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ON A QUADRATICALLY CONVERGENT ITERATIVE METHOD USING DIVIDED DIFFERENCES OF ORDER ONE

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ABSTRACT. We introduce a new two-point iterative method to approximate solutions of nonlinear operator equations. The method uses only divided differences of order one, and two previous iterates. However in contrast to the Secant method which is of order 1.618..., our method is of order two. A local and a semilocal convergence analysis is provided based on the majorizing principle. Finally the monotone convergence of the method is explored on partially ordered topological spaces. Numerical examples are also provided where our results compare favorably to earlier ones [1], [4], [5], [19].

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on an open subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = B(x)$ (for some suitable operator B), where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems.

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The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

The secant method is the most popular iterative procedure using two previous iterates and divided differences of order one for approximating x^*. The order of the Secant method is $1.618\ldots$ [1]–[5]. In the elegant paper of F.A. Potra [18] a three point method was used and divided differences of order one (see also (26) and (27)). This method is of order $1.839\ldots$. More recently Secant-like methods of order between $1.618\ldots$ and $1.839\ldots$ were introduced in the works of Amat, Hernandez, Gutierrez et al. [1]–[4], [11]–[13]. The question arises if it is then possible to realize an iterative method using two previous iterates and divided differences of only order one with at least quadratic convergence.

It turns out that this is possible. Indeed we introduce the method

$$x_{n+1} = x_n - [2x_n - x_{n-1}, x_{n-1}]^{-1} F(x_n) \quad (x_{-1}, x_0 \in D) \quad (n \geq 0) \tag{1.2}$$

for approximating x^*. Here, a linear operator from X into Y, denoted by $[x, y; F]$ or simply $[x, y]$ which satisfies the condition

$$[x, y](x - y) = F(x) - F(y), \tag{1.3}$$

is called a divided difference of order one [8], [14], [16]. Iteration (1.2) has a geometrical interpretation similar to the Secant method in the scalar case.

In Sections 2 and 3, respectively, we provide a local and semilocal convergence analysis for method (1.2) using Lipschitz-type conditions and the majorant principle [14] as in [18]. The monotone convergence of method (1.2) is examined on partially ordered topological spaces in Section 4 [8], [19], [21]. Numerical examples are also provided where our results compare favorably to earlier ones [1], [4], [5], [19].

2. LOCAL CONVERGENCE ANALYSIS OF METHOD (1.2)

We can show the following local convergence result for method (1.2).

Theorem 2.1. *Let F be a nonlinear operator defined on an open subset D of a Banach space X with values in a Banach space Y.*

Assume:

the equation $F(x) = 0$ has a solution $x^* \in D$ at which the Fréchet derivative $F'(x^*)$ exists, and is invertible;

the operator F is Fréchet-differentiable with divided difference of order one on $D_0 \subseteq D$ satisfying the Lipschitz conditions:

$$\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \leq a\|x - x^*\|, \tag{2.1}$$

$$\|F'(x^*)^{-1}([x, y] - [x, x^*])\| \leq b\|y - x^*\| \tag{2.2}$$

and

$$\|F'(x^*)^{-1}([y, y] - [2y - x, x])\| \leq c\|y - x\|^2; \tag{2.3}$$

the ball

$$U^* = U(x^*, r^*) = \{x \in X \mid \|x - x^*\| < r^*\} \subseteq D_0, \tag{2.4}$$

where

$$r^* = \frac{4}{a + b + \sqrt{(a + b)^2 + 32c}}; \tag{2.5}$$

$$\textit{for all } x, y \in D_0 \Rightarrow 2y - x \in D_0. \tag{2.6}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by method (1.2) is well defined, remains in $U(x^*, r^*)$ for all $n \geq 0$ and converges to x^* provided that

$$x_{-1},\ x_0 \textit{ belong in } U(x^*, r^*). \tag{2.7}$$

Moreover the following estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{b\|x_n - x^*\| + c\|x_{n-1} - x_n\|^2}{1 - a\|x_n - x^*\| - c\|x_{n-1} - x_n\|^2}\|x_n - x^*\|. \tag{2.8}$$

