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WEAK BI-IDEALS OF NEAR-RINGS

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ABSTRACT. The notion of bi-ideals in near-rings was effectively used to characterize the near-fields. Using this notion, various generalizations of regularity conditions have been studied. In this paper, we generalize further the notion of bi-ideals and introduce the notion of weak bi-ideals in near-rings and obtain various characterizations using the same in left self distributive near-rings.

1. INTRODUCTION

Throughout this paper by a near-ring we mean a right near-ring. For basic definitions one may refer to Pilz [3]. Tamizh Chelvam and Ganesan [4] introduced the notion of bi-ideals in near-rings. Further Tamizh Chelvam [5] introduced the concept of b-regular near-rings and obtained equivalent conditions for regularity in terms of bi-ideals. In this paper the notion weak bi-ideals has been introduced and studied to the extent possible.

2. PRELIMINARIES

Let N be a right near-ring. For two subsets A and B of N, $AB = \{ab \mid a \in A, b \in B\}$ and $A * B = \{a_1(a_2 + b) - a_1a_2 \mid a_1, a_2 \in A$ and $b \in B\}$. A subgroup B of $(N, +)$ is said to be a *bi-ideal* of N if $BNB \cap (BN) * B \subseteq B$ [4]. In the case of a zero-symmetric near-ring, a subgroup B of $(N, +)$ is a bi-ideal if and only if $BNB \subseteq B$ [4]. A subgroup Q of $(N, +)$ is called a *quasi-ideal* of N if $QN \cap NQ \cap N * Q \subseteq Q$ [7]. If N is zero-symmetric, a subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $QN \cap NQ \subseteq Q$.

A near-ring N is said to be *left (right)-unital* if $a \in Na (a \in aN)$ for all $a \in N$. A near-ring N is said to be *unital* if it is both left as well as right unital. An

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element $a \in N$ is said to be *regular* if $a = aba$ for some $b \in N$. A near-ring N is said to be *regular* if every element in N is regular. It may be noted that a regular near-ring is a unital near-ring, but not the converse. An element $a \in N$ is said to be *strongly regular* if $a = ba^2$, for some $b \in N$. A near-ring N is called *strongly regular* if every element in N is strongly regular. N is said to satisfy IFP (Insertion of Factors Property) if $ab = 0$ implies $axb = 0$ for all $x \in N$. A near-ring is called *left bi-potent* if $Na = Na^2$ for $a \in N$. A subgroup M of $(N, +)$ is said to be a *left (right) N-subgroup* if $NM \subseteq M\,(MN \subseteq M)$. A near-ring N is said to be *two sided* if every left N-subgroup is a right N-subgroup and vice versa. A near-ring N is called *b-regular near-ring* if $a \in (a)_r N (a)_l$ for every $a \in N$, where $(a)_r$ ($(a)_l$ is the right (left) N-subgroup generated by $a \in N$ [6]. Note that every regular near-ring is b-regular. A near-ring N is said to be *left self distributive* if $abc = abac$ for all $a, b, c \in N$. Let E be the set of all idempotents of N and L the set of all nilpotent elements of N.

3. WEAK BI-IDEALS

In this section, we introduce weak bi-ideals and obtain some of the properties of the same.

Definition 3.1. A subgroup B of (N, +) is said to be a *weak bi-ideal* if $B^3 \subseteq B$.

Example 3.2. Every bi-ideal is a weak bi-ideal, but the converse is not true. For, consider the near-ring N constructed on the Kleins 4-group according to the scheme (0,0,2,1) (see [3, p. 408]). In this near-ring, one can check that $\{0, b\}$ and $\{0, c\}$ are weak bi-ideals. Note that $\{0, b\}N\{0, b\} = \{0, c, b\}$ and hence $\{0, b\}$ is not a bi-ideal of N.

Proposition 3.3. *The set of all weak bi-ideals of a near-ring N form a Moore system on N.*

Proposition 3.4. *If B is a weak bi-ideal of a near-ring N and S is a sub near-ring of N, then $B \cap S$ is a weak bi-ideal of N.*

Proof. Let $C = B \cap S$. Now $C^3 = (B \cap S)((B \cap S)(B \cap S)) \subseteq (B \cap S)(BB \cap SS) \subseteq (B \cap S)BB \cap (B \cap S)SS \subseteq BBB \cap SSS = B^3 \cap SS \subseteq B \cap S = C$, i.e., $C^3 \subseteq C$. Therefore C is a weak bi-ideal of N. □

