d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f dg_1 + \beta \int_a^b f dg_2.$$

Corollary 1. *Let $g : [a,b] \to R$ be an bounded variation function and f be continuous. Then f is C-Stieltjes integrable with respect to g on $[a,b]$.*

Proof. Since $g : [a,b] \to R$ is an bounded variation function, we may assume that g is nondecreasing on $[a,b]$ and by the definition of the C-Stieltjes integral and continuity of f, f is C-Stieltjes integrable with respect to g on $[a,b]$. □

Lemma 1 (Saks-Henstock). *Let $f : [a,b] \to X$ be C-Stieltjes integrable with respect to g on $[a,b]$. Then for every $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a,b] \to R^+$ such that*

$$\left\| S(f,g,D) - \int_a^b f dg \right\| < \varepsilon$$

for each δ-fine C-partition $D = \{(I,\xi)\}$ of $[a,b]$.

Particularly, if $D' = \{([u_i,v_i],\xi_i)\}_{i=1}^m$ is an arbitrary δ-fine partial C-partition of $[a,b]$, we have

$$\left\| S(f,g,D') - \sum_{i=1}^{m} \int_{u_i}^{v_i} f(\xi_i)dg \right\| \leq \varepsilon.$$

Proof. The proof is similar to the proof of Henstock-Stieltjes integral, see Lemma 2.5 in [3]. □

Theorem 4. *Let $g : [a,b] \to R$ be an increasing function and $g \in C^1([a,b])$. If $f = \theta$ almost everywhere on $[a,b]$, then f is C-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b fdg = \theta$.*

Proof. Since $g \in C^1([a,b])$, there exists a number $M > 0$ such that $|g'(\xi)| \leq M$ for each $\xi \in [a,b]$. From the mean-valued theorem we know that there exists $\xi_i' \in [u_i, v_i]$ such that

$$g(v_i) - g(u_i) = g'(\xi_i')(v_i - u_i).$$

Assume $E = \{\xi \in [a,b] : f(\xi) \neq \theta\}$ and $E = \bigcup_n E_n \subset [a,b]$, where $E_n = \{\xi \in [a,b] : n-1 \leq \|f(\xi)\| < n\}$. Obviously, $\mu(E) = 0$ and therefore $\mu(E_n) = 0$. Then there are an open sets $G_n \subset [a,b]$ such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\varepsilon}{n \cdot 2^n \cdot M}$. We choose a positive function $\delta(\xi) : I_0 \to R^+$ as follows: for each $\xi \in E_n$, $B(\xi, \delta(\xi)) \subset G_n$ and $\delta(\xi)$ is arbitrary for $\xi \in [a,b] \setminus E$. For each δ-fine C-partition $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$\begin{aligned} \|S(f,g,D) - \theta\| &= \left\| \sum_{n=1}^{\infty} \sum_{\xi_i \in E_n} f(\xi_i)[g(v_i) - g(u_i)] \right\| \\ &= \left\| \sum_{n=1}^{\infty} \sum_{\xi_i \in E_n} f(\xi_i) g'(\xi_i')(v_i - u_i) \right\| \\ &< \sum_{n=1}^{\infty} n \cdot M \cdot \frac{\varepsilon}{n \cdot 2^n \cdot M} = \varepsilon. \end{aligned}$$

Hence, f is C-Stieltjes integrable with respect to g on $[a,b]$ and

$$\int_a^b fdg = \theta.$$

□

Corollary 2. *Let $f_1 : [a,b] \to X$ be C-Stieltjes integrable with respect to g on $[a,b]$. If $f_1 = f_2$ almost everywhere on $[a,b]$, then f_2 is C-Stieltjes integrable with respect to g on $[a,b]$ and $\int_a^b f_1 dg = \int_a^b f_2 dg$.*

3. The C-Stieltjes Representable Operators

Definition 5. A continuous linear operator $T : L_1[a,b] \to X$ is *C-Stieltjes representable* with respect to g if there exists a scalar essentially bounded C-Stieltjes integrable function $h : [a,b] \to X$ with respect to g such that $T(f) = \int_a^b fhdg$, for every $f \in L_1[a,b]$.

