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NUMBER OF VERTICES FOR POLYGONAL FUNCTIONS UNDER ITERATION

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ABSTRACT. Being complicated in computation, iteration of a nonlinear 1-dimensional mapping makes many interesting problems, one of which is about the change of the number of vertices under iteration. In this paper we investigate iteration of polygonal functions which each have only one vertex and give conditions under which the number of vertices either does not increase or has a bound under iteration.

1. INTRODUCTION

Iteration is repetition of the same operation. In mathematical sense, for a fixed integer $n \geq 1$, the n-th *iterate* of a mapping $f : E \to E$, where E is a nonempty set, is defined by

$$f^n = f \circ f^{n-1}, \; f^0 = \mathbf{id}$$

inductively, where $\circ$ denotes the composition of mappings and **id** denotes the identity mapping. If f is a bijection, i.e., one-to-one and onto, then the index n of iteration can be extended to the whole set $\mathbf{Z}$ of integers.

Iteration is often observed in mathematics, science, engineering and daily life, but its calculation is complicated even in the one-dimensional case [12, 14]. Although conjugacy between mappings, i.e., $f \sim g$ if and only if $f = h^{-1} \circ g \circ h$ for a certain bijection $h : E \to E$, may reduce many functions to those whose general iteration can be given easily, only a few classes of functions can be calculated for their general iteration even if a computer is used [12, 14].

General iteration of mappings often gives useful ideas and methods for the study of iterative roots [5, 10], in particular, of some special classes of functions. It is considered for polynomials and formal power series in [7, 8, 9]. Properties of general iteration are investigated for strictly piecewise monotone functions (abbreviated by

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PM functions) in [2, 11, 13]. It is shown in [13] that the number $N(f^n)$ of *non-monotone points* of the n-th iterate of a PM function f is nondecreasing.

Polygonal function is an elementary class of one-dimensional maps and recently increasing interests were made to it in various fields [1, 3, 6]. A polygonal function $f : I = [a, b] \to I$ is a continuous piecewise linear function, which is of the general form

$$f(t) = f_j(t) := \kappa_j t + \beta_j, \quad \text{as } a_j \le t \le a_{j+1}, \tag{1.1}$$

where (a_j) is a strictly increasing sequence in I and $\kappa_j \ne \kappa_{j+1}$ for each j. By continuity we see for each j that $\kappa_{j-1}a_j + \beta_{j-1} = \kappa_j a_j + \beta_j$, i.e.,

$$(\kappa_j - \kappa_{j-1})a_j + (\beta_j - \beta_{j-1}) = 0, \ \forall j. \tag{1.2}$$

Here all a_j's are vertices of this polygonal function and we let $V(f)$ consist of all a_j's, which can be either a finite set or an infinite set. For simplicity, we also refer to a polygonal function f with $V(f) = n$ as a *n-polygonal function.* Although polygonal function is of an elementary form, computing its iterates is not easy because the concerned different lines on distinct intervals may interact each other in iteration. To our best knowledge, only the second order iterate of polygonal functions was investigated in [4].

In this paper we discuss iteration of 1-polygonal functions on an interval. More concretely, we consider functions $f : I = [a, b] \to I$ of the form

$$f(t) = \begin{cases} f_1(t) := \kappa_1 t + \beta_1, & \text{as } a \le t \le t_0, \\ f_2(t) := \kappa_2 t + \beta_2, & \text{as } t_0 < t \le b, \end{cases} \tag{1.3}$$

where $\kappa_1 \ne \kappa_2$. From (1.2) we know that the vertex is (t_0, x_0), where

$$x_0 = \kappa_1 t_0 + \beta_1, \ t_0 = -\frac{\beta_2 - \beta_1}{\kappa_2 - \kappa_1}.$$

We find conditions of κ_1 and κ_2 under which the number of vertices either does not increase or has a bound under iteration. As observed in what follows, our results highly depend on the position of the vertex (t_0, x_0) to the diagonal line $x = t$ (i.e., either $t_0 < x_0$, or $t_0 = x_0$, or $t_0 > x_0$) in the (t, x)-coordinate plane.