Proposition 3.5. *Let B be a weak bi-ideal of N. Then Bb and $b'B$ are the weak bi-ideals of N where $b, b' \in B$ and b' is a distributive element.*

Proof. Clearly Bb is a subgroup of $(N,+)$. Also $(Bb)^3 = BbBbBb \subseteq BBBb \subseteq B^3b \subseteq Bb$. Since b' is distributive, $b'B$ is a subgroup of $(N,+)$ and $(b'B)^3 = b'Bb'Bb'B \subseteq b'BBB = b'B^3 \subseteq b'B$. Hence Bb and $b'B$ are weak bi-ideals of N. □

Corollary 3.6. *Let B be a weak bi-ideal of N. For $b, c \in B$, if b is distributive, then bBc is a weak bi-ideal of N.*

Proposition 3.7. *Let N be a left self-distributive left-unital near-ring. Then $B^3 = B$ for every weak bi-ideal B of N if and only if N is strongly regular.*

Proof. Let B be a weak bi-ideal of N. If N is strongly regular, then N has no non-zero nilpotent elements. This further implies that N has IFP. Let $x \in N$ and $x = ax^2$ for $a \in N$. Now $(xax - x)x = 0$ and so $x(xax - x) = 0$ as N has IFP. Hence $(xax - x)^2 = 0$ and so $xax - x = 0$, i.e., x is regular and N is regular. Let $b \in B$. Since N is regular, $b = bab$ for some $a \in N$. By our assumption that N is left self-distributive, we have $bab = babb$. Thus $b = bab = babb = babb^2 = bb^2 = b^3 \subseteq B^3$, i.e., $B \subseteq B^3$. Hence $B = B^3$ for every weak bi-ideal B of N. Conversely let $a \in N$. Since Na is a weak bi-ideal of N and N is a left-unital near-ring, we get $a \in Na = (Na)^3 = NaNaNa \subseteq NaNa$, i.e., $a = n_1an_2a$. Since N is left self-distributive, $a = n_1an_2a^2$, i.e., N is strongly regular. □

Proposition 3.8. *Let N be a left self-distributive left unital near-ring. Then $B = NB^2$ for every strong bi-ideal B of N if and only if N is strongly regular.*

Proof. Assume that $B = NB^2$ for every strong bi-ideal B of N. Since Na is a strong bi-ideal of N and N is a left unital near-ring, we have $a \in Na = N(Na)^2 = NNaNa \subseteq NaNa$, i.e., $a = n_1an_2a$. Since N is a left self-distributive near-ring, $a = n_1an_2a = n_1an_2a^2 \in Na^2$, i.e., N is strongly regular. Conversely, let B be a strong bi-ideal of N. Since N is strongly regular, for $b \in B$, $b = nb^2 \in NB^2$, i.e., $B \subseteq NB^2$. Hence $NB^2 = B$ for every strong bi-ideal B of N □

Theorem 3.9. *Let N be a left self-distributive left unital near-ring. Then $B^3 = B$ for every weak bi-ideal B of N if and only if $NB^2 = B$ for every strong bi-ideal B of N.*

Proof. Follows from Propositions 3.7 and 3.8. □

Proposition 3.10. *Let N be a left self-distributive left-unital near-ring. Then $B = BNB$ for every bi-ideal B of N if and only if N is regular.*

Proof. Let B be a bi-ideal of N. If N is regular, then $B = BNB$ for every bi-ideal B of N. Conversely, let $B = BNB$ for every bi-ideal B of N. Since Na is a bi-ideal of N and N is a left-unital near-ring, we have $a \in Na = NaNNa$, i.e., $a = n_1an_2a$ for some $n_1, n_2 \in N$. Since N is a left self-distributive near-ring, $a = n_1an_2a^2 \subseteq Na^2$, i.e., N is strongly regular and as in the proof of Proposition 3.7, N is regular. □

Proposition 3.11. *Let N be a left self-distributive left-unital near-ring. Then $B = B^3$ for every weak bi-ideal B of N if and only if $A \cap C = AC$ for any two left N-subgroups A and C of N.*

Proof. Assume that $B = B^3$ for every weak bi-ideal B of N. By the Proposition 3.7, N is strongly regular. Therefore N is regular. Let A and C be any two left N-subgroups of N. Let $x \in A \cap C$. Since N is regular, $x = xax$ for some $a \in N$. Therefore $(xa)x \in ANC \subseteq AC$ which implies that $A \cap C = AC$. On the other hand, let $x \in AC$. Since N is strongly regular, $L = 0$ and so $en = ene$ for all $e \in E$. Then $x = yz \in AC$ with $y \in A$ and $z \in C$. Now $x = yz = (yby)z$. Since by is an idempotent element $(by)z = (by)z(by)$. Thus $x = yz = y(by)z = y(by)z(by) \in NA \subseteq A$. Thus $x \in A \cap C$. From the two inclusions proved above, we get that $AC = A \cap C$.