Proof. Let us denote by $L = L(x, y)$ the linear operator

$$L = [2y - x, x]. \tag{2.9}$$

Assume $x, y \in U(x^*, r^*)$. We shall show L is invertible on $U(x^*, r^*)$, and

$$\|L^{-1}F'(x^*)\| \leq [1 - a\|y - x^*\| - c\|x - y\|^2]^{-1} \leq [1 - ar^* - 4c(r^*)^2]^{-1}. \tag{2.10}$$

Using (2.1) and (2.3), we obtain in turn:

(2.11)

$$\begin{aligned} \|F'(x^*)^{-1}[F'(x^*) - L]\| &= \|F'(x^*)^{-1}[([x^*, x^*] - [y, y]) + ([y, y] - [2y - x, x])] \\ &\leq a\|y - x^*\| + c\|y - x\|^2 \\ &\leq ar^* + c[\|y - x^*\| + \|x^* - x\|]^2 \\ &\leq ar^* + 4c(r^*)^2 < 1 \end{aligned}$$

by the choice of r^*.

It follows from the Banach lemma on invertible operators [14] and (2.11) that L^{-1} exists on $U(x^*, r^*)$, so that estimate (2.10) holds. We can also have by (2.2) and (2.3):

(2.12)

$$\begin{aligned} \|F'(x^*)^{-1}([y, x^*] - L)\| &= \|F'(x^*)^{-1}[([y, x^*] - [y, y]) + ([y, y] - L)]\| \\ &\leq \|F'(x^*)^{-1}([y, x^*] - [y, y])\| + \|F'(x^*)^{-1}([y, y] - L)\| \\ &\leq b\|y - x^*\| + c\|y - x\|^2 \\ &\leq br^* + 4c(r^*)^2. \end{aligned}$$

Moreover, by (1.2) we get for $y = x_n$, $x = x_{n-1}$

$$\begin{aligned} \|x_{n+1} - x^*\| &= \| - L_n^{-1}([x_n, x^*] - L_n)(x_n - x^*)\| \\ &\leq \|L_n^{-1}F'(x^*)\| \cdot \|F'(x^*)^{-1}([x_n, x^*] - L_n)\| \cdot \|x_n - x^*\|. \end{aligned} \tag{2.13}$$

Estimate (2.8) now follows from (2.10), (2.11) and (2.13). Furthermore from (2.10), (2.11) and (2.13) we get

$$\|x_{n+1} - x^*\| < \|x_n - x^*\| < r^* \quad (n \geq 0). \tag{2.14}$$

Hence, sequence $\{x_n\}$ $(n \geq -1)$ is well defined, remains in $U(x^*, r^*)$ for all $n \geq -1$ and converges to x^*. That completes the proof of Theorem 2.1. □

Let $x, y, z \in D_0$, and define the divided difference of order two of operator F at the points x, y and z denoted by $[x, y, z]$ by

$$[x, y, z](y - z) = [x, y] - [x, z]. \tag{2.15}$$

Remark 2.2. In order for us to compare method (1.2) with others [18] using divided differences of order one, consider the condition

$$\|F'(x^*)^{-1}([x, y] - [u, v])\| \leq \bar{a}(\|x - u\| + \|y - v\|) \tag{2.16}$$

instead of (2.1) and (2.2). Note that (2.16) implies (2.1) and (2.2). Moreover we have:

$$a \leq 2\bar{a} \tag{2.17}$$

and

$$b \leq 2\bar{a}. \tag{2.18}$$

Therefore stronger but more popular condition (2.16) can replace (2.1) and (2.2) in Theorem 2.1.