2. CASE: $\kappa_1 > 0, \kappa_2 < 0$

Proposition 1. *Let f be the polygonal function of (1.3), where $\kappa_1 > 0, \kappa_2 < 0$. If $x_0 \le t_0$ (Fig. 1) then the number of vertices under iteration does not increase, i.e., $V(f^n) = 1$ for all $n \in \mathbf{N}$, and the vertex of f^n locates at $(t_0, \kappa_1^n t_0 + \beta_1(1 - \kappa_1^n)/(1 -$*

κ_1)). In addition, $f^n(t) \to \beta_1/(1-\kappa_1)$ as $n \to \infty$ for each $t \in I$ when $0 < \kappa_1 < 1$. If $x_0 > t_0$ then $V(f^n) = 1$ for all $n \in \mathbf{N}$ when $f_1(a) \geq t_0, f_2(b) \geq t_0$ (Fig. 2); otherwise, $V(f^n) \geq 2$ for all integers $n \geq 2$ (Fig. 3).

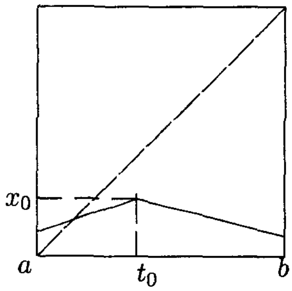

Fig.1: $x_0 < t_0$.

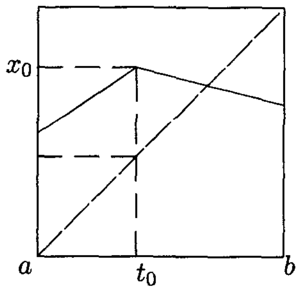

Fig.2: $x_0 > t_0$ and $f_1(a) > t_0, f_2(b) > t_0$

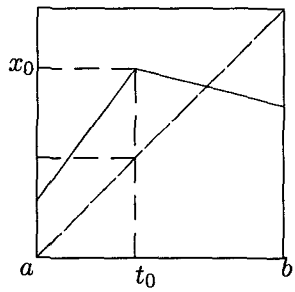

Fig.3: $x_0 > t_0$ and $f_1(a) < t_0, f_2(b) > t_0$

Proof. First of all, in the case of $x_0 \leq t_0$, we note that

$$a \leq f(t) \leq t_0, \quad \forall t \in [a, b], \tag{2.4}$$

because f_1 is increasing and f_2 is decreasing, which implies that $x_0 = f(t_0)$ is the maximum of f. Thus, $f(f(t)) = f_1(f(t))$ and the iterate of f can be presented as

$$f^n(t) = \begin{cases} f_1^n(t) = \kappa_1^n t + \beta_1(\frac{1-\kappa_1^n}{1-\kappa_1}), & \text{as } a \leq t \leq t_0, \\ f_1^{n-1}(f_2(t)) = \kappa_1^{n-1}\kappa_2 t + \kappa_1^{n-1}\beta_2 + \beta_1(\frac{1-\kappa_1^{n-1}}{1-\kappa_1}), & \text{as } t_0 < t \leq b, \end{cases}$$

It implies that $V(f^n) = 1$ for all $n \in \mathbf{N}$ because the left derivative $Df^n(t_0 - 0) = \kappa_1^n$ and the right derivative $Df^n(t_0+0) = \kappa_1^{n-1}\kappa_2$ are different. In addition, if $0 < \kappa_1 < 1$ then $f^n(t) \to \beta_1/(1-\kappa_1)$ as $n \to \infty$, because $\kappa_1^n \to 0$ as $n \to \infty$.

We discuss the case that $x_0 > t_0$ in the following 4 subcases: (i) $f_1(a) \geq t_0, f_2(b) \geq t_0$; (ii) $f_1(a) \geq t_0, f_2(b) < t_0$; (iii) $f_1(a) < t_0, f_2(b) \geq t_0$; and (iv) $f_1(a) < t_0, f_2(b) < t_0$.

In the subcase (i), as in (2.4), we have $t_0 \leq f(t) \leq b$ for all $t \in [a, b]$. Thus, $f(f(t)) = f_2(f(t))$ and

$$f^n(t) = \begin{cases} f_2^{n-1}(f_1(t)) = \kappa_2^{n-1}\kappa_1 t + \kappa_2^{n-1}\beta_1 + \beta_2(\frac{1-\kappa_2^{n-1}}{1-\kappa_2}), & \text{as } a \leq t \leq t_0, \\ f_2^n(t) = \kappa_2^n t + \beta_2(\frac{1-\kappa_2^n}{1-\kappa_2}), & \text{as } t_0 < t \leq b, \end{cases}$$

which implies that $V(f^n) = 1$ for all $n \in \mathbf{N}$.