Theorem 6. *Assume that X, Y are real Banach spaces, $g : [a,b] \to R$ is an increasing function and $f : [a,b] \to X$ is C-Stieltjes integrable with respect to g. If $T : X \to Y$ is a continuous linear operator, then $T(f)$ is C-Stieltjes integrable with respect to g such that*

$$T\left(\int_a^b fdg\right) = \int_a^b T(f)dg \quad \textit{for all } f \in L_1[a,b].$$

Proof. Since $T : X \to Y$ is a continuous linear operator, there exists a number $M > 0$ such that $\|Tx\| \le M\|x\|$ for each $x \in X$. Since $f : [a,b] \to X$ is C-Stieltjes integrable with respect to g on $[a,b]$, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left\| S(f,g,D) - \int_a^b fdg \right\| < \frac{\varepsilon}{M}$$

for each δ-fine C-*partition* $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[a,b]$, where

$$S(f,g,D) = \sum_{i=1}^n f(\xi_i)|g(v_i) - g(u_i)|.$$

Hence we have

$$\begin{aligned}\left\| S(Tf,g,D) - T\left(\int_a^b fdg\right) \right\| &= \left\| T\left(S(f,g,D) - \int_a^b fdg\right) \right\| \\ &\le M \cdot \left\| S(f,D) - \int_a^b f \right\| \\ &< M \cdot \frac{\varepsilon}{M} = \varepsilon.\end{aligned}$$

So,

$$T\left(\int_a^b fdg\right) = \int_a^b T(f)dg.$$

□

Theorem 7. *If $T : L_1[a,b] \to X$ is C-Stieltjes representable with respect to g and $S : X \to Y$ is any continuous linear operator. Then $S(T) : L_1[a,b] \to Y$ is C-Stieltjes representable with respect to g.*

Proof. The proof is similar to Theorem 2.4 in [5]. □

Theorem 8. *Assume that $T, G : L_1[a,b] \to X$ are C-Stieltjes representable with respect to g. Then k_1T+k_2G is C-Stieltjes representable with respect to g for arbitrary $k_1, k_2 \in R$.*

Proof. We will prove that kT and $T+G$ are C-Stieltjes representable with respect to g.

(1) Suppose that a bounded linear operator $T : L_1[a,b] \to X$ is C-Stieltjes representable with respect to g, there exists a scalarly essentially bounded C-Stieltjes integrable function $h : [a,b] \to X$ with respect to g such that

$$T(f) = \int_a^b fhdg.$$

Since $T : L_1[a,b] \to X$ is bounded linear operator, $kT : L_1[a,b] \to X$ is a bounded linear operator for arbitrary k in R and T is a C-Stieltjes representable with respect to g. Hence,

$$(kT)(f) = \int_a^b k(fh)dg = \int_a^b f(kh)dg.$$

Thus, $kT : L_1[a,b] \to X$ is C-Stieltjes representable with respect to g.

(2) Since the bounded linear operators T and G are C-Stieltjes representable with respect to g, there exist scalar essentially bounded C-Stieltjes integrable function $h_1 : L_1[a,b] \to X$ and $h_2 : L_1[a,b] \to X$ with respect to g such that

$$T(f) = \int_a^b fh_1dg, \quad G(f) = \int_a^b fh_2dg$$

for all $f \in L_1[a,b]$. Since T, G are bounded linear operators, $T+G$ is also a bounded linear operator and $h_1 + h_2$ is scalar essentially bounded C-Stieltjes representable with respect to g. Hence

$$(T+G)(f) = T(f) + G(f) = \int_a^b fh_1dg + \int_a^b fh_2dg = \int_a^b f(h_1+h_2)dg.$$

This means that $T+G$ is C-Stieltjes representable with respect to g. Therefore, $k_1T + k_2G : L_1[a,b] \to X$ is C-Stieltjes representable with respect to g. □

