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A STUDY ON FUZZY TOPOLOGY ASSOCIATED WITH A LATTICE

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ABSTRACT. In this paper we define a topology (analogous to Chang-type fuzzy topology) and a fuzzy topology (analogous to Höhle-type fuzzy topology) associated with a lattice and study some of their properties.

0. INTRODUCTION

After the introduction of fuzzy subsets by Lotfi Zadeh [20], various generalizations of the concept of fuzzy subsets such as L-fuzzy sets, rough sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, intuitionistic fuzzy rough sets etc. were studied by many authors. For references see [1], [4], [10], [15], etc. On the otherhand, Chang [2] was the first to introduce the concept of a fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X, where he referred to each member of T as an open set. In his definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset was absent. The fundamental idea of a fuzzy topology with fuzziness in the topology, i.e., a topology being a fuzzy subset of a power set was first appeared in a paper of Höhle [6] in 1980. Subsequently, different authors such as Kubiak [7], Šostak [16], Samanta et. al. [12], Ying [18], Ramadan [11] developed this idea idependently. Fuzzy topologies were also studied in intuitionistic fuzzy sets [3], interval-valued fuzzy sets [13], interval-valued intuitionistic fuzzy sets [14], L-fuzzy sets, etc. It is to be noted that the collections of all fuzzy subsets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, rough sets, intuitionistic fuzzy rough sets, L-fuzzy sets etc. form lattices w.r.t. the inclusion relation. This initiates us to study topological structure (both Chang-type and Höhle-type fuzzy topology) on a

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lattice so that fuzzy topologies on various types of fuzzy sets could be obtained as a particular choice of the lattice. In this connection it is worth mentioning the work of G. Nöbeling [9] and P. Hamburg & L. Florescu [5].

In Section 1 of this paper, we give some preliminary results on a lattice.

In Section 2, we define a topology (analogous to Chang-type fuzzy topology) associated with a lattice and study its properties.

In Section 3, we define a fuzzy topology (analogous to Höhle-type fuzzy topology) associated with a lattice and study different properties on this structure.

In Section 4, we define subspaces of a fuzzy topological lattice space and study its properties.

In Section 5, we define gradation preserving maps and study its properties.

In Section 6, we define fuzzy lattice closure of an element in a fuzzy topological lattice space and establish some characteristic properties of the fuzzy lattice closure operator.

In Section 7, we show that the category of topological lattice spaces and continuous maps is a bireflective full subcategory of the category of fuzzy topological lattice spaces and gp-maps in our sense. Further, it is observed that the category of fuzzy topological lattice spaces and gp-maps in our sense is a topological category.

In Section 8, the concept of fuzzy gradation of openness as introduced by us [17] is developed in this lattice setting.

1. NOTATIONS AND PRELIMINARIES

Let $(S, \leq)$ be a lattice with 0_S and 1_S as the least element and the greatest element respectively and $'$ be an order-reversing involution. For all $a \in S$, a' is called the complement of a. Denote by S_c the set $\{b(\neq 0_S) \in S \ ; \ b \wedge b' = 0_S\}$. For $a, b \in S$, define $a < b$ if $a \leq b$ and $a \neq b$.

Definition 1.1. A lattice S is said to be *dense* if for any $r, s \in S$ with $r < s$, $\exists t \in S$ such that $r < t < s$.

Theorem 1.2. *If a complete lattice S is dense then $r = \vee\{s; s < r\}$.*

Proof. Let $r_o = \vee\{s; s < r\}$. Then $r_o \leq r$. If possible, let $r_o \neq r$. Then $r_o < r$. By denseness of S, $\exists s \in S$ s.t. $r_o < s < r$, a contradiction, since $r_o = \vee\{s; s < r\}$. Hence $r = \vee\{s; s < r\}$. □

For a complete lattice S and for $P \subset S$, denote $\vee\{a; a \in P\}$ by $\vee P$.

Definition 1.3. Let S be a complete lattice. Then S is said to possess *sup property* if for any $P \subset S$, $\vee P > s \Rightarrow \exists\, p \in P$ s.t. $p > s$.

Theorem 1.4. *Let S be complete and dense, $r_o, t_o \in S$. If $s < t_o \Rightarrow s \leq r_o$ then $r_o \geq t_o$.*

Proof. Since $s < t_o \Rightarrow s \leq r_o$, it follows that r_o is an upper bound of $\{s; s < t_o\}$. Since S is dense, $t_o = \vee\{s; s < t_o\}$. So, $t_o \leq r_o$. □

Definition 1.5. Let S be complete and for $a\ (\neq 0_S) \in S$, let $\mathcal{S}(a) = \wedge\{b \geq a; b \wedge b' = 0_S\}$. Then $\mathcal{S}(a)$ is called the *support* of a.

Note 1.6. If S is complete then $\mathcal{S}(a) \geq a$.

Definition 1.7 ([19]). Let S be a complete lattice. S is called *infinitely distributive* (briefly ID), if S satisfies both following two conditions, called the 1st infinitely distributive law and the 2nd infinitely distributive law respectively:
(ID1) $\forall a \in S,\ \forall B \subset S,\ a \wedge (\vee B) = \vee_{b \in B}(a \wedge b)$,
(ID2) $\forall a \in S,\ \forall B \subset S,\ a \vee (\wedge B) = \wedge_{b \in B}(a \vee b)$.

Note 1.8. If S is complete and ID then $\mathcal{S}(a) \in S_c$.

Definition 1.9. $a \in S_c$ is said to be a *minimal element* if $b \in S_c$, $b \leq a$ implies $b = a$. The set of all minimal elements is denoted by $X(S)$.

Definition 1.10. Let S be complete and ID. Then $a\ (\neq 0_S) \in S$ is said to be a *fuzzy point* (briefly FP) of S if $\mathcal{S}(a) \in X(S)$. The set of all FPs of S is denoted by $P(S)$.

Remark 1.11. Let S be complete and ID. Then $X(S) \subset P(S)$. In fact $a \in X(S) \Rightarrow a \in S_c \Rightarrow a = \mathcal{S}(a) \Rightarrow a \in P(S)$.

Definition 1.12. Let S be complete. Then $X(S)$ is said to be *generative* if $b \in S_c \Rightarrow \exists A = \{p_i; i \in \Delta\} \subset X(S)$ such that $b = \vee_{i \in \Delta} p_i$.

Definition 1.13. Let S be complete and ID. Then S is said to be *fuzzy point generative* (briefly FP-generative) if for $a\ (\neq 0_S) \in S$, $\exists\ \{b_i; i \in \Delta\} \subset P(S)$ such that $a = \vee_{i \in \Delta} b_i$.

Theorem 1.14. *Let S be complete and ID. If $X(S)$ is generative then S is FP-generative.*

Proof. $a\ (\neq 0_S)\ \in S \Rightarrow \mathcal{S}(a) \in S_c \Rightarrow \exists\{p_i; i \in \Delta\} \subset X(S)$ such that $\mathcal{S}(a) = \vee_{i\in\Delta} p_i$ (since $X(S)$ is generative). Then $a = a \wedge \mathcal{S}(a) = \vee_{i\in\Delta}(a \wedge p_i) = \vee_{i\in\Delta} b_i$, where $b_i = a \wedge p_i,\ i \in \Delta$. Without loss of generality, assume $b_i \neq 0_S$. Now, for $i \in \Delta,\ b_i \leq p_i$ and $p_i \in X(S) \subset S_c$. So $\mathcal{S}(b_i) \leq p_i,\ i \in \Delta$, since $\mathcal{S}(b_i) \neq 0_S$ and $p_i \in X(S)$, it follows that $\mathcal{S}(b_i) = p_i \in X(S)$, which implies $b_i \in P(S),\ i \in \Delta$. Hence S is FP-generative.

Definition 1.15. A fuzzy point p is said to *belong to* $a \in S$ if $p \leq a$. We denote it by $p\ \tilde{\in}\ a$.

Remark 1.16. If S is complete, ID and $X(S)$ is generative, then for $a, b \in S,\ a = b$ iff $p\ \tilde{\in}\ a \Leftrightarrow p\ \tilde{\in}\ b$.

Definition 1.17. A FP a is said q.c. to $b \in S$ if $a \not\leq b'$. It is denoted by aqb. The set $\{a \in P(S); aqb\}$ is denoted by $Q(b)$.

Definition 1.18. Let S be a lattice and $\alpha \in S$. Then α is called *coprime* if $\alpha > 0_S$ and $\forall a, b \in S,\ \alpha \leq a \vee b \Rightarrow \alpha \leq a$ or $\alpha \leq b$.

Theorem 1.19. *Let S be complete and ID. Then*

(1) $aq1_S,\ a \not{q} 0_S,\ \forall\ a \in P(S)$,
(2) $aqb, b \leq c \Rightarrow aqc,\ \forall\ a \in P(S),\ \forall\ b, c \in S$,
(3) $aqb_1, aqb_2 \Rightarrow aq(b_1 \wedge b_2), \forall\ a \in P(S)$ *such that a is coprime and* $b_1, b_2 \in S$.

The proof is straightforward.