In the subcase (ii), the continuity of f_2 guarantees the existence of $t_2^* \in (t_0, b)$ such that

$$f_2(t_2^*) = t_0 \tag{2.5}$$

because $f_2(b) < t_0$ and $f_2(t_0) > t_0$. Thus $t_2^* = f_2^{-1}(t_0)$ and

$$f^2(t) = \begin{cases} f_2(f_1(t)), & \text{as } a \le t \le t_0, \\ f_2^2(t), & \text{as } t_0 < t \le t_2^*, \\ f_1(f_2(t)), & \text{as } t_2^* < t \le b. \end{cases}$$

It follows that $Df^2(t_0 - 0) = \kappa_2\kappa_1 \neq \kappa_2^2 = Df^2(t_0 + 0)$, and $Df^2(t_2^* - 0) = \kappa_2^2 \neq \kappa_2\kappa_1 = Df^2(t_2^* + 0)$, implying that $V(f^2) = 2$. Moreover, t_0 and t_2^* are both non-monotone points of f^2 as defined in [13] and stated in section 9. By Lemma 2.2 in [13] and its proof, a non-monotone point of a PM function f is also a non-monotone point of f^n and therefore the number $N(f^n)$ of non-monotone points is nondecreasing in n. On the other hand, for polygonal functions each non-monotone point is a vertex (but the converse is not true). Thus $V(f^n) \geq N(f^n)$, implying that $V(f^n) \geq 2$ for all integers $n \geq 2$.

The subcase (iii) is similar to the subcase (ii). As in (2.5),

$$f_1(t_1^*) = t_0 \tag{2.6}$$

for some $t_1^* \in (a, t_0)$. Thus $t_1^* = f_1^{-1}(t_0)$ and f^2 has two non-monotone points t_1^* and t_0, implying that $V(f^n) \geq 2$ for all integers $n \geq 2$.

The subcase (iv) is a combination of (ii) and (iii), where f^2 has three non-monotone points t_1^*, t_0, t_2^*. We similarly see that $V(f^n) \geq 3$ for all integers $n \geq 2$ and the proof is completed. □

3. CASE: $\kappa_1 > 0$, $\kappa_2 > 0$

In this section we will see that the growth of the number $V(f^n)$ depends on the relation between $f_1^n(a)$ (or $f_2^n(b)$) and t_0. For a simple statement, an increasing (resp. decreasing) sequence (c_n) in $[a, b]$ is said to *cross* $t_0 \in (a, b)$ *in m-th order* if $c_1 < t_0$ (resp. $c_1 > t_0$) and $m \geq 2$ is the least integer such that $c_i < t_0$ (resp. $c_i > t_0$) for all $i = 1, 2, \cdots m - 1$ and $c_m \geq t_0$ (resp. $c_m \leq t_0$).

Proposition 2. *Let f be the polygonal function of (1.3), where $\kappa_1 > 0, \kappa_2 > 0$.*

- (i) *If $x_0 = t_0$ (Fig.4) then the number of vertices does not increase, i.e., $V(f^n) = 1$ for all $n \in \mathbf{N}$.*
- (ii) *If $x_0 < t_0$ then either $V(f^n) = 1$ for all $n \in \mathbf{N}$ when $f_2(b) \leq t_0$ (Fig.5) or $V(f^i) = i$ for all $i = 1, 2, ..., m - 1$ and $V(f^n) = m$ for all $n \geq m$ when the sequence $(f_2^n(b))$ crosses t_0 in m-th order.*

(iii) *If $x_0 > t_0$ then either $V(f^n) = 1$ for all $n \in \mathbf{N}$ when $f_1(a) \geq t_0$ (Fig.6) or $V(f^i) = i$ for all $i = 1, 2, ..., m-1$ and $V(f^n) = m$ for all $n \geq m$ when the sequence $(f_1^n(a))$ crosses t_0 in m-th order.*