Theorem 9. *Let $f : [a,b] \to X$ be C-integrable on $[a,b]$ and $F(x) = \int_a^x f$ for each $x \in [a,b]$. If $G : [a,b] \to R$ is of bounded variation on $[a,b]$, then fG is C-integrable on $[a,b]$ and*

$$\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

Proof. Let $\varepsilon > 0$. Since f is C-integrable on $[a,b]$, there exists a positive function δ_1 defined on $[a,b]$ such that

$$\left\| S(f,D_1) - \int_a^b f \right\| < \varepsilon$$

whenever $D_1 = \{(u_i,v_i),\xi_i\}_{i=1}^n$ is a δ_1-fine C-partition of $[a,b]$. F is the primitive of f, then F is continuous and therefore uniformly continuous on $[a,b]$. We claim that F is C-Stieltjes integrable on $[a,b]$ with respect to G, the proof is similar to [13, Theorem 3.3.2]. Then there exists a positive function $\delta < \delta_1$ such that

$$\left\| \sum_{k=1}^n F(c_i)(G(x_i) - G(x_{i-1})) - \int_a^b FG' \right\| < \varepsilon.$$

By the Saks-Henstock Lemma, we have

$$\left\| \sum_{k=1}^n f(c_i)(x_i - x_{i-1}) - F(x_i) \right\| < \varepsilon$$

whenever $D = \{([x_{i-1},x_i],c_i)\}_{i=1}^n$ is a δ-fine C-partition of $[a,b]$. Let

$$D = \{([x_{k-1},x_k],c_k)\}_{k=1}^n$$

be a δ-fine C-partition of $[a,b]$ and assume that each tag c_k occurs only once. Note that $c_1 = a$ and that $c_n = b$. By the Saks-Henstock Lemma and Abel transform formula, we obtain

$$\begin{aligned}
&\left\| \sum_{k=1}^n f(c_k)G(c_k)(x_k - x_{k-1}) - \left(F(b)G(b) - \int_a^b FG'\right) \right\| \\
&= \left\| \sum_{k=1}^{n-1} \left(\sum_{i=1}^k f(c_i)(x_i - x_{i-1})(G(c_k) - G(c_{k+1})) \right) \right. \\
&\quad \left. + \sum_{i=1}^n f(c_i)(x_i - x_{i-1})G(c_n) - (F(b)G(b) - \int_a^b FG') \right\| \\
&\le \sum_{k=1}^{n-1} |G(c_k) - G(c_{k+1})| \left\| \sum_{i=1}^k (f(c_i)(x_i - x_{i-1}) - F(x_k)) \right\| \\
&\quad + \left\| \sum_{k=1}^{n-1} F(x_k)(G(c_{k+1}) - G(c_k)) - \int_a^b FG' \right\| \\
&\quad + |G(b)| \left\| \sum_{i=1}^n (f(c_i)(x_i - x_{i-1}) - F(b)) \right\| \\
&< \varepsilon V(G,[a,b]) + \varepsilon + \varepsilon |G(b)|
\end{aligned}$$

$$= \varepsilon(V(G,[a,b]) + 1 + |G(b)|).$$

This completes the proof. □

Remark 1. In fact, B. Bongiorno discussed theorem 3.5 in [12, Theorem 4.2] for the case of real valued functions. Here, we extend this result to Banach-valued functions.

We can easily get the following corollary.

Corollary 3. *Let $f : [a,b] \to X$ be C-integrable on $[a,b]$ and $F(x) = \int_a^x f$ for each $x \in [a,b]$. If $G : [a,b] \to R$ is absolutely continuous on $[a,b]$, then fG is C-integrable on $[a,b]$ and*

$$\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

4. CONVERGENCE THEOREMS

Definition 10. Let $g : [a,b] \to R$ be an increasing function. A sequence $\{f_k\}$ is *C-Stieltjes equi-integrable* with respect to g on $[a,b]$ if each f_k is C-Stieltjes integrable with respect to g and for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a,b] \to R^+$ such that

$$\left\| S(f_k, g, D) - \int_a^b f_k dg \right\| < \varepsilon \quad \forall k \in N$$

for each δ-fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$.