Definition 1.20. Let S and S_1 be two complete lattices and $f : S \to S_1$ be a mapping such that $f(a) = 0_{S_1}$ iff $a = 0_S$. Define $\hat{f}^{-1} : S_1 \to S$ by $\hat{f}^{-1}(b) = \vee\{a \in S; f(a) \leq b\}$. If $b \notin \mathrm{Img}(S)$, then define $\hat{f}^{-1}(b) = 0_S$. f is called *order-preserving* if $\forall\ a, b\ \in S,\ a \leq b \Rightarrow f(a) \leq f(b)$. f is called *arbitrary join-preserving* if $\forall\ a_i \in S,\ i \in \Delta,\ f(\vee_{i\in\Delta} a_i) = \vee_{i\in\Delta} f(a_i)$. $\hat{f}^{-1}$ is called *complement preserving* if $\forall\ b\ \in S_1,\ \hat{f}^{-1}(b') = (\hat{f}^{-1}(b))'$. $\hat{f}^{-1}$ is called *arbitrary join-preserving (finite meet-preserving)* if $\hat{f}^{-1}(\vee_{i\in\Delta} b_i) = \vee_{i\in\Delta} \hat{f}^{-1}(b_i),\ \forall\ b_i \in S_1,\ i \in \Delta\ (\hat{f}^{-1}(\wedge_{i=1}^{n} b_i) = \wedge_{i=1}^{n} \hat{f}^{-1}(b_i),\ \forall\ b_i \in S_1,\ i = 1, 2, ..., n)$.

Theorem 1.21. *Let S and S_1 be two complete lattices and $f : S \to S_1$ be a mapping such that $f(a) = 0_{S_1}$ iff $a = 0_S$. Then we have*

(1) $f\hat{f}^{-1}(b) \leq b,\ \forall b \in S_1,$ *if* f *is arbitrary join-preserving,*
(2) $a \leq \hat{f}^{-1}f(a),\ \forall\ a \in S,$
(3) $b_1 \leq b_2 \Rightarrow \hat{f}^{-1}(b_1) \leq \hat{f}^{-1}(b_2),\ \forall\ b_1, b_2 \in S_1,$
(4) $f(\wedge_{i\in\Delta} a_i) \leq \wedge_{i\in\Delta} f(a_i),\ \forall\ a_i \in S,\ i \in \Delta,$ *if* f *is order-preserving,*
(5) $\hat{f}^{-1}(1_{S_1}) = 1_S,\ \hat{f}^{-1}(0_{S_1}) = 0_S.$

The proof is straightforward.

2. Topology Associated with a Lattice

In this section we shall define topology associated with a lattice and study some of its properties.

Definition 2.1. Let S be a complete lattice and $T = \{a_i \in S\ ;\ i \in \Delta\}$ be a subcollection of S. Then T is said to *form a topology* if the following three axioms hold:

(1) $0_S,\ 1_S \in T,$
(2) $a_i \in T,\ i \in \Delta \Rightarrow \vee_{i\in\Delta} a_i \in T,$
(3) $a,\ b \in T \Rightarrow a \wedge b \in T.$

T is called a *topology associated with* S and (S, T) is called a *topological lattice space* (briefly TL-space). Each element of T is called an *open member*. An element of S is called a *closed member* with respect to T if its complement belongs to T. We shall denote the collection of all topologies associated with S by $C(S)$. If $T_1,\ T_2 \in C(S)$, then we say T_1 is *coarser* than T_2 or say T_2 is *finer* than T_1 if $T_1 \subset T_2$.

Remark 2.2. This topology is analogous to Chang-type fuzzy topology. In fact if S = Lattice of all fuzzy subsets of a nonempty set X with respect to inclusion relation '$\leq$' then T is a Chang-type fuzzy topology.

Definition 2.3. Let $T \in C(S)$. A subcollection B of T is said to be a *base* for T if $a \in T$ can be expressed as $a = \vee_{i\in\Delta} a_i,\ a_i \in B,\ \forall\ i \in \Delta.$

Definition 2.4. Let $T \in C(S)$. A subcollection S of T is said to be a *subbase* for T if the family of all finite meets of members of S forms a base for T.

Definition 2.5. Let $T \in C(S)$. Then the *closure* of $a \in S$, denoted by $cl(a)$, is defined by

$$cl(a) = \wedge\{b \ ; \ b \geq a, \ a' \in T\}.$$

Theorem 2.6. *Let $T \in C(S)$. Then we have*

(1) $cl(0_S) = 0_S, \ cl(1_S) = 1_S$,
(2) $cl(a) \geq a, \ \forall a \in S$,
(3) $cl(a \vee b) = cl(a) \vee cl(b)$,
(4) $cl(cl(a)) = cl(a)$.

The proof is straightforward.

Definition 2.7. Let the lattice S be complete and ID. Let $T \in C(S), \ p \in P(S)$, $a \in S$. Then a is said to be a *neighborhood* (briefly nbd) of p if $\exists \ g \in T$ such that $p\tilde{\in}g \leq a$. The collection of all nbds of p is denoted by $N(p)$.

Theorem 2.8. *Let the lattice S be complete, ID, FP-generative and $T \in C(S)$. Then $a \in T$ iff $a \in N(p), \ \forall \ p\tilde{\in}a$.*

Proof. Obviously $a \in T \Rightarrow a \in N(p), \ \forall \ p\tilde{\in}a$. Conversely, suppose $a \in N(p), \ \forall \ p\tilde{\in}a$. Then $\forall \ p\tilde{\in}a, \ \exists \ b_p \in T$ such that $p\tilde{\in}b_p \leq a$. Since S is FP-generative, $a = \vee\{p \in P(S) \ ; \ p\tilde{\in}a\} \leq \vee\{b_p \ ; \ p\tilde{\in}a\} \leq a$. Hence $a = \vee\{b_p \ ; \ p\tilde{\in}a\} \in T$. □

Theorem 2.9. *Let the lattice S be complete, ID, FP-generative and $T \in C(S)$. Let $p \in P(S)$. Then*

(N1) $p\tilde{\in}a, \ \forall \ a \in N(p)$,
(N2) $b \geq a \in N(p) \Rightarrow b \in N(p)$,
(N3) $b_1, \ b_2 \in N(p) \Rightarrow b_1 \wedge b_2 \in N(p)$,
(N4) $b \in N(p) \Rightarrow \exists \ a \in N(p)$ *such that* $a \in N(q), \ \forall \ q\tilde{\in}a$.

The proof is straightforward.

Theorem 2.10. *Let the lattice S be complete, ID, FP-generative and FPs of S be coprime. Let for $p \in P(S)$, $\hat{N}(p)(\neq \phi) \subset S$ be such that (N1)-(N4) of Theorem 2.9 holds. Let*

$$T = \{a \in S \ ; \ a \in \hat{N}(p), \ \forall \ p\tilde{\in}a\}.$$

Then $T \in C(S)$ and for $p \in P(S)$, $a \in \hat{N}(p)$ iff a is a nbd of p with respect to T.

Proof. (1) Obviously $0_S, \ 1_S \in T$.
(2) Let $a_i \in T, \ i \in \Delta$ and $a = \vee_{i\in\Delta}a_i$. Let $p\tilde{\in}a$. Since p is coprime, $p\tilde{\in}a_{i_o}$, for some $i = i_o \in \Delta$. Since $a_{i_o} \in T, \ a_{i_o} \in \hat{N}(p)$ and hence $a \in \hat{N}(p)$. This implies that $a \in T$.

(3) Let $a_i \in T$, $i = 1, 2$ and $a = a_1 \wedge a_2$. Then $p\tilde{\in}a \Rightarrow p\tilde{\in}a_i$, $i = 1, 2 \Rightarrow a_i \in \hat{N}(p)$, $i = 1, 2 \Rightarrow a = a_1 \wedge a_2 \in \hat{N}(p) \Rightarrow a \in T$. Hence $T \in C(S)$. Clearly $a \in \hat{N}(p)$ iff a is a nbd of p with respect to T. □

Definition 2.11. Let S be a complete, ID lattice and $T \in C(S)$. Then $b \in S$ is said to be a *q-nbd* of $p \in P(S)$ if $\exists\ c \in T$ such that $pqc \leq b$. The set of all q-nbds of p is denoted by $N_q(p)$.

Theorem 2.12. *Let the lattice S be complete, ID, FP-generative and $T \in C(S)$. Then*

($\bar{N}1$) $1_S \in N_q(p)$, $0_S \notin N_q(p)$, $\forall\ p \in P(S)$,
($\bar{N}2$) $c \geq b \in N_q(p) \Rightarrow c \in N_q(p)$,
($\bar{N}3$) $b_1, b_2 \in N_q(p) \Rightarrow b_1 \wedge b_2 \in N_q(p)$, *if p is coprime,*
($\bar{N}4$) $c \in N_q(p) \Rightarrow \exists\ b(\leq c)$ *q.c. to p such that* $b \in N_q(t)$, $\forall\ t \in P(S)$ *q.c. to b.*

The proof is straightforward.

Theorem 2.13. *Let the lattice S be complete, ID, FP-generative and $a \in P(S)$ is coprime. Let for each $p \in P(S)$, $\hat{N}_q(p)(\neq \phi) \subset S$ satisfying $(\bar{N}1) - (\bar{N}4)$ of Theorem 2.12. Let*

$$T = \{a \in S\ ;\ a \in \hat{N}_q(p),\ \forall\ p \in P(S) \text{ q.c. to } a\}.$$

Then $T \in C(S)$ and $a \in \hat{N}_q(p)$ iff a is a q-nbd of p with respect to T.