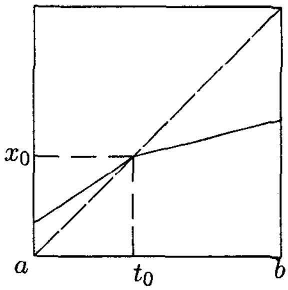

Fig. 4: $x_0 = t_0$

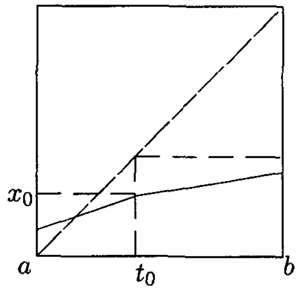

Fig.5: $x_0 < t_0, f_2(b) < t_0$

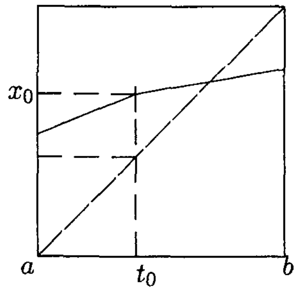

Fig.6: $x_0 > t_0, f_1(a) > t_0$

Proof. In the case of $x_0 = t_0$, since f_1 is increasing we see that

$$a \leq f_1(t) \leq t_0, \ \forall t \in [a, t_0], \tag{3.7}$$

Similar to (3.7), we also have $t_0 \leq f_2(t) \leq b$ for all $t \in [t_0, b]$. It follows that

$$f^n(t) = \begin{cases} f_1^n(t), & \text{as } a \leq t \leq t_0, \\ f_2^n(t). & \text{as } t_0 < t \leq b, \end{cases}$$

implying that $V(f^n) = 1$ as $\kappa_1 \neq \kappa_2$ for all $n \in \mathbf{N}$.

In the case of $x_0 < t_0$, as in (3.7), we also have $a \leq f_1(t) < t_0$ for all $t \in [a, t_0]$. Moreover, for $t \in (t_0, b]$ we have $a \leq f_2(t) \leq t_0$ if $f_2(b) \leq t_0$. Thus

$$f^n(t) = \begin{cases} f_1^n(t) = \kappa_1^n t + \beta_1(\frac{1-\kappa_1^n}{1-\kappa_1}), & \text{as } a \leq t \leq t_0, \\ f_1^{n-1}(f_2(t)) = \kappa_1^{n-1}\kappa_2 t + \kappa_1^{n-1}\beta_2 + \beta_1(\frac{1-\kappa_1^{n-1}}{1-\kappa_1}), & \text{as } t_0 < t \leq b, \end{cases}$$

implying that $V(f^n) = 1$ for all $n \in \mathbf{N}$ since $\kappa_1^n \neq \kappa_1^{n-1}\kappa_2$.

Suppose that $x_0 < t_0$ and $f_2^i(b) > t_0$ for all $i = 1, 2, \cdots m-1$ but $f_2^m(b) \leq t_0$. We first prove that

$$V(f^i) = i \ \text{ for all } i = 1, ..., m-1, \tag{3.8}$$

Use the notations $t_j^* := f_2^{-j}(t_0)$, $j = 0, ..., m-1$. Since f_1, f_2 are both increasing, we see that $t_0 < t_{j-1}^* < t_j^* < b$. We claim that

$$f^k(t) = \begin{cases} f_1^k(t), & \text{as } t \in [a, t_0^*], \\ f_1^{k-j}(f_2^j(t)), & \text{as } t \in (t_{j-1}^*, t_j^*], \ j = 1, ..., k-1, \\ f_2^k(t), & \text{as } t \in (t_{k-1}^*, b], \end{cases} \tag{3.9}$$

for $k = 2, 3, ..., m$. Obviously,

$$f^2(t) = \begin{cases} f_1^2(t), & \text{as } a \le t \le t_0, \\ f_1(f_2(t)), & \text{as } t_0 < t \le f_2^{-1}(t_0), \\ f_2^2(t), & \text{as } f_2^{-1}(t_0) < t \le b, \end{cases}$$

implying that $V(f^2) = 2$. For an inductive proof, assume that (3.9) holds for an integer $k \in \{2, ..., m-1\}$. In this case, $f_1([a, t_0]) \subset [a, t_0]$. Thus $f^{k+1}(t) = f(f_1^k(t)) = f_1^{k+1}(t)$ for $t \in [a, t_0]$ and