Theorem 11. *Assume that $g : [a,b] \to R$ be an increasing function and $f_k : [a,b] \to X$ be C-Stieltjes equi-integrable with respect to g on $[a,b]$ such that*

$$\lim_{k\to\infty} f_k(\xi) = f(\xi) \quad \forall \xi \in [a,b].$$

Then the function $f : [a,b] \to X$ is C-integrable with respect to g on $[a,b]$ and

$$\lim_{k\to\infty} \int_a^b f_k dg = \int_a^b f dg.$$

Proof. We will prove that $\int_a^b f_k dg$ has the limit A and $\int_a^b f dg = A$.

(1) Let $\varepsilon > 0$. Since $\{f_k\}$ is C-Stieltjes equi-integrable on $[a,b]$, there exists a $\delta(\xi) > 0$ for any δ-fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$,

$$\left\| S(f_k, g, D) - \int_a^b f_k dg \right\| < \varepsilon$$

for all k. Since $\{f_k\}$ converges point-wise on $[a,b]$, there exists a positive integer $N \in \mathbb{N}$ such that

$$\left\| S(f_k, g, D) - S(f_l, g, D) \right\| < \varepsilon$$

for all $k, l > N$. Then we have

$$\begin{aligned}\left\| \int_a^b f_k dg - \int_a^b f_l dg \right\| &\le \left\| S(f_k, g, D) - \int_a^b f_k dg \right\| \\ &+ \left\| S(f_k, g, D) - S(f_l, g, D) \right\| \\ &+ \left\| S(f_l, g, D) - \int_a^b f_l dg \right\| < 3\varepsilon\end{aligned}$$

for all $k, l > N$

Hence, the sequence $\left\{ \int_a^b f_k dg \right\}$ of elements of X is a Cauchy sequence. Let A be the limit of this sequence. Then

$$\lim_{k\to\infty} \int_a^b f_k dg = A \in X.$$

(2) Since $\lim_{k\to\infty} \int_a^b f_k dg = A$, for each $\varepsilon > 0$ there is a $m \in \mathbb{N}$ such that

$$\left\| \int_a^b f_k dg - A \right\| < \varepsilon$$

for all $k > m$. We will prove that $\int_a^b f dg = A$.

Take any δ-fine C-partition $D = \{([u, v], \xi)\}$ of $[a, b]$. Since $\lim_{k\to\infty} f_k(\xi) = f(\xi)$, there is a $k > m$ such that

$$\left\| S(f_k, g, D) - S(f, g, D) \right\| < \varepsilon.$$

Thus, we have

$$\begin{aligned}\left\| S(f, g, D) - A \right\| &\le \left\| S(f, g, D) - S(f_k, g, D) \right\| \\ &+ \left\| S(f_k, g, D) - \int_a^b f_k dg \right\| \\ &+ \left\| \int_a^b f_k dg - A \right\| < 3\varepsilon.\end{aligned}$$

This means that f is C-Stieltjes integrable with respect to g on $[a, b]$ and

$$\lim_{k\to\infty} \int_a^b f_k dg = \int_a^b f dg.$$

□

Definition 12. Let $F : [a, b] \to X$ and let E be a subset of $[a, b]$.

(a) F is said to be AC_δ on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : [a,b] \to R^+$ such that $\|\sum_i F([u_i, v_i])\| < \varepsilon$ for each δ - fine partial partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ satisfying the endpoints of $[u_i, v_i]$ belonging to E and $\sum_i |v_i - u_i| < \eta$.

(b) F is said to be AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : [a,b] \to R^+$ such that $\sum_i \|F([u_i, v_i])\| < \varepsilon$ for each δ-fine partial C-partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ satisfying the endpoints of $[u_i, v_i]$ belonging to E and $\sum_i |v_i - u_i| < \eta$.

(c) F is said to be ACG_δ if F is continuous on E and E can be expressed as a union of countable sets on which F is AC_δ .