Proof. (1) Obviously 0_S, $1_S \in T$.
(2) Let $a_i \in T$, $i \in \Delta$ and $a = \vee_{i \in \Delta} a_i$. Then $p \in Q(a) \Rightarrow p \in Q(a_{i_o})$, for some $i = i_o \in \Delta \Rightarrow a_{i_o} \in N_q(p)$, since $a_{i_o} \in T \Rightarrow a \in N_q(p)$, by $(\bar{N}2) \Rightarrow a\ (= \vee_{i \in \Delta} a_i) \in T$. Hence $T \in C(S)$.
(3) Let $a_1, a_2 \in T$ and $a = a_1 \wedge a_2$. Let $p \in Q(a)$. Then $p \in Q(a_i)$, $i = 1, 2$. Since $a_i \in T$, $a_i \in \hat{N}_q(p)$, $i = 1, 2$ and hence $a_1 \wedge a_2 \in \hat{N}_q(p)$ (by $(\bar{N}3)$) which implies $a_1 \wedge a_2 = a \in T$.
Obviously $a \in \bar{N}_q(p)$ iff a is a q-nbd of p with respect to T. □

Definition 2.14. Let (S_1, T_1) and (S_2, T_2) be two TL-spaces and $f : S_1 \to S_2$ be a mapping such that $f(a) = 0_{S_2}$ iff $a = 0_{S_1}$. Then f is said to be *continuous* if $\hat{f}^{-1}(b) \in T_1$, $\forall b \in T_2$.

Theorem 2.15. *Let (S_1, T_1) and (S_2, T_2) be two TL-spaces and $f : S_1 \to S_2$ be a mapping. Then the following statements are equivalent.*

(1) *The mapping f is continuous,*
(2) $\hat{f}^{-1}(b)$ *is a closed member in* S_1, $\forall$ *closed member* b *in* S_2,
(3) $f(cl(a)) \subset cl(f(a))$.

The proof is straightforward.

From now on, by the lattices S and L we mean two complete and ID lattices with 1_S, 1_L respectively as their greatest elements and 0_S, 0_L respectively as their least elements and $' : S \to S$, $c : L \to L$ be order reversing involutions. By L_0 we mean $L \setminus \{0_L\}$.

3. Gradation of Openness Associated with a Lattice

Definition 3.1. A mapping $\tau : S \to L$ satisfying the following three axioms is said to be a *gradation of openness* (shortly GO) or fuzzy topology associated with S:

(GO1) $\tau(1_S) = \tau(0_S) = 1_L$,
(GO2) $\tau(\vee_i a_i) \geq \wedge_i \tau(a_i)$, $\forall a_i \in S$, $i \in \Delta$,
(GO3) $\tau(a \wedge b) \geq \tau(a) \wedge \tau(b)$, $a, b \in S$.

For $a \in S$, $\tau(a)$ is called the *degree of openness* of a. (S, τ) is called a *fuzzy topological lattice space* (shortly FTL-space). Denote the collection of all GOs associated with S by $m(S)$.

Definition 3.2. Let $\tau_1, \tau_2 \in m(S)$. Then τ_1 is said to be *stronger* than τ_2 if $\tau_1(a) \geq \tau_2(a)$, $\forall a \in S$.

Definition 3.3. Let $F : S \to L$ be a mapping satisfying the conditions:

(GC1) $F(1_S) = F(0_S) = 1_L$,
(GC2) $F(\wedge_i a_i) \geq \wedge_i F(a_i)$, $\forall a_i \in S$, $i \in \Delta$,
(GC3) $F(a \vee b) \geq F(a) \wedge F(b)$, $a, b \in S$.

Then F is called a *gradation of closedness* (shortly GC) associated with S.

Theorem 3.4. *Let τ (F) be a GO (GC) associated with S. Let $\tau_F : S \to L$ $(F_\tau : S \to L)$ be such that $\tau_F(a) = F(a')$ $(F_\tau(a) = \tau(a'))$, $\forall a \in S$. Then τ_F (F_τ) is a GO (GC) associated with S.*

The proof is straightforward.

Corollary 3.5. *Let τ and F be GO and GC associated with S respectively. Then* $\tau_{F_\tau} = \tau, \ F_{\tau_F} = F$.

Theorem 3.6. *Arbitrary intersection of GOs is a GO.*

The proof is straightforward.

Note 3.7. Define $\tau_0, \ \tau_1 : S \to L$ by $\tau_0(1_s) = \tau_0(0_S) = 1_L, \ \tau_0(a) = 0_L, \ \forall a \in S \setminus \{0_S, 1_S\}$ and $\tau_1(a) = 1_L, \ \forall a \in S$. Then $\tau_0, \ \tau_1 \in m(S)$ such that for any $\tau \in m(S), \ \tau_0(a) \leq \tau(a) \leq \tau_1(a), \ \forall a \in S$.

Theorem 3.8. *$(m(S), \leq)$ is a complete lattice with τ_0 and τ_1 as the smallest and the greatest element.*

The proof follows from Theorem 3.6 and Note 3.7.

Let $\tau \in m(S)$ and $r \in L_0$. Define $\tau_r = \{a \in S \ ; \ \tau(a) \geq r\}$.

Theorem 3.9. *τ_r is a topology associated with S.*

The proof is straightforward.

Definition 3.10. For each $r \in L_0, \ \tau_r$ is called *r-level topology* associated with S.

Theorem 3.11. *Let F be a GC associated with S. Define $F_r = \{a \in S; F(a) \geq r\}, \ r \in L_0$. Then F_r satisfies*

(1) $0_S, \ 1_S \in F_r$,
(2) $a_i \in F_r \Rightarrow \wedge_i a_i \in F_r, \ \forall i \in \Delta, \ a_i \in S$,
(3) $a_1, a_2 \in F_r \Rightarrow a_1 \vee a_2 \in F_r, \ \forall a_1, a_2 \in S$.

The proof is straightforward.

Theorem 3.12. *Let τ and F be GO and GC associated with S, respectively, such that $F(a) = \tau(a'), \ \forall a \in S$. Then $a \in \tau_r$ iff $a' \in F_r, \ \forall r \in L$.*

The proof is straightforward.

Theorem 3.13. *Let $\tau \in m(S)$, L be dense and $\{\tau_r \ ; \ r \in L_0\}$ be the family of all r-level topology associated with S. Then this family is a descending family and for each $r \in L_0, \ \tau_r = \cap_{s<r} \tau_s$.*

Proof. If $r \geq s$, then obviously, $\tau_r \subset \tau_s$. Hence $\{\tau_r \ ; \ r \in L_0\}$ is a descending family of topologies associated with S. Clearly, $\tau_r \subset \cap_{s<r} \tau_s, \ r \in L_0$.

Also, if $a \notin \tau_r$, then $\tau(a) < r$ implies $\exists\ s \in L_0$ such that $\tau(a) < s < r$ (by denseness of L). So, $a \notin \tau_s$, for some $s < r$. Hence $a \notin \cap_{s<r}\tau_s$. Consequently $\cap_{s<r}\tau_s \subset \tau_r$. Hence $\tau_r = \cap_{s<r}\tau_s$. □

Theorem 3.14. *Let $\{T_r\ ;\ r \in L_0\}$ be a descending family of topologies associated with S and let $\tau : S \to L$ be a mapping defined by*

$$\tau(a) = \vee\{r \in L_0\ ;\ a \in T_r\}.$$

Let L be dense and possess sup property. Then τ is a GO associated with S. If further for any $r \in L_0$, $T_r = \cap_{s<r}T_s$ then $\tau_r = T_r$, $\forall r \in L_0$.

Proof. From the definition of τ, it is clear that $\tau(0_S) = \tau(1_S) = 1_L$. Let $a_i \in S$ and $\tau(a_i) = k_i$, $i = 1,2$. If $k_i = 0_L$ for some i, then obviously, $\tau(a_1 \wedge a_2) \geq \tau(a_1) \wedge \tau(a_2)$. Next suppose, $k_i > 0_L$, $i = 1,2$. Let $s < \tau(a_1) \wedge \tau(a_2)$. Therefore $\vee\{r\ :\ a_i \in T_r\} > s$, $i = 1,2$. Then by sup property, $\exists\ r_i \in L_0$ such that $r_i > s$ and $a_i \in T_{r_i}$, $i = 1,2$. Let $r = r_1 \wedge r_2$. Then $a_i \in T_r$ ($T_r \supset T_{r_1} \cap T_{r_2}$, because $r \leq r_1, r_2 \Rightarrow T_r \supset T_{r_1}, T_{r_2}$) $\Rightarrow a_1 \wedge a_2 \in T_r \Rightarrow \tau(a_1 \wedge a_2) \geq r \geq s$. By Theorem 1.4, $\tau(a_1 \wedge a_2) \geq \tau(a_1) \wedge \tau(a_2)$. Next, let $a_i \in S$ and $l_i = \tau(a_i)$, $i \in \Delta$. Let $l = \wedge_{i\in\Delta} l_i$. If $l = 0_L$, then obviously, $\tau(\vee_{i\in\Delta} a_i) \geq \wedge_{i\in\Delta}\tau(a_i)$. So consider the case when $l > 0_L$. By denseness of L, $\exists\ n \in L$ such that $l > n > 0_L$. $l_i \geq l > n \Rightarrow \vee\{r \in L_0\ ;\ a_i \in T_r\} > n \Rightarrow a_i \in T_{r_i}$ for some $n < r_i \in L$, $i \in \Delta$. Let $r_o = \wedge_{i\in\Delta} r_i$. Then $a_i \in T_{r_o}$, $i \in \Delta$. So, $\vee_{i\in\Delta} a_i \in T_{r_o} \Rightarrow \tau(\vee_{i\in\Delta} a_i) \geq r_o \geq n$. Hence $\tau(\vee_{i\in\Delta} a_i) \geq l = \wedge_{i\in\Delta}\tau(a_i)$ (by Theorem 1.4). Since $a \in T_r \Rightarrow \tau(a) \geq r \Rightarrow a \in \tau_r$, we have $T_r \subset \tau_r$. Next let $a \in \tau_r$. Then $\tau(a) \geq r \Rightarrow \vee\{s \in L_0\ ;\ a \in T_S\} \geq r$. Let $t < r$. Then by denseness of L, $\exists\ s \in L$ such that $s > t$ and $a \in T_S$. So, $a \in T_t$. Therefore, $a \in \cap_{t<r}T_t$. Then, $a \in T_r$. So, $\tau_r \subset T_r$. Hence $T_r = \tau_r$. □

Note 3.15. The GO τ so obtained in the Theorem 3.14 will be referred to as the GO generated by the descending family of topologies $\{T_r\ :\ r \in L_0\}$ associated with S.