$$f^{k+1}(t) = f(f_1^{k-j}(f_2^j(t))) = f_1^{k+1-j}(f_2^j(t)), \text{ as } t \in (t_{j-1}^*, t_j^*], \tag{3.10}$$

for $j = 1, ..., k-1$. Moreover, $f_2^k(b) > t_0$, that is, $t_k^* := f_2^{-k}(t_0) \in (t_{k-1}^*, b)$. Thus for $t \in (t_{k-1}^*, t_k^*]$ we have $f^{k+1}(t) = f(f_2^k(t)) = f_1(f_2^k(t))$ because $f_2^k(t) \in (x_0, t_0] \subset [a, t_0]$ and, on the other hand, for $t \in (t_k^*, b]$ we have $f_2^k(t) \in (t_0, f_2^k(b)] \subset (t_0, b]$, implying that $f^{k+1}(t) = f(f_2^k(t)) = f_2^{k+1}(t)$. This proves that (3.9) holds for $k+1$. Since $\kappa_1 \neq \kappa_2$ we can see that the left derivative is different from the right derivative at each junction, i.e., (3.9) implies that $V(f^k) = k$ and (3.8) is proved by induction.

Furthermore, using (3.9) for $k = m$, we have that $f^{m+1}(t) = f_1^{m+1}(t)$ for $t \in [a, t_0]$, $f^{m+1}(t) = f_1^{m+1-j}(f_2^j(t))$ for $t \in (t_{j-1}^*, t_j^*]$, where $j = 1, ..., m-1$. Note that $f_2^m(b) \le t_0$, that is, $t_{m-1}^* < b \le t_m^* := f_2^{-m}(t_0)$. Thus for $t \in (t_{m-1}^*, b]$ we have $f_2^m(t) \in (x_0, t_0]$ and therefore $f^{m+1}(t) = f(f_2^m(t)) = f_1(f_2^m(t))$. This implies that $V(f^{m+1}) = m$. In comparison with f^k for $k \le m-1$ we observe that iteration makes a new vertex in the last sub-interval $(t_{k-1}^*, b]$, but none of new vertex arises there when $k \ge m$. Actually, for any natural number n we can calculate $f^{m+n}(t) = f_1^{m+n}(t)$ on $[a, t_0]$, $f^{m+n}(t) = f_1^{m+n-j}(f_2^j(t))$ on $(t_{j-1}^*, t_j^*]$, where $j = 1, ..., m-1$, and for $t \in (t_{m-1}^*, b]$ we have

$$f^{m+n}(t) = f^n(f_2^m(t)) = f_1^n(f_2^m(t)).$$

Therefore, $V(f^{m+n}) = m$ for all natural numbers n.

The proof of result (iii) is similar to that of (ii). □

In Proposition 2 we show that $V(f^n)$ is bounded for some classes of f. In other cases, i.e., $f_2^i(b) > t_0$ or $f_2^i(a) < t_0$ holds for all natural numbers i, the proof of Proposition 2 implies that (3.9) holds for all $k \in \mathbf{N}$. Thus $V(f^i) = i$ for all $i \in \mathbf{N}$ and $V(f^n) \to \infty$ as $n \to \infty$.

4. Case: $\kappa_1 < 0,\ \kappa_2 > 0$

Proposition 3. *Let f be the polygonal function of (1.3), where $\kappa_1 < 0, \kappa_2 > 0$. If $x_0 \geq t_0$ (Fig. 7) then the number of vertices does not increase under iteration, i.e., $V(f^n) = 1$ for all $n \in \mathbf{N}$, and the vertex of f^n locates at $(t_0, \kappa_2^n t_0 + \beta_2(1 - \kappa_2^n)/(1 - \kappa_2))$. In addition, $f^n(t) \to \beta_2/(1 - \kappa_2)$ as $n \to \infty$ for each $t \in I$ when $0 < \kappa_2 < 1$. If $x_0 < t_0$ then $V(f^n) = 1$ for all $n \in \mathbf{N}$ when $f_1(a) \leq t_0, f_2(b) \leq t_0$ (Fig. 8); otherwise, $V(f^n) \geq 2$ for all integers $n \geq 2$ (Fig. 9).*