(d) F is said to be ACG_c on E if F is continuous on E and E can be expressed a union of countable sets on which F is AC_c.

Theorem 13. *Assume that $g : [a,b] \to R$ is an increasing function and $g \in C^1[a,b]$. If functions $f_n : [a,b] \to X$ are C-Stieltjes integrable with respect to g such that*

1) *$f_n(x) \to f(x)$ for all $x \in [a,b]$;*
2) *there exists a real-valued function h that is C-Stieltjes integrable with respect to g on $[a,b]$ and such that $\|f_n - f_m\| \le h$ for each n, m.*

Then f is C-Stieltjes integrable with respect to g on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg.$$

Proof. We will prove this theorem by three steps.

(1) Assume $E_j = \{\xi \in [a,b] : j - 1 \le |h(\xi)| < j\}$ for each natural number j. Then $[a,b] = \bigcup_j E_j$. Let $\varepsilon > 0$ and $H(x) = \int_a^x hdg$. We claim that $H(x)$ is ACG_c on $[a,b]$. By Saks-Henstock lemma, for the given $\varepsilon > 0$, there is a positive function δ such that

$$\sum |h(\xi_i)(g(v_i) - g(u_i)) - H(u_i, v_i)| < \frac{\varepsilon}{2}$$

for each δ-fine partial C-partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$, whenever $\xi_i \in E_j$, $H(u_i, v_i) = \displaystyle\int_{u_i}^{v_i} hdg$.

Let M be a bound of the function g' on $[a,b]$. By the Mean Value Theorem, for each i, there exists $x_i \in (u_i, v_i)$ such that

$$g(v_i) - g(u_i) = g'(x_i)(v_i - u_i) \le M(v_i - u_i). \tag{1}$$

Choose $\eta < \frac{\varepsilon}{2Mn(b-a)}$ and let $\sum_i (v_i - u_i) < \eta$, then we have

$$
\begin{aligned}
\Big|\sum_i H(u_i,v_i)\Big| &\le \sum_i |h(\xi_i)(g(v_i)-g(u_i)) - H(u_i,v_i)| \\
&\quad + \sum_i |h(\xi_i)| g'(x_i)(v_i-u_i) \qquad (2)\\
&< \frac{\varepsilon}{2} + Mn\sum_i (v_i-u_i) < \varepsilon
\end{aligned}
$$

Hence, $H(x)$ is AC_c on E_j and therefore H is ACG_c on $[a,b]$.

(2) Since $H(x)$ is AC_c on E_j for each j, there exists $\eta_j > 0$ such that

$$\sum_i |H(v_i,u_i)| < \varepsilon\cdot 2^{-j}$$

whenever $\{[u_i,v_i]\}$ is a finite collection of non-overlapping intervals in $[a,b]$ satisfying $\sum_i |v_i-u_i| < \eta_j$ and $u_i, v_i \in E_j$. Since $h(x)$ is C-Stieltjes integrable with respect to g on $[a,b]$, there is a choice $\delta_h > 0$ such that

$$\left|\sum_i \left[h(\xi)(g(v_i)-g(u_i)) - \int_{u_i}^{v_i} hdg\right]\right| < \varepsilon$$

for each δ_h- fine C-partition $D_h = \{([u_i,v_i],\xi)\}$ of $[a,b]$.

Let $D_0 = \{([u_i,v_i],\xi)\}$ be a δ_h- fine partial C-partition of $[a,b]$ and $u_i, v_i \in E_j$, $\sum_{\xi\in E_j} |v_i-u_i| < \eta_j$. Then for each n, m, we have

$$
\begin{aligned}
\left\|\sum_i \int_{u_i}^{v_i} f_n dg - \sum_i \int_{u_i}^{v_i} f_m dg\right\| &\le \sum_i \int_{u_i}^{v_i} \|f_n - f_m\| dg \\
&\le \sum_i \int_{u_i}^{v_i} hdg \qquad (3)\\
&= \sum_{j=1}^{\infty}\sum_{\xi\in E_j} \int_{u_i}^{v_i} hdg < \varepsilon.
\end{aligned}
$$