Assuming F has divided differences of order two, condition (2.3) can be replaced by the stronger

$$\|F'(x^*)^{-1}([y,x,y] - [2y-x,x,y])(y-x)\| \leq \bar{c}\|y-x\|^2, \tag{2.19}$$

or the even stronger

$$\|F'(x^*)^{-1}([u,x,y] - [v,x,y])(y-x)\| \leq \bar{\bar{c}}\|u-v\|^2. \tag{2.20}$$

Note also that

$$\bar{c} \leq \bar{\bar{c}} \tag{2.21}$$

and we can set

$$c = \bar{c} \tag{2.22}$$

despite the fact that $\bar{c}$ (or $\bar{\bar{c}}$) is more difficult to compute since we use divided differences of order two (instead of one). Conditions (2.16) and (2.20) were used in [18] to show method

$$y_{n+1} = y_n - ([y_n, y_{n-1}] + [y_{n-2}, y_n] - [y_{n-2}, y_{n-1}])^{-1}F(y_n) \quad (n \geq 0) \tag{2.23}$$

converges to x^* with order $1.839\ldots$ which is the solution of the scalar equation

$$t^3 - t^2 - t - 1 = 0. \tag{2.24}$$

Potra in [18] has also shown how to compute the Lipschitz constants appearing here in some cases.

It follows from (2.8) that there exist a constant c_0, and N a sufficiently large integer such that:

$$\|x_{n+1} - x^*\| \leq c_0\|x_n - x^*\|^2 \quad \text{for all } n \geq N. \tag{2.25}$$

Hence the order of convergence for method (1.2) is essentially at least two, which is higher than $1.839\ldots$. Note also that the radius of convergence r^* given by (2.5) is larger than the corresponding one given in [18, estimate (22)]. This observation is very important since it allows a wider choice of initial guesses x_{-1} and x_0.

It turns out that our convergence radius r^* given by (2.8) can even be larger than the one given by Rheinboldt [19] (see, e.g., [18, Remark 4.2]) for Newton's method and Example 2.3 that follows. Indeed under condition (2.17) radius r_R^* is given by

$$r_R^* = \frac{1}{3\overline{a}}. \tag{2.26}$$

We showed in [7] that $\frac{\overline{a}}{a}$ (or $\frac{\overline{a}}{b}$) can be arbitrarily large. Hence we can have:

$$r_R^* < r^*. \tag{2.27}$$

In [7] we also showed that r_R^* is enlarged under the same hypotheses and computational cost as in [19].

We note that condition (2.6) suffices to hold only for x, y being iterates of method (1.2) (see, e.g., Example 3.4).

Condition (2.6) can be removed if $D_0 = X$. In this case (2.4) is also satisfied.

Finally delicate condition (2.6) can also be replaced by a stronger but more practical one which we decided not to introduce originally in Theorem 2.1, so we can leave the result as uncluttered-general as possible.

Indeed, define ball U_1 by

$$U_1 = U(x^*, R^*) \quad \text{with} \quad R^* = 3r^*. \tag{2.28}$$

If $x_{n-1}, x_n \in U^*$ $(n \geq 0)$ then we conclude $2x_n - x_{n-1} \in U_1$ $(n \geq 0)$. This is true since it follows from the estimates

$$\begin{aligned} \|2x_n - x_{n-1} - x^*\| &\leq \|x_n - x^*\| + \|x_n - x_{n-1}\| \\ &\leq 2\|x_n - x^*\| + \|x_{n-1} - x^*\| < 3r^* = R^* \quad (n \geq 0). \end{aligned} \tag{2.29}$$

Hence the proof of Theorem 2.1 goes through if both conditions (2.4), (2.6) are replaced by

$$U_1 \subseteq D_0. \tag{2.30}$$

We complete this section with a numerical example to justify estimate (2.27).

Example 2.3. Let $X = Y = \mathbf{R}$, $x^* = 0$, $D = U(0,1)$ and define function F on D by

$$F(x) = e^x - 1. \tag{2.31}$$

Using (2.1)–(2.3), (2.16) and (2.31), we obtain

$$a = b = e - 1, \quad c = e \quad \text{and} \quad \overline{a} = \frac{e}{2}. \tag{2.32}$$

In view of (2.5) and (2.26), we have

$$r_R = .24525296 < .299040145 = r^*. \tag{2.33}$$

We can also set $R^* = 3r^* = .897120435$.

3. Semilocal