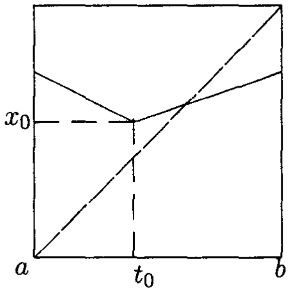

Fig.7: $x_0 > t_0$

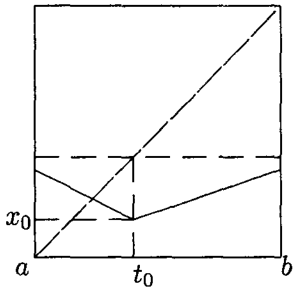

Fig.8: $x_0 < t_0$ and $f_1(a) < t_0, f_2(b) < t_0$

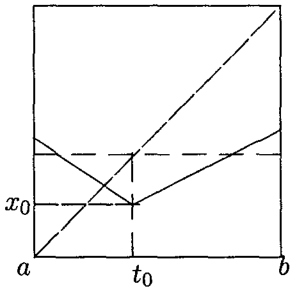

Fig.9: $x_0 < t_0$ and $f_1(a) > t_0, f_2(b) > t_0$

Proof. First of all, in the case of $x_0 \geq t_0$, we note that

$$t_0 \leq f(t) \leq b, \ \forall t \in [a, b], \tag{4.11}$$

because f_1 is decreasing and f_2 is increasing. Thus $f(f(t)) = f_2(f(t))$ and the iterate of f can be presented as

$$f^n(t) = \begin{cases} f_2^{n-1}(f_1(t)) = \kappa_2^{n-1}\kappa_1 t + \kappa_2^{n-1}\beta_1 + \beta_2\left(\frac{1-\kappa_2^{n-1}}{1-\kappa_2}\right), & \text{as } a \leq t \leq t_0, \\ f_2^n(t) = \kappa_2^n t + \beta_2\left(\frac{1-\kappa_2^n}{1-\kappa_2}\right), & \text{as } t_0 < t \leq b, \end{cases}$$

implying that $V(f^n) = 1$ for all $n \in \mathbf{N}$ because $Df^n(t_0 - 0) = \kappa_2^{n-1}\kappa_1$ and $Df^n(t_0 + 0) = \kappa_2^n$ are different. In addition, if $0 < \kappa_2 < 1$ then $f^n(t) \to \beta_2/(1 - \kappa_2)$ as $n \to \infty$, because $\kappa_2^n \to 0$ as $n \to \infty$.

We omit the proof in the case of $x_0 < t_0$, which is similar to that of Proposition 1 and given separately in the following four subcases: (i) $f_1(a) \leq t_0, f_2(b) \leq t_0$; (ii) $f_1(a) \leq t_0, f_2(b) > t_0$; (iii) $f_1(a) > t_0, f_2(b) \leq t_0$; and (iv) $f_1(a) > t_0, f_2(b) > t_0$. □

5. CASE: $\kappa_1 < 0, \kappa_2 < 0$

Proposition 4. *Let f be the polygonal function of (1.3), where $\kappa_1 < 0, \kappa_2 < 0$. Then*

(i) $V(f^{2m}) = 0$ *and* $V(f^{2m+1}) = 1$ *for all* $m \in \mathbf{N}$ *if* $x_0 = t_0$ *(Fig. 10).*

(ii) *If* $x_0 < t_0$ *then* $V(f^n) = 1$ *for all* $n \in \mathbf{N}$ *when* $f_1(a) \leq t_0$ *(Fig. 11); otherwise,* $V(f^n) \geq 2$ *for all integers* $n \geq 2$.

(iii) *If* $x_0 > t_0$ *then* $V(f^n) = 1$ *for all* $n \in \mathbf{N}$ *when* $f_2(b) \geq t_0$ *(Fig. 12); otherwise,* $V(f^n) \geq 2$ *for all integers* $n \geq 2$.