Corollary 3.16. *Two GOs τ and τ' associated with S are equal iff $\tau_r = \tau'_r$, $\forall\ r \in L_0$.*

The proof is straightforward.

Theorem 3.17. *Let T be a topology associated with S. Define for each $r \in L_0$, a mapping $T^r : S \to L$ by*

$$T^r(0_S) = T^r(1_S) = 1_L,\ \ T^r(a) = \begin{cases} r & \text{if } a \in T - \{0_S, 1_S\}, \\ 0_L & \text{otherwise} \end{cases}$$

Then T^r is a GO associated with S such that $(T^r)_r = T$.

Proof. (GO1) is obvious. For (GO2), take $a_i \in S$, $i \in \Delta$. Let $a = \vee_{i\in\Delta} a_i$. If $a \notin T$, then $\exists\ i = i_o \in \Delta$ such that $a_{i_o} \notin T$. Thus $T^r(a) = 0_L = \wedge_{i\in\Delta} T^r(a_i)$. If $a \in T$, then either $a \in \{0_S, 1_S\}$ or $a \in T - \{0_S, 1_S\}$. If $a \in \{0_S, 1_S\}$, then obviously $T^r(a) = 1_L \geq \wedge_{i\in\Delta} T^r(a_i)$. If $a \in T - \{0_S, 1_S\}$, then $a_i \neq 0_S$, $\forall\ i \in \Delta$ and $\exists\ i = i_o \in \Delta$ such that $a_{i_o} \neq 1_S$ and hence $a_{i_o} \in T - \{0_S, 1_S\}$. Then $T^r(a) = r \geq T^r(a_{i_o}) \geq \wedge_{i\in\Delta} T^r(a_i)$. Similarly (GO3) can be verified. Thus T^r is a GO associated with S where $(T^r)_r = \{a \in S\ ;\ T^r(a) \geq r\} = T$. □

Definition 3.18. If T is a topology associated with S then T^r is called an *r-th gradation* associated with S.

4. Subspaces

Theorem 4.1. *Let L be dense and satisfies sup property. Let τ be a GO associated with S and $a \in S$. Let*

$$S_a = \{a \wedge b\ ;\ b \in S\}.$$

Then S_a is a complete sublattice of S with a and 0_S respectively as the greatest and the least element. Define a mapping $\tau_{S_a} : S_a \to L$ by

$$\tau_{S_a}(c) = \vee\{\tau(b)\ ;\ c = a \wedge b,\ b \in S\}.$$

Then τ_{S_a} is a GO associated with S_a.

Proof. (GO1) is obvious. For (GO2), let $a_i \in S_a$, $i \in \Delta$ and $l = \wedge_{i\in\Delta} \tau_{S_a}(a_i) = \wedge_{i\in\Delta}(\vee\{\tau(b)\ ;\ a_i = a \wedge b, b \in S\})$. Let $r \in L$ be such that $r < l$. Then $\tau_{S_a}(a_i) > r$, $\forall\ i \in \Delta$. By sup property, $\exists\ b_i \in S$ with $a \wedge b_i = a_i$ such that $\tau(b_i) > r$. Now, $\vee_{i\in\Delta} b_i \in S$ and $a \wedge (\vee_{i\in\Delta} b_i) = \vee_{i\in\Delta}(a \wedge b_i) = \vee_{i\in\Delta} a_i$. Further, $\tau(\vee_{i\in\Delta} b_i) \geq \wedge_{i\in\Delta} \tau(b_i) \geq r$. So, $\tau_{S_a}(\vee_{i\in\Delta} a_i) \geq \tau(\vee_{i\in\Delta} b_i) \geq r$. By Theorem 1.4, $\tau_{S_a}(\vee_{i\in\Delta} a_i) \geq \wedge_{i\in\Delta} \tau_{S_a}(a_i)$. For (GO3), let $s = \tau_{S_a}(a_1) \wedge \tau_{S_a}(a_2)$. Let $r < s$. Then $\exists\ b_i \in S$ such that $a_i = a \wedge b_i$ and $\tau(b_i) > r$, $i = 1, 2$ (by sup property). Now, $b_1 \wedge b_2 \in S$ and $a \wedge (b_1 \wedge b_2) = (a \wedge b_1) \wedge (a \wedge b_2) = a_1 \wedge a_2$. Therefore $\tau_{S_a}(a_1 \wedge a_2) \geq \tau(b_1 \wedge b_2) \geq \tau(b_1) \wedge \tau(b_2) > r$. By Theorem 1.4, $\tau_{S_a}(a_1 \wedge a_2) \geq \tau_{S_a}(a_1) \wedge \tau_{S_a}(a_2)$. Hence τ_{S_a} is a GO associated with S_a. □

Definition 4.2. The FTL-space (S_a, τ_{S_a}) so determined is called a *subspace* of the FTL-space (S, τ) and τ_{S_a} is called the *induced GO* associated with S_a from (S, τ).

Theorem 4.3. *Let L be dense and satisfies sup property. Let F be a GC associated with S and $S_a \subset S$. Define a mapping $F_{S_a} : S_a \to L$ by*

$$F_{S_a}(d) = \vee\{F(b)\ ;\ d = a \wedge b, b \in L\}.$$

Then F_{S_a} is a GC associated with S_a.

The proof is similar to that of Theorem 4.1.

Definition 4.4. Let $a \wedge a' = 0_S$. For $c = a \wedge b \in S_a$, we define the complement of c, denoted by $c^\dagger$, by $c^\dagger = a \wedge b'$.

Property 4.5. The complement of $c \in S_a$ is unique.

Proof. Let $c = a \wedge b_1 = a \wedge b_2$. Let $c_1 = a \wedge b_1'$, $c_2 = a \wedge b_2'$. We shall show that $c_1 = c_2$. Now, $c' = (a \wedge b_1)' = (a \wedge b_2)' = a' \vee b_1' = a' \vee b_2'$. Therefore $a \wedge c' = a \wedge (a' \vee b_1') = a \wedge (a' \vee b_2') = (a \wedge a') \vee (a \wedge b_1') = (a \wedge a') \vee (a \wedge b_2')$. If $a \wedge a' = 0_S$, then $a \wedge c' = a \wedge b_1' = c_1 = a \wedge b_2' = c_2$. This proves the uniqueness of $c^\dagger$. □

Property 4.6. The complementary operator $\dagger$ on S_a is order-reversible.

Proof. Let $c_1 \leq c_2$, where $c_1 = a \wedge b_1$ and $c_2 = a \wedge b_2$. Therefore $a \wedge b_1 \leq a \wedge b_2 \Rightarrow (a \wedge b_1)' \geq (a \wedge b_2)' \Rightarrow a' \vee b_1' \geq a' \vee b_2' \Rightarrow a \wedge (a' \vee b_1') \geq a \wedge (a' \vee b_2') \Rightarrow (a \wedge a') \vee (a \wedge b_1') \geq (a \wedge a') \vee (a \wedge b_2')$. If $a \wedge a' = 0_S$, we get $a \wedge b_1' \geq a \wedge b_2'$. Therefore $c_1^\dagger \geq c_2^\dagger$. Hence $\dagger$ is order-reversible. □

Property 4.7. The complementary operator $\dagger$ on S_a is idempotent.

Proof. Let $c = a \wedge b$. Then $(c^\dagger)^\dagger = (a \wedge b')^\dagger = a \wedge (b')' = a \wedge b = c$. □

Theorem 4.8. *Let τ and F be GO and GC associated with S respectively and let $a \in S$ be such that $a \wedge a' = 0_S$. If $\tau(x) = F(x')$, $\forall\ x \in S$, then $\tau_{S_a}(d) = F_{S_a}(d^\dagger)$ and $F_{S_a}(d) = \tau_{S_a}(d^\dagger)$, $\forall\ d \in S_a$.*

Proof. For $d \in S_a$,

$$\begin{aligned}\tau_{S_a}(d) &= \vee\{\tau(b)\ ;\ d = a \wedge b, b \in S\}\\ &= \vee\{F(b')\ ;\ d = a \wedge b, b \in S\}\\ &= \vee\{F(b')\ ;\ d^\dagger = a \wedge b', b' \in S\}\ (\text{since } d = a \wedge b \Leftrightarrow d^\dagger = a \wedge b')\\ &= \vee\{F(g)\ ;\ d^\dagger = a \wedge g, g \in S\} = F_{S_a}(d^\dagger).\end{aligned}$$

Similarly, $F_{S_a}(d) = \tau_{S_a}(d^{\dagger}),\ \forall\ d \in S_a$. □

5. Gradation preserving maps

Definition 5.1. Let S and S_1 be two complete lattices and $f : S \to S_1$ be a mapping such that $f(a) = 0_{S_1}$ iff $a = 0_S$. Then f is called *join admissible* (*meet admissible*) if $\hat{f}^{-1}$ is arbitrary join-preserving (finite meet-preserving). f is called *admissible* if both these properties hold.

Definition 5.2. Let $f : S \to S_1$ and $g : S_1 \to S_2$ be two admissible mappings. Then the mapping $h : S \to S_2$ defined by $h(a) = g(f(a)),\ a \in S$ is called the *composition* of f with g and is denoted by gof.