Conversely let $a \in N$. Since Na is a left N-subgroup of N, from the assumption we get that $Na = Na \cap Na = NaNa$. But $Na = Na \cap N = NaN$ implies that $Naa = NaNa$. Therefore $Na = Na^2$. Since N is a left-unital near-ring, $a \in Na = Na^2$, i.e., N is strongly regular. By the Proposition 3.7, $B = B^3$ for every weak bi-ideal B of N. □

Theorem 3.12. *Let N be a left self-distributive left unital near-ring. Then the following conditions are equivalent:*

(i) *$B = B^3$ for every weak bi-ideal B of N.*
(ii) *N is regular and $NxNy = NyNx$ for all $x, y \in N$.*
(iii) *$NxNy = Nxy$ for all $x, y \in N$.*
(iv) *N is left bi-potent.*
(v) *N is Boolean.*

Proof. (i)⇒(ii) Assume that $B = B^3$ for every weak bi-ideal B of N. By Proposition 3.7, N is strongly regular and so N is regular. Again by Proposition 3.11, $A \cap B = AB$

for two left N-sub groups A and B of N. Let $x, y \in N$. Since Nx and Ny are left N-sub groups of N, from the above fact we get that $NxNy = Nx \cap Ny = Ny \cap Nx = NyNx$.

(ii)$\Rightarrow$(iii) Let $x, y \in N$. Let A be a left N-subgroup of N. Trivially, $A^2 \subseteq A$. Since N is regular, for any $a \in N, a = aba$ for some $b \in N$. Hence $a = a(ba) \in A(NA) \subseteq AA = A^2$. Thus $A = A^2$. Since $Nx \cap Ny$ is a left N-sub group of N, $Nx \cap Ny = (Nx \cap Ny)^2 \subseteq NxNy \subseteq Ny$. Again by the assumption, $NxNy = NyNx \subseteq Nx$. Therefore $Nx \cap Ny = NxNy$. Now $Nx = Nx \cap N = NxN$ and from this we get that $Nxy = NxNy$. Therefore $Nxy = Nx \cap Ny$ for all $x, y \in N$.

(iii)$\Rightarrow$(iv) Let $a \in N$. Then $Na = Na \cap Na = Naa = Na^2$, i.e., N is left bi-potent near-ring.

(iv)$\Rightarrow$(v) By the assumption that $a \in Na = Na^2, N$ is strongly regular and so N is regular. Let $x \in N$. Then $x = xyx = xyxx = x^2$, i.e., N is Boolean.

(v)$\Rightarrow$(vi) Let B be a weak bi-ideal of N. Let $x \in B$. By the assumption, $x = x^2 = x^3 \in B^3$. Therefore $B \subseteq B^3$ and hence $B = B^3$.

□

Theorem 3.13. *Let N be a left self-distributive left unital near-ring. Then the following conditions are equivalent:*

(i) *$Q = QNQ$ for every quasi-ideal Q of N.*
(ii) *$B = B^3$ for every weak bi-ideal B of N.*
(iii) *$NB^2 = B$ for every strong bi-ideal B of N.*
(iv) *N is regular.*
(v) *$B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for every pair of bi-ideals B_1, B_2 of N.*
(vi) *$Q_1 \cap Q_2 = Q_1Q_2 \cap Q_2Q_1$ for every pair of quasi-ideals Q_1, Q_2 of N.*
(vii) *$Q^2 = Q$ for every quasi-ideal Q of N.*
(viii) *$B^2 = B$ for every bi-ideal B of N.*
(ix) *N is strongly regular.*
(x) *N is left bi-potent.*
(xi) *$B = BNB$ for every bi-ideal B of N.*

Proof. (i)$\Rightarrow$(ii) Let $a \in N$. Since Na is a quasi-ideal of N and N is a left-unital near-ring, we have $a \in Na = NaNNa \subseteq NaNa$, i.e., $a = n_1an_2a$. Since N is left self-distributive, $a = n_1an_2a = n_1an_2a^2 \in Na^2$, i.e., N is strongly regular. Therefore, by Proposition 3.7, $B^3 = B$ for every weak bi-ideal B of N.