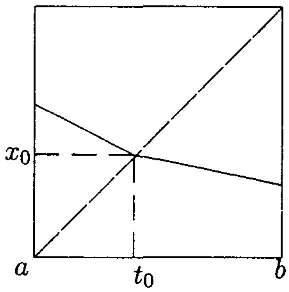

Fig.10: $x_0 = t_0$

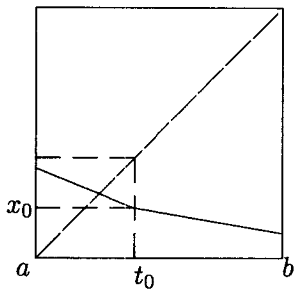

Fig.11: $x_0 < t_0, f_1(a) < t_0$

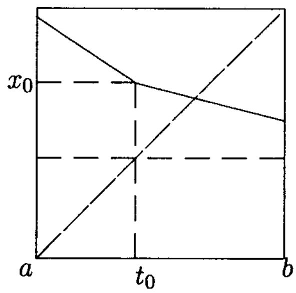

Fig.12: $x_0 > t_0, f_2(b) > t_0$

Proof. In the case of $x_0 = t_0$, since f_1 is decreasing, we have $t_0 \leq f_1(t) \leq b$ for all $t \in [a, t_0]$. Similarly, we also know that $a \leq f_2(t) \leq t_0$ for all $t \in [t_0, b]$. Thus

$$f^2(t) = \begin{cases} f_2(f_1(t)), & \text{as } a \leq t \leq t_0, \\ f_1(f_2(t)), & \text{as } t_0 < t \leq b, \end{cases}$$

$$f^3(t) = \begin{cases} f_1(f_2(f_1(t))), & \text{as } a \leq t \leq t_0, \\ f_2(f_1(f_2(t))), & \text{as } t_0 < t \leq b. \end{cases}$$

One can inductively prove that for all $m \in \mathbf{N}$

$$f^{2m}(t) = \begin{cases} (f_2 f_1)^m(t), & \text{as } a \leq t \leq t_0, \\ (f_1 f_2)^m(t), & \text{as } t_0 < t \leq b, \end{cases}$$

$$f^{2m+1}(t) = \begin{cases} f_1((f_2 f_1)^m(t)), & \text{as } a \leq t \leq t_0, \\ f_2((f_1 f_2)^m(t)), & \text{as } t_0 < t \leq b. \end{cases}$$

Since $Df^{2m}(t_0 - 0) = \kappa_1^m \kappa_2^m = Df^{2m}(t_0 + 0)$, we see that t_0 is no longer a vertex of f^{2m} and therefore $V(f^{2m}) = 0$. Similarly, $V(f^{2m+1}) = 1$ for all $m \in \mathbf{N}$ because $Df^{2m+1}(t_0 - 0) = \kappa_1^{m+1} \kappa_2^m \neq \kappa_1^m \kappa_2^{m+1} = Df^{2m+1}(t_0 + 0)$.

The case (iii) can be discussed by the same arguments as in the case (ii). We discuss the case (ii), i.e., $x_0 < t_0$, separately in the subcases: (ii-a) $f_1(a) \leq t_0$ and (ii-b) $f_1(a) > t_0$. In the subcase (ii-a), since f is decreasing, we have

$$f^n(t) = \begin{cases} f_1^n(t), & \text{as } a \le t \le t_0, \\ f_1^{n-1}(f_2(t)), & \text{as } t_0 < t \le b, \end{cases}$$

implying that $V(f^n) = 1$ for all $n \in \mathbf{N}$ since $\kappa_1^n \neq \kappa_1^{n-1}\kappa_2$.

In the subcase (ii-b), the iterate f^2 is presented as

$$f^2(t) = \begin{cases} f_2(f_1(t)), & \text{as } a \le t \le f_1^{-1}(t_0), \\ f_1^2(t), & \text{as } f_1^{-1}(t_0) < t \le t_0, \\ f_1(f_2(t)), & \text{as } t_0 < t \le b. \end{cases} \tag{5.12}$$

If $\kappa_1 = -1$ we have

$$f^2(t_0) = f_1^2(t_0) = \kappa_1^2 t_0 + \beta_1(1+\kappa_1) = t_0, \tag{5.13}$$

but $f(t_0) = x_0 < t_0$, implying that t_0 is 2-periodic point of f. Moreover, for each $m \in \mathbf{N}$, $a \le f_1^{2m}(t) \le f_1^{2m}(t_0) = t_0$ and $a \le f_1^{2m+1}(t) < f_1^{2m+1}(f_1^{-1}(t_0)) = t_0$ for all $t \in (f_1^{-1}(t_0), t_0]$ since f_1 is decreasing. It follows that