Theorem 5.3. *Let $f : S \to S_1$ and $g : S_1 \to S_2$ be two admissible mappings. Then*

(1) $(g\hat{o}f)^{-1} \leq (\hat{f})^{-1}o(\hat{g})^{-1}$,

(2) $(g\hat{o}f)^{-1} = (\hat{f})^{-1}o(\hat{g})^{-1}$, *if g is order-preserving.*

Proof. For $b \in S_2$, $(g\hat{o}f)^{-1}(b) = \vee\{a \in S\ ;\ g(f(a)) \leq b\}$ and $((\hat{f})^{-1}o(\hat{g})^{-1})(b) = (\hat{f})^{-1}(\vee\{c \in S_1\ ;\ g(c) \leq b\}) = \vee\{(\hat{f})^{-1}(c)\ ;\ g(c) \leq b\} = \vee\{\vee\{a \in S\ ;\ f(a) \leq c\}\ ;\ g(c) \leq b\}$. If $a \in S$, $f(a) \leq c$ and $g(c) \leq b$ then putting $f(a) = c$ we get that $f(a) = c \leq c$ and $g(f(a)) = g(c) \leq b$. So, $(g\hat{o}f)^{-1}(b) \leq ((\hat{f})^{-1}o(\hat{g})^{-1})(b)$ i.e. (1) holds. If g is order-preserving then for $a \in S$, $f(a) \leq c$, $g(c) \leq b$, we get $g(f(a)) \leq g(c) \leq b$. So, $((\hat{f})^{-1}o(\hat{g})^{-1})(b) \leq (g\hat{o}f)^{-1}(b)$. Thus (2) holds. □

Definition 5.4. Let (S, τ) and (S_1, τ_1) be two FTL-spaces and $f : S \to S_1$ be an admissible mapping. Then f is said to be a *gradation preserving map* (gp-map) if for each $b \in S_1$, $\tau(\hat{f}^{-1}(b)) \geq \tau_1(b)$.

Note 5.5. We shall consider continuous mapping or gp-map for admissible mappings only.

Theorem 5.6. *Let (S, τ) and (S_1, τ') be two FTL-spaces and $f : S \to S_1$ be a mapping. Then f is a gp-map iff $f : (S, \tau_r) \to (S_1, \tau'_r)$ is continuous, $\forall\ r \in L_0$.*

Proof. Suppose f is a gp-map. Then $\forall\ b \in S_1$, $\tau(\hat{f}^{-1}(b)) \geq \tau'(b)$. Let $b \in \tau'_r$. Then $\tau'(b) \geq r \Rightarrow \tau(\hat{f}^{-1}(b)) \geq r$ and hence $\hat{f}^{-1}(b) \in \tau_r$. Hence f is continuous. Conversely, let $f : (S, \tau_r) \to (S_1, \tau'_r)$ be continuous, $\forall\ r \in L_0$, Let $b \in S_1$. If

$\tau'(b) = 0_{S_1}$, then $\tau(\hat{f}^{-1}(b)) \geq \tau_1(b)$ holds. If $\tau'(b) = r \in L_0$, then $b \in \tau'_r$. By continuity of f, $\hat{f}^{-1}(b) \in \tau_r \Rightarrow \tau(\hat{f}^{-1}(b)) \geq r = \tau'(b)$. Hence f is a gp-map. □

Theorem 5.7. *Let (S,T) and (S_1,T_1) be two TL-spaces and $f : S \to S_1$ be a mapping. Then $f : (S,T) \to (S_1,T_1)$ is continuous iff $f : (S,T^r) \to (S_1,T_1^r)$ is a gp-map $\forall\ r \in L_0$.*

Proof. Suppose $f : (S,T) \to (S_1,T_1)$ is continuous. Take $b \in S_1$. Then we have three possibilities: (1) $b = 0_{S_1}$ or 1_{S_1}, (2) $b \in T_1 - \{0_{S_1}, 1_{S_1}\}$, (3) $b \notin T_1$.

In case (1), $\hat{f}^{-1}(0_{S_1}) = 0_S$, $\hat{f}^{-1}(1_{S_1}) = 1_S$ and hence $T_1^r(b) \leq T^r(\hat{f}^{-1}(b))$. In case (2), $b \in T_1 - \{0_{S_1}, 1_{S_1}\} \Rightarrow T_1^r(b) = r$. By continuity of f, $\hat{f}^{-1}(b) \in T$, and hence $T^r(\hat{f}^{-1}(b)) \geq r$. So, $T_1^r(b) \leq T^r(\hat{f}^{-1}(b))$. In case (3), $b \notin T_1 \Rightarrow T_1^r(b) = 0_{S_1}$, and hence $0_{S_1} = T_1^r(b) \leq T^r(\hat{f}^{-1}(b))$. Hence $f : (S,T^r) \to (S_1,T_1^r)$ is a gp-map $\forall\ r \in L_0$. Conversely, let $f : (S,T^r) \to (S_1,T_1^r)$ is a gp-map $\forall\ r \in L_0$. Then $\forall\ b \in S_1$, $T^r(\hat{f}^{-1}(b)) \geq T_1^r(b)$. Now, take $b \in T_1 = (T_1^r)r$, by Theorem 3.17. This gives $T_1^r(b) \geq r \Rightarrow T^r(\hat{f}^{-1}(b)) \geq r \Rightarrow \hat{f}^{-1}(b) \in (T^r)_r = T$ i.e., $b \in T_1 \Rightarrow \hat{f}^{-1}(b) \in T$. Hence $f : (S,T) \to (S_1,T_1)$ is continuous. □

Theorem 5.8. *Let (S,τ) be a FTL-space and $f : S \to S_1$ be a mapping. Let $\{\tau'_r\ ;\ r \in L_0\}$ be a descending family of topologies associated with S_1. Let τ_1 be the GO associated with S_1 generated by this family. Further suppose, for each $r \in L_0$, $\mathcal{B}_r$ is a base and $\mathcal{S}_r$ is a subbase of τ'_r. Then the following results hold:*

(1) *$f : (S,\tau) \to (S_1,\tau_1)$ is a gp-map iff $\tau(\hat{f}^{-1}(b)) \geq r$, $\forall\ b \in \tau'_r$, $\forall\ r \in L_0$,*

(2) *$f : (S,\tau) \to (S_1,\tau_1)$ is a gp-map iff $\tau(\hat{f}^{-1}(b)) \geq r$, $\forall\ b \in \mathcal{B}_r$, $\forall\ r \in L_0$,*

(3) *$f : (S,\tau) \to (S_1,\tau_1)$ is a gp-map iff $\tau(\hat{f}^{-1}(b)) \geq r$, $\forall\ b \in \mathcal{S}_r$, $\forall\ r \in L_0$.*

Proof. (1) Let $f : (S,\tau) \to (S_1,\tau_1)$ be a gp-map. Then for $r \in L_0$, $b \in \tau'_r \Rightarrow \tau(\hat{f}^{-1}(b)) \geq \tau_1(b) \geq r$. Conversely, suppose the condition holds. Let $b \in S_1$ and $\tau_1(b) = r > 0_L$. Then $b \in \tau'_r$. Therefore $\tau(\hat{f}^{-1}(b)) \geq r = \tau_1(b)$. Hence f is a gp-map. (2) Let $f : (S,\tau) \to (S_1,\tau_1)$ be a gp-map. Then for $r \in L_0$, $b \in \mathcal{B}_r \Rightarrow b \in \tau'_r$, and hence $\tau(\hat{f}^{-1}(b)) \geq \tau_1(b) \geq r$. Conversely, assume that the condition holds. Let $b \in S_1$ and $\tau_1(b) = r > 0_L$. Then $b \in \tau'_r$. Therefore $\exists\ b_i \in \mathcal{B}_r$, $i \in \Delta$, such that $b = \vee_{i\in\Delta} b_i$. Now $\hat{f}^{-1}(b) = \hat{f}^{-1}(\vee_i b_i) = \vee_i \hat{f}^{-1}(b_i)$. Therefore $\tau(\hat{f}^{-1}(b)) = \tau(\vee_i \hat{f}^{-1}(b_i)) \geq \wedge_i(\tau(\hat{f}^{-1}(b_i))) \geq r$ (by the given condition) $= \tau_1(b)$. Hence f is a gp-map.

(3) Suppose the condition holds. Let $b \in S_1$ and $\tau_1(b) = r > 0_L$. Then $b \in \tau'_r$. Therefore $\exists\ b_i \in \mathcal{B}_r,\ i \in \Delta$, such that $b = \vee_i b_i$. Since $b_i \in \mathcal{B}_r,\ i \in \Delta,\ \exists\ d_{i,j} \in \mathcal{S}_r,\ j = 1, 2, ..., n_i$ such that $b_i = \wedge_{j=1}^{n_i} d_{i,j}$. Then $b = \vee_{i\in\Delta}(\wedge_{j=1}^{n_i} d_{i,j})$. Now $\hat{f}^{-1}(b) = \hat{f}^{-1}(\vee_{i\in\Delta}(\wedge_{j=1}^{n_i} d_{i,j})) = \vee_{i\in\Delta}(\hat{f}^{-1}(\wedge_{j=1}^{n_i} d_{i,j})) = \vee_{i\in\Delta}(\wedge_{j=1}^{n_i} \hat{f}^{-1}(d_{i,j}))$. Therefore

$$\begin{aligned}\tau(\hat{f}^{-1}(b)) &= \tau(\vee_{i\in\Delta}(\wedge_{j=1}^{n_i} \hat{f}^{-1}(d_{i,j}))) \\ &\geq \wedge_{i\in\Delta}\tau(\wedge_{j=1}^{n_i} \hat{f}^{-1}(d_{i,j})) \\ &\geq \wedge_{i\in\Delta}(\wedge_{j=1}^{n_i} \tau(\hat{f}^{-1}(d_{i,j}))) \\ &\geq r, \text{ by the given condition} \\ &= \tau_1(b).\end{aligned}$$

Hence f is a gp-map. The converse part is obvious.