(ii)$\Rightarrow$(iii) Let B be a strong bi-ideal of N. Every strong bi-ideal is a bi-ideal and so

weak bi-ideal. By the assumption $B = B^3 = BBB = BB^2 \subseteq NB^2$. i.e., $B \subseteq NB^2$ and so $B = NB^2$ for every strong bi-ideal B of N.
(iii)⇒(iv) By Proposition 3.8, N is strongly regular and so N is regular.
(iv)⇒(v) Let B_1 and B_2 be a pair of bi-ideals of N. Let $x \in B_1B_2 \cap B_2B_1$. Then $x = b_1b_2$ and $x = c_2c_1$ for some $b_1, c_1 \in B_1$ and $b_2, c_2 \in B_2$. Since N is regular, $b_1 = b_1a_1b_1$ and $b_2 = b_2a_2b_2$ for some $a_1, a_2 \in N$. From this $x = b_1b_2 = b_1a_1b_1b_2 = b_1a_1c_2c_1 \in B_1NB_1 \subseteq B_1$. i.e., $B_1B_2 \cap B_2B_1 \subseteq B_1$. Similarly $B_1B_2 \cap B_2B_1 \subseteq B_2$. Hence $B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2$. On the other hand if $x \in B_1 \cap B_2$, then $x = b_1 = b_2$ for some $b_1 \in B_1$ and $b_2 \in B_2$. Since B_1 is a bi-ideal and N is regular, $B_1 = B_1NB_1$, and so $b_1 = b_1nb_1$ for some $n \in N$. Since N is a left self-distributive near-ring, $x = b_1 = b_1nb_1 = b_1nb_1b_1 = b_1nb_1b_2 \in B_1NB_1B_2 \subseteq B_1B_2$. Therefore $B_1 \cap B_2 \subseteq B_1B_2$. Similarly one can prove that $B_1 \cap B_2 \subseteq B_2B_1$. Hence $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for every pair of bi-ideals B_1, B_2 of N.
(v)⇒(vi) Since every quasi-ideal is also a bi-ideal of N, we have $Q_1 \cap Q_2 = Q_1Q_2 \cap Q_2Q_1$ for every pair of quasi-ideals Q_1 and Q_2 of N.
(vi)⇒(vii) Take $Q_1 = Q_2 = Q$. By the assumption $Q = Q \cap Q = Q^2 \cap Q^2 = Q^2$. Hence $Q^2 = Q$ for every quasi-ideal Q of N.
(vii)⇒(viii) Let B be a bi-ideal of N and $x \in B$. Since N is a left unital near-ring and Nx is a quasi-ideal of N, $x \in Nx = (Nx)^2 = NxNx$, i.e., $x = n_1xn_2x$ for some $n_1, n_2 \in N$. Since N is a left self-distributive near-ring, $x = n_1xn_2x = n_1xn_2x^2 \in Nx^2$. From this N is strongly regular and so N is regular. Let $a \in BN \cap NB$. Then $a = xn = n_1x_1$ for some $x, x_1 \in B$ and $n, n_1 \in N$. Since x is regular, $x = xyx$ for some $y \in N$. Hence $a = xn = (xyx)n = (xy)(n_1x_1) \in BNB \subseteq B$, i.e., $BN \cap NB \subseteq B$. Therefore B is a quasi-ideal of N and so $B^2 = B$ for every bi-ideal B of N.
(viii)⇒(ix) For $x \in N$, Nx is a bi-ideal of N. Since N is a left unital near-ring $x \in Nx = (Nx)^2$, i.e., $x = n_1xn_2x$ for some $n_1, n_2 \in N$. Since N is a left self-distributive near-ring, $x = n_1xn_2x = n_1xn_2x^2 \in Nx^2$, i.e., N is strongly regular.
(ix)⇒ (x) From the assumption, N is regular. Let $a \in N, a = aba$ for some $b \in N$. For $x \in Na, x = na = naba = nabaa \in Na^2$. From this $Na = Na^2$, i.e., N is left bi-potent.
(x)⇒(xi) Since N is a left unital near-ring, $a \in Na = Na^2$, N is strongly regular and so N is regular. Hence $B = BNB$ for every bi-ideal B of N.
(xi)⇒(i) Let Q be a quasi-ideal of N. Since every quasi-ideal is a bi-ideal of N, $Q = QNQ$. □

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