Since $\{f_n\}$ is C-Stieltjes integrable with respect to g on $[a,b]$, for the given $\varepsilon > 0$, there exists δ_n and $\delta_{n+1} < \delta_n$ such that

$$\left\|\sum_i f_n(g(v_i)-g(u_i)) - \sum_i \int_{u_i}^{v_i} f_n dg\right\| < \varepsilon\cdot 2^{-n} \qquad (4)$$

for each δ_n-fine C-partition $D_n = \{([u,v],\xi)\}$ of $[a,b]$. For each $\xi \in E_j$, choose $m(\xi) \in \mathbb{N}$ for all $n, m > m(\xi)$ such that

$$\|f_n(\xi) - f_m(\xi)\| < \varepsilon. \qquad (5)$$

(3) In the following, we will prove $\{f_n\}$ is C-Stieltjes equi-integrable with respect to g on $[a,b]$.

Let $\delta(\xi)=\min\{\delta_{m(\xi)}(\xi),\delta_h(\xi)\}, \xi\in E_j, j=1,2\cdots$. Take any $\delta-$ fine C-partition $D=\{([u_i,v_i],\xi)\}$ of $[a,b]$, splitting the sum $\sum$ over D into two partial sums over D_1 and D_2 with $m(\xi)\geq n$ and $m(\xi)<n$ respectively. When $m(\xi)\geq n$, the sum over D_1 has finite terms, so,

$$\left\|\sum_{D_1}\left[f_n(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_n dg\right]\right\|<\varepsilon.$$

For the sum $\sum$ over D we have

$$\begin{aligned}
&\left\|\sum_{D}\left[f_n(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_n dg\right]\right\|\\
&\leq\left\|\sum_{D_1}\left[f_n(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_n dg\right]\right\|\\
&\quad+\left\|\sum_{D_2}\left[f_n(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_n dg\right]\right\|\\
&<\varepsilon+\left\|\sum_{D_2}(f_n-f_{m(\xi)})(g(v_i)-g(u_i))\right\|\\
&\quad+\left\|\sum_{D_2}[f_{m(\xi)}(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_{m(\xi)}dg]\right\|\\
&\quad+\left\|\sum_{D_2}\left[\int_{u_i}^{v_i}f_{m(\xi)}dg-\int_{u_i}^{v_i}f_n dg\right]\right\|.
\end{aligned}\tag{6}$$

From the formula (5), we obtain

$$\left\|\sum_{D_2}(f_n-f_{m(\xi)})(g(v_i)-g(u_i))\right\|<\varepsilon(b-a).$$

By (4)

$$\left\|\sum_{D_2}\left[f_{m(\xi)}(g(v_i)-g(u_i))-\int_{u_i}^{v_i}f_{m(\xi)}dg\right]\right\|<\varepsilon$$

and by (3),

$$\left\|\sum_{D_2}\left[\int_{u_i}^{v_i}f_{m(\xi)}dg-\int_{u_i}^{v_i}f_n dg\right]\right\|<\varepsilon.$$

Therefore, from (6) and the above inequalities we have that

$$\begin{aligned}&\left\|\sum_D [f_n(g(v_i)-g(u_i)) - \int_{u_i}^{v_i} f_n dg]\right\| \\ &< \varepsilon + \varepsilon(b-a) + \varepsilon + \varepsilon \\ &= \varepsilon(b-a+3).\end{aligned}$$

Then for all $n \in N$, $\{f_n\}$ is C-Stieltjes equi-integrable. By Theorem 4.2, f is C-Stieltjes integrable with respect to g on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg.$$

□

Remark 2. The previous theorem holds for the Ap-Henstock-Stieltjes integral [3]. We prove that it also holds for the C-Stieltjes integral.

Acknowledgement. The authors are grateful to the referee for his or her careful reading of the manuscript and for valuable and helpful suggestions.

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