Theorem 5.9. *Let (S, τ), (S_1, τ_1) and (S_2, τ_2) be three FLT-spaces. If $f : (S, \tau) \to (S_1, \tau_1)$ and $g : (S_1, \tau_1) \to (S_2, \tau_2)$ be gp-maps then $gof : (S, \tau) \to (S_2, \tau_2)$ is a gp-map.*

The proof follows from Theorem 5.3.

6. Fuzzy Lattice Closure Operator

Definition 6.1. Let F be a GC associated with S. For each $r \in L_0$ and for each $a \in S$ we define *fuzzy lattice closure* (briefly L-closure) by

$$cl(a, r) = \wedge\{b \in S\ ;\ b \geq a,\ F(b) \geq r\}.$$

Theorem 6.2. *Let F be a GC associated with S, L be dense and satisfies sup property and let $cl : S \times L_0 \to S$ be the L-closure operator in (S, F). Then*

(1) $cl(0_S, r) = 0_S,\ cl(1_S, r) = 1_S,\ \forall\ r \in L_0$,
(2) $cl(a, r) \geq a,\ \forall\ a \in S$,
(3) $cl(a, r) \leq cl(a, r')$, *if* $r \leq r'$,
(4) $cl(a \vee b, t) = cl(a, t) \vee cl(b, t),\ \forall\ t \in L_0$,
(5) $cl(cl(a, t), t) = cl(a, t)$,
(6) *if* $r = \vee\{t \in L_0\ ;\ cl(a, t) = a\}$, *then* $cl(a, r) = a$.

Proof. (1) $cl(0_S, r) = \wedge\{b\ ;\ b \geq 0_S,\ F(b) \geq r\}$, since $0_S \geq 0_S,\ F(0_S) = 1_L \geq r$. Hence $cl(0_S, r) = 0_S$. $cl(1_S, r) = \wedge\{b\ ;\ b \geq 1_S,\ F(b) \geq r\} = 1_S$, since $1_S \geq 1_S,\ F(1_S) = 1_L \geq r$. The proof of (2) and (3) are obvious.

(4) If $a_1 \leq a_2$, then

$$\begin{aligned} cl(a_2, r) &= \wedge\{b \; ; \; b \geq a_2, F(b) \geq r\} \\ &\geq \wedge\{b \; ; \; b \geq a_1, F(b) \geq r\} \\ &= cl(a_1, r). \end{aligned}$$

Using this, we have $cl(a \vee b, t) \geq cl(a, t) \vee cl(b, t)$. Again, $cl(a, t) \vee cl(b, t) \geq a \vee b$, using (2). As F is GC, so $F(cl(a, t) \vee cl(b, t)) \geq F(cl(a, t)) \wedge F(cl(b, t))$. Also $F(cl(a, t)) = F(\wedge\{b \; ; \; b \geq a, F(b) \geq t\}) \geq \wedge\{F(b) \; ; \; b \geq a, F(b) \geq t\} \geq t$. So, $F(cl(a, t) \vee cl(b, t)) \geq t$. Therefore, $cl(a, t) \vee cl(b, t) \geq cl(a \vee b, t)$. Hence $cl(a \vee b, t) = cl(a, t) \vee cl(b, t), \; \forall \; t \in L_0$.
(5) $cl(cl(a, t), t) = \wedge\{b \; ; \; b \geq cl(a, t), F(b) \geq t\} \leq cl(a, t)$ (since $cl(a, s) \geq cl(a, s)$ and $F(cl(a, s)) \geq s$). By (2), $cl(cl(a, t), t) \geq cl(a, t)$. Hence $cl(cl(a, t), t) = cl(a, t)$.
(6) Let $r = \vee\{t \in L_0 \; ; \; cl(a, t) = a\}$. Take $t < r$. By sup property, $\exists \; p \in L_0$ with $cl(a, p) = a$ such that $t < p$. Now, we have $a \geq a$ and $F(a) = F(cl(a, p)) \geq p > t$. Therefore $F(a) \geq r$ (by Theorem 1.4). Then $a \leq cl(a, r) \leq a$. Hence $cl(a, r) = a$. □

Theorem 6.3 *Let L be dense and satisfies sup property. Let $cl : S \times L_0 \rightarrow S$ be a mapping satisfies* (1) – (4) *of Theorem* 6.2. *Let $F : S \rightarrow L$ be a mapping defined by*

$$F(a) = \vee\{r \in L_0 \; ; \; cl(a, r) = a\}, \; a \in S.$$

Then F is a GC associated with S. Further, $cl = cl_F$ iff (5) *and* (6) *of Theorem* 6.2 *are satisfied by cl.*

Proof. (GC1) : $F(0_S) = \vee\{r \in L_0 \; ; \; cl(0_S, r) = 0_S\} = 1_L$ by (1). Similarly $F(1_S) = 1_S$.
(GC2) : Let $a = \wedge_{i \in \Delta} a_i$. Then $cl(a, r) \leq cl(a_i, r), \; \forall \; r \in L_0, \; \forall \; i \in \Delta$ by (4). So, $cl(a, r) \leq \wedge_{i \in \Delta} cl(a_i, r)$. Let $\alpha < \wedge_{i \in \Delta} F(a_i)$. Then $F(a_i) > \alpha, \; \forall \; i \in \Delta$. By sup property, $\exists \; s_i \in L_0$ such that $cl(a_i, s_i) = a_i$ and $s_i > \alpha, \; \forall \; i \in \Delta$. Let $s = \wedge_{i \in \Delta} s_i$. Then by (2) and (3), $cl(a_i, s) = a_i, \; \forall \; i \in \Delta$. So, by (4), $cl(a, s) \leq cl(a_i, s) = a_i, \; \forall \; i \in \Delta$, i.e., $cl(a, s) \leq \wedge_{i \in \Delta} a_i = a$, i.e., $cl(a, s) = a$. So, $F(a) \geq s \geq \alpha$. Hence, by Theorem 1.4, $F(\wedge_{i \in \Delta} a_i) \geq \wedge_{i \in \Delta} F(a_i)$.
(GC3) : Take $\alpha < F(a_1) \wedge F(a_2)$, where $a_1, \; a_2 \in S$. Then by sup property, $\exists \; s_i$ such that $s_i > \alpha$ and $cl(a_i, s_i) = a_i, \; i = 1, 2$. Let $s = s_1 \wedge s_2$. Then $a_i \leq cl(a_i, s) \leq cl(a_i, s_i) = a_i, \; i = 1, 2$, by (2) and (3) i.e., $cl(a_i, s) = a_i, \; i = 1, 2$. By (4), $cl(a_1 \vee a_2, s) = cl(a_1, s) \vee cl(a_2, s) = a_1 \vee a_2$. So, $F(a_1 \vee a_2) \geq s > \alpha$. By Theorem 1.4, $F(a_1 \vee a_2) \geq F(a_1) \wedge F(a_2)$.

Now to prove the next part, suppose cl satisfies conditions (5) and (6) in addition to the conditions (1)-(4) of Theorem 6.2. $cl_F(a,r) = \wedge\{b \geq a \ ; \ F(b) \geq r\}$. If $s < r$ then $F(b) \geq r \Rightarrow F(b) > s \Rightarrow \vee\{t \in L_0 \ ; \ cl(b,t) = b\} > s \Rightarrow \exists\, t \in L_0$ with $t > s$ such that $cl(b,t) = b$ (by sup property) $\Rightarrow b \leq cl(b,s) \leq cl(b,t) = b \Rightarrow cl(b,s) = b$ —(A) $\Rightarrow \wedge\{b \geq a \ ; \ F(b) \geq r\} \leq cl(b,s)$. Therefore $cl_F(a,r) \leq cl(b,s)$, where $s < r$ and $b \geq a$ with $F(b) \geq r$. Now $a \geq a$ and $F(cl(a,r)) = \vee\{t \ ; \ cl(cl(a,r),t) = cl(a,r)\}$. Therefore $F(cl(a,r)) \geq r$ (since $cl(cl(a,r),r) = cl(a,r)$). Therefore $cl(a,r) \geq a$ and $F(cl(a,r)) \geq r$. So,

$$\begin{aligned} cl_F(a,r) &\leq cl(cl(a,r),s), \text{ where } s < r \\ &\leq cl(cl(a,r),r) \\ &= cl(a,r). \end{aligned}$$

Next $F(b) \geq r \Rightarrow cl(b,s) = b$ (by (A)), for $s < r \Rightarrow cl(a,r) = b$, by (6). Therefore $cl_F(a,r) = \wedge\{b \geq a \ ; \ F(b) \geq r\} \geq cl(a,r)$. Hence $cl_F(a,r) = cl(a,r)$. The converse part is obvious. □

Remark 6.4. If $cl : S \times L_0 \to S$ is a L-closure operator on S, then for each $r \in L_0$, $cl_r : S \to S$ defined by $cl_r(\lambda) = cl(\lambda, r)$ is a closure operator on S.