$$f^n(t) = f_1^n(t)\ \forall t \in (f_1^{-1}(t_0), t_0], \tag{5.14}$$

implying that the left derivative $Df^n(t_0 - 0) = Df_1^n(t_0 - 0) = \kappa_1^n$. However, in general, for a given $t \in I$ there exists an integer $j = 0, ..., n$ such that $Df^n(t) = \kappa_1^{n-j}\kappa_2^j$. From (5.12) we see that the right derivative $Df^n(t_0 + 0)$ has at least a factor κ_2, i.e., $j \ge 1$. It implies that $Df^n(t_0 - 0) \neq Df^n(t_0 + 0)$. Similarly, we have $Df^n(f_1^{-1}(t_0) - 0) \neq Df^n(f_1^{-1}(t_0) + 0)$. Thus, f^n has at least two vertices at t_0 and $f_1^{-1}(t_0)$, and therefore $V(f^n) \ge 2$ for all integers $n \ge 2$.

If $\kappa_1 < -1$, from (5.13), we have

$$f^2(t_0) = \kappa_1^2 t_0 + \beta_1\left(\frac{1-\kappa_1^2}{1-\kappa_1}\right) > \kappa_1^2 t_0 + \left(\frac{1-\kappa_1^2}{1-\kappa_1}\right)(1-\kappa_1)t_0 = t_0$$

because $f_1(t_0) < t_0$, i.e., $\beta_1 < (1-\kappa_1)t_0$. It follows that

$$\begin{aligned} \cdots < f^{2m+1}(t_0) < f^{2m-1}(t_0) < \ldots < f^3(t_0) < f(t_0) < t_0 \\ < f^2(t_0) < f^4(t_0) < \ldots < f^{2m}(t_0) < f^{2m+2}(t_0) < \ldots \end{aligned}$$

since f^2 is increasing. Thus from (5.12) we see that $f^3(t) = f_1^3(t)$ for all $t \in (f_1^{-1}(t_0), f_1^{-2}(t_0)]$. One can inductively prove that

$$f^{2m}(t) = f_1^{2m}(t)\ \forall t \in (f_1^{-(2m-1)}(t_0), f_1^{-(2m-2)}(t_0)], \tag{5.15}$$

$$f^{2m+1}(t) = f_1^{2m+1}(t)\ \forall t \in (f_1^{-(2m-1)}(t_0), f_1^{-2m}(t_0)] \tag{5.16}$$

for all $m \in \mathbf{N}$. Hence for even $n = 2m$, similarly to the discussion after (5.14), we can show that the left derivative and the right one of f^n at $f_1^{-(2m-1)}(t_0)$ are different. So are the derivatives of f^n at $f_1^{-(2m-2)}(t_0)$. It follows that $V(f^n) \geq 2$ for all even $n \geq 2$. For the same reason, $V(f^n) \geq 2$ for all odd $n \geq 3$.

If $-1 < \kappa_1 < 0$ then $f^2(t_0) < \kappa_1^2 t_0 + ((1-\kappa_1^2)/(1-\kappa_1))(1-\kappa_1)t_0 = t_0$. It follows that $f(t_0) < f^3(t_0) < \ldots < f^{2m+1}(t_0) < f^{2m+3}(t_0) < \ldots$ and $\ldots < f^{2m}(t_0) < f^{2m-2}(t_0) < \ldots < f^2(t_0) < t_0$ and therefore $f^n(t_0) < t_0$ for all $n \in \mathbf{N}$ since f^2 is increasing. Thus from (5.12)

$$f^n(t) = f_1^n(t)\ \forall t \in (f_1^{-1}(t_0), t_0]$$

As discussed above, $f_1^{-1}(t_0)$ and t_0 are both vertices of f^n, implying that $V(f^n) \geq 2$ for all integers $n \geq 2$. The proof is completed. □

Using the same proof, we can also consider the case that $\kappa_1 = 0$ or $\kappa_2 = 0$ and give the similar conclusions. Not mentioning the computation of general iteration, the law on the change of vertices under iteration is much more complicated if a polygonal function has more than one vertices. For a general polygonal function we want to know: Is the number of vertices not increasing or bounded under iteration? More concretely, what m-polygonal function f satisfies that $V(f^n) \leq m$ or $V(f^n) < \infty$ for all integers $n \geq 2$. This is a direction of our further efforts.

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