Theorem 6.5. *Let L be dense and satisfies sup property. An operator $cl : S \times L_0 \to S$ is a L-closure operator for the FTL-space (S,τ) iff $cl_r : S \to S$ is a closure operator for the TL-space (S,τ_r), $r \in L_0$.*

Proof. Obviously, cl is a L-closure operator for the FTL-space (S,τ) implies that cl_r is a closure operator for the TL-space (S,τ_r), $\forall r \in L_0$. Conversely, assume that cl_r is a closure operator for the TL-space (S,τ_r), $\forall r \in L_0$. Then (1), (2), (4) and (5) of Theorem 6.2 are satisfied by cl. Since $\tau_r \supset \tau_{r'}$ for $r < r'$, (3) of Theorem 6.2 follows. Finally, to verify the condition (6) of Theorem 6.2, let $r = \vee\{s \in L_o \ ; \ cl(a,s) = a\}$. For $t < r$, by sup property, $\exists\ s > t$ such that $cl(a,s) = a \Rightarrow a \leq cl(a,t) \leq cl(a,s) = a \Rightarrow a = cl(a,t) \Rightarrow a = \wedge\{b \in L_0 \ ; \ b \geq a, F(b) \geq t\} \Rightarrow F(a) = F(\wedge\{b \in L_0 \ ; \ b \geq a, F(b) \geq t\}) \geq \wedge F\{b \in L_0 \ ; \ b \geq a, F(b) \geq t\} \geq t$. So, $\forall t < r$, $a \in F_t \Rightarrow a' \in \tau_t \Rightarrow a' \in \cap_{t<r}\tau_t \Rightarrow a' \in \tau_r$ (by Theorem 3.13) $\Rightarrow a \in F_r \Rightarrow cl(a,r) = a$. □

Theorem 6.6. *Let $f : (S,\tau) \to (S_1,\tau')$ be a mapping. Then f is a gp-map iff $f(cl(b,r)) \leq cl(f(b),r)$, $\forall r \in L_o$, $\forall b \in S$.*

Proof. By Theorem 5.6, f is a gp-map iff $f : (S, \tau_r) \to (S_1, \tau_r')$ is continuous $\forall r \in L_o$, i.e., iff $f(cl(b,r)) \leq cl(f(b), r)$, $\forall r \in L_o$, $\forall b \in S$ (by Theorem 2.5 and Theorem 2.15). □

7. Category of FTL-spaces

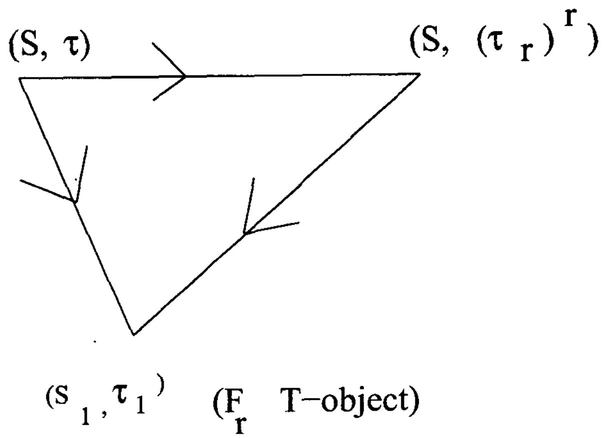
Let Sc denote the category of all TL-spaces and continuous functions; FT denote the category of all FTL-spaces and gp-maps; for each $r \in L_o$, F_rT denote the category of r-th graded FTL-spaces and gp-maps.

Theorem 7.1. (1) *F_rT is a full subcategory of FT,*

(2) *For each $r \in L_o$, Sc and F_rT are isomorphic,*

(3) *F_rT is a bireflective full subcategory of FT, $\forall r \in L_o$.*

Proof. The results (1) and (2) follow from the following facts: $(\tau_r)^r = \tau$ if τ is an r-th gradation of openness; $(T^r)_r = T$ if $T \in C(S)$ and $f : (S,T) \to (S_1, T_1)$ is continuous iff $f : (S, T^r) \to (S_1, T_1^r)$ is a gp-map, $\forall r \in L_o$. In order to prove (3), let us take a member (S,τ) of FT. Then for $r \in L_o$, $(S, (\tau_r)^r)$ is a member of F_rT and also $I_S : (S,\tau) \to (S, (\tau_r)^r)$ is a gp-map. Let (S_1, τ_1) be a member of F_rT and $f : (S,\tau) \to (S_1, \tau_1)$ be a gp-map. To complete the proof of (3), we need to check only that $f : (S, (\tau_r)^r) \to (S_1, \tau_1)$ is a gp-map. In fact, $\tau_1(0_{S_1}) = \tau(f^{-1}(0_{S_1})) = \tau(0_S) = (\tau_r)^r(0_S) = (\tau_r)^r(f^{-1}(0_{S_1})) = 1_L$. Similarly, $\tau_1(1_{S_1}) = (\tau_r)^r(f^{-1}(1_{S_1}))$. In case $\tau_1(a) = 0_L$, then clearly, $\tau_1(a) \leq (\tau_r)^r(f^{-1}(a))$. If $\tau_1(a) = r$, then $\tau_1(a) \leq \tau(f^{-1}(a)) \Rightarrow f^{-1}(a) \in \tau_r$, and hence $(\tau_r)^r(f^{-1}(a)) \geq r = \tau_1(a)$. Thus $f : (S, (\tau_r)^r) \to (S_1, \tau_1)$ is a gp-map. □



Remark 7.2. Because of (2) and (3), henceforth Sc may be called a bireflective full subcategory of FT.

Theorem 7.3. *Let $\{(S_i, \tau_i')\ ;\ i \in \Delta\}$ be a class of FTL-spaces, S be a lattice and $f_i : S \to S_i$ is an admissible and order-preserving map for each $i \in \Delta$. Then $\exists$ a GO τ associated with S such that the following conditions hold:*

(1) *for each $i \in \Delta$, $f_i : (S.\tau) \to (S_i, \tau_i')$ is a gp-map,*

(2) *if (S', τ') is a FTL-space then an admissible map $g : (S', \tau') \to (S, \tau)$ is a gp-map iff $f_i og$ is a gp-map, $\forall i \in \Delta$.*

Proof. For each $r \in L_o$ and for each $j \in J$, define $T_{j,r} = \{\hat{f}_j^{-1}(b)\ ;\ b \in (\tau_j')_r\}$. Recall that $(\tau_j')_r = \{b \in S_j\ ;\ \tau_j'(b) \geq r\}$ is the r-level topology associated with S_j with respect to τ_j'. It can be proved that each $T_{j,r}$ is a topology associated with S. It is clear that $\{T_{j,r}\ ;\ r \in L_o\}$ is a descending chain of topologies associated with S. For each $r \in L_o$, define $\hat{S}_r = \cup_{j \in J} T_{j,r}$, and let T_r be the topology associated with S generated by $\hat{S}_r$ as a subbase. It can be verified that $\{T_r\ ;\ r \in L_o\}$ is a descending chain of topologies associated with S. Now, from Theorem 3.14 we have a GO τ associated with S with respect to $\{T_r\ ;\ r \in L_o\}$, where $\tau(a) = \vee\{r \in L_o\ ;\ a \in T_r\}$. Firstly, we show that for each $j \in J$, $f_j : (S.\tau) \to (S_j, \tau_j')$ is a gp-map. For this, let $b \in S_j$ and let $\tau_j'(b) = r$, $r > 0_L$. Then $\hat{f}_j^{-1}(b) \in T_{j,r} \subset \hat{S}_r \subset T_r$. Thus, $\tau(\hat{f}_j^{-1}(b)) \geq r = \tau_j'(b)$. Hence $f_j : (S, \tau) \to (S_j, \tau_j')$ is a gp-map. So (1) is true. Next, suppose $g : (S', \tau') \to (S, \tau)$ is a gp-map. Since for each $j \in J$, $f_j : (S, \tau) \to (S_j, \tau_j')$ is a gp-map, so by Theorem 5.9, $f_j og : (S', \tau') \to (S_j, \tau_j')$ is a gp-map. Conversely, suppose for each $j \in J$, $f_j og : (S', \tau') \to (S_j, \tau_j')$ is a gp-map. We shall show that $g : (S', \tau') \to (S, \tau)$ is a gp-map. In order to show this it is sufficient to check that (by Theorem 5.8) $\tau'(\hat{g}^{-1}(b)) \geq r$, $\forall b \in \hat{S}_r$, $\forall r \in L_o$. Let $r \in L_o$, $b \in \hat{S}_r$. Then $b \in T_{j,r}$, for some $j \in J$. So, there is $a \in (\tau_j')_r$ such that $\hat{f}_j^{-1}(a) = b$. Since for each $j \in J$, $\{(\tau_j')_r\ ;\ r \in L_o\}$ is a family of topologies associated with S_j with respect to τ_j' and since $f_j og : (S', \tau') \to (S_j, \tau_j')$ is a gp-map, by Theorem 5.8, $\tau'((\widehat{f_j og})^{-1}(a)) \geq r$ i.e., $\tau'(\hat{g}^{-1}(\hat{f}_j^{-1}(a))) \geq r$ and thus $\tau'(\hat{g}^{-1}(b)) \geq r$. This completes the proof of (2). □

Remark 7.4. Thus from the categorical results so obtained we observe that the concept of FTL-spaces as introduced by us is a meaningful fuzzification of topological spaces.

8. Fuzzy Gradation of Openness

In this section, firstly we introduce a notion of S-family which is derived from the notion of fuzzy family as introduced by Šostak [16]. The operations of union, intersection and complementation of an S-family is defined. Using these notion an idea of fuzzy gradation of openness is introduced in lattice setting.

Definition 8.1. A mapping $\mathcal{G} : S \to L$ is called an *S-family*.

Notation 8.2. For an S-family $\mathcal{G}$, we denote $\mathcal{S}(\mathcal{G}) = \{a \in S;\ \mathcal{G}(a) > 0_L\}$.

Definition 8.3. We take two functions $\dot\wedge$, $\dot\vee : L \times S \to S$ satisfying the conditions $(\alpha\dot\wedge a)' = \alpha'\dot\vee a'$ and $(\alpha\dot\vee a)' = \alpha'\dot\wedge a'$, $\forall\ (\alpha, a) \in L \times S$. Then we define for an S-family $\mathcal{G}$ the operations $\wedge$ and $\vee$ (with respect to $\dot\wedge$ and $\dot\vee$ respectively) as $\vee\mathcal{G} = \vee_{a\in\mathcal{S}(\mathcal{G})}\{\mathcal{G}(a)\dot\wedge a\}$ and $\wedge\mathcal{G} = \wedge_{a\in\mathcal{S}(\mathcal{G})}\{\mathcal{G}(a)'\dot\vee a\}$.

Definition 8.4. Let $\mathcal{G}$ be an S-family. Then an S-family $\mathcal{G}^*$ is defined by $\mathcal{G}^*(a) = \mathcal{G}(a'), \forall a \in S$.

Proposition 8.5. *For an S-family $\mathcal{G}$ we have*

(a) $[\vee\mathcal{G}]' = \wedge\mathcal{G}^*$,
(b) $[\wedge\mathcal{G}]' = \vee\mathcal{G}^*$.

The proof is straightforward.

Definition 8.6. A mapping $\tau : S \to L$ is said to be *fuzzy gradation of openness* (shortly FGO) if it satisfies the following axioms:

(FGO1) $\tau(1_S) = \tau(0_S) = 1_L$,
(FGO2) For any S-family $\mathcal{G}$

$$\tau(\vee\mathcal{G}) \geq \wedge_{a\in\mathcal{S}(\mathcal{G})}(\tau(a) \wedge \mathcal{G}(a)),$$

(FGO3) For any finite S-family $\mathcal{B} = \{\frac{a_1}{p_1}, \ldots \frac{a_n}{p_n}\}$

$$\tau(\wedge\mathcal{B}) \geq \wedge_{i=1}^{n}(\tau(a_i) \wedge \mathcal{B}(a_i)).$$

Theorem 8.7. *A GO $\tau : S \to L$ is a FGO iff*

(c_1) $\tau(\alpha\dot\wedge a) \geq \alpha \wedge \tau(a),\ \forall\ \alpha \in L_{0,1}$,
(c_2) $\tau(\alpha\dot\vee a) \geq \alpha' \wedge \tau(a),\ \forall\ \alpha \in L_{0,1}$.

Proof. Suppose τ is a GO satisfying (c_1) and (c_2) and $\mathcal{G}$ be an S-family. Then

$$\begin{aligned}\tau(\vee\mathcal{G}) &= \tau[\vee_{a\in S(\mathcal{G})}(\mathcal{G}(a)\dot{\wedge}a)]\\ &\geq \wedge_{a\in S(\mathcal{G})}\tau(\mathcal{G}(a)\dot{\wedge}a)\\ &\geq \wedge_{a\in S(\mathcal{G})}(\mathcal{G}(a)\wedge\tau(a)),\ \text{by } (c_1).\end{aligned}$$

For a finite S-family $\mathcal{B}=\{\frac{a_1}{p_1},\dots\frac{a_n}{p_n}\}$,

$$\begin{aligned}\tau(\wedge\mathcal{B}) &= \tau[\wedge_{i=1}^n(p_i'\dot{\vee}a_i)]\\ &\geq \wedge_{i=1}^n\tau(p_i'\dot{\vee}a_i)\\ &\geq \wedge_{i=1}^n(p_i\wedge\tau(a_i)),\ \text{by } (c_2)\\ &= \wedge_{i=1}^n(\mathcal{B}(a_i)\wedge\tau(a_i)).\end{aligned}$$

Therefore τ is a FGO. Conversely, let τ is a FGO. Then τ is a GO. Let $a\in S$ and $\alpha\in L_{0,1}$. Define an S-family $\mathcal{G}_a : S\to L$ by $\mathcal{G}_a(a)=\alpha,\ \mathcal{G}_a(b)=0_L$ if $b(\neq a)\in S$. Then

$$\begin{aligned}&\tau(\vee\mathcal{G}_a)\geq\wedge_{d\in S(\mathcal{G}_a)}(\mathcal{G}_a(d)\wedge\tau(d))\\ &\Leftrightarrow\tau(\mathcal{G}_a(a)\dot{\wedge}a)\geq\mathcal{G}_a(a)\wedge\tau(a)\\ &\Leftrightarrow\tau(\alpha\dot{\wedge}a)\geq\alpha\wedge\tau(a).\end{aligned}$$

Again,

$$\begin{aligned}&\tau(\wedge\mathcal{G}_a)\geq\mathcal{G}_a(a)\wedge\tau(a)\\ &\Leftrightarrow\tau(\mathcal{G}_a'(a)\dot{\vee}a)\geq\alpha\wedge\tau(a)\\ &\Leftrightarrow\tau(\alpha'\dot{\vee}a)\geq\alpha\wedge\tau(a).\end{aligned}$$

Thus τ is a GO satisfying (c_1) and (c_2). □

Definition 8.8. Let $\tau : S\to L$ be a mapping. For, $r\in L_0$, define $\tau_r=\{a\in S\ ;\ \tau(a)\geq r\}$.

Theorem 8.9. *Let τ be a fuzzy gradation of openness. Then $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type) satisfying*

(1) $\tau_{\vee_{i\in\Delta}\alpha_i}=\cap_{i\in\Delta}\tau_{\alpha_i}$,
(2) $a\in\tau_r\Rightarrow\alpha\dot{\wedge}a\in\tau_{\alpha\wedge r}$ *and* $\alpha\dot{\vee}a\in\tau_{\alpha'\wedge r}$, $\forall\ \alpha\in L_{0,1}$.

Proof. Since τ is a fuzzy gradation of openness, it is a gradation of openness and hence $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type). Further,

$$a \in \cap_{i\in\Delta}\tau_{\alpha_i} \Rightarrow \tau(a) \geq \alpha_i, \ \forall\, i \in \Delta$$
$$\Rightarrow \tau(a) \geq \vee_{i\in\Delta}\alpha_i$$
$$\Rightarrow a \in \tau_{\vee_{i\in\Delta}\alpha_i}.$$

So, $\cap_{i\in\Delta}\tau_{\alpha_i} \subset \tau_{\vee_{i\in\Delta}\alpha_i}$. Obviously, $\tau_{\vee_{i\in\Delta}\alpha_i} \subset \cap_{i\in\Delta}\tau_{\alpha_i}$. Hence $\tau_{\vee_{i\in\Delta}\alpha_i} = \cap_{i\in\Delta}\tau_{\alpha_i}$.

Next $a \in \tau_r \Rightarrow \tau(a) \geq r$. Then, by (c_1), we have

$$\tau(\alpha\dot{\wedge}a) \geq \alpha \wedge \tau(a) \geq \alpha \wedge r \Rightarrow \alpha\dot{\wedge}a \in \tau_{\alpha\wedge r}, \ \forall\, \alpha \in L_{0,1},$$

and by (c_2), we have

$$\tau(\alpha\dot{\vee}a) \geq \alpha' \wedge \tau(a) \geq \alpha' \wedge r \Rightarrow \alpha\dot{\vee}a \in \tau_{\alpha'\wedge r}, \ \forall\, \alpha \in L_{0,1}.$$

Hence $\{\tau_r\}_{r\in L_0}$ is a descending family of L-fuzzy topologies (Chang-type) satisfying (1) and (2). □

Theorem 8.10. *Let $\{T_r \ : \ r \in L_0\}$ be a descending family of L-fuzzy topologies (Chang-type) satisfying conditions (1) and (2) of Theorem 8.9, then the mapping $\tau : S \to L$ defined by $\tau(a) = \vee\{r \ : \ a \in T_r\}$ is a FGO such that $\tau_r = T_r$, $r \in L_0$.*

Proof. From Proposition 2.2 of [8], τ is a gradation of openness and it satisfies $\tau_r = T_r$, $\forall\, r \in L_0$. Let $\tau(a) = s$. If $s = 0_L$, then obviously (c_1) and (c_2) hold. If $s > 0_L$, then

$$a \in \tau_s \Rightarrow a \in T_s$$
$$\Rightarrow \alpha\dot{\wedge}a \in T_{\alpha\wedge s}, \ \forall\, \alpha \in L_{0,1}$$
$$\Rightarrow \alpha\dot{\wedge}a \in \tau_{\alpha\wedge s}, \ \forall\, \alpha \in L_{0,1} \text{ (as } \tau_{\alpha\wedge s} = T_{\alpha\wedge s})$$
$$\Rightarrow \tau(\alpha\dot{\wedge}a) \geq \alpha \wedge s, \ \forall\, \alpha \in L_{0,1}$$
$$\Rightarrow \tau(\alpha\dot{\wedge}a) \geq \alpha \wedge \tau(a), \ \forall\, \alpha \in L_{0,1}.$$

Again,

$$a \in \tau_s \Rightarrow a \in T_s$$
$$\Rightarrow \alpha\dot{\vee}a \in T_{\alpha'\wedge s}, \ \forall\, \alpha \in L_{0,1}$$
$$\Rightarrow \alpha\dot{\vee}a \in \tau_{\alpha'\wedge s}, \ \forall\, \alpha \in L_{0,1} \text{ (as } \tau_{\alpha'\wedge s} = T_{\alpha'\wedge s})$$
$$\Rightarrow \tau(\alpha\dot{\vee}a) \geq \alpha' \wedge s, \ \forall\, \alpha \in L_{0,1}$$
$$\Rightarrow \tau(\alpha\dot{\vee}a) \geq \alpha' \wedge \tau(a), \ \forall\, \alpha \in L_{0,1}.$$

Hence by Theorem 8.7, τ is a FGO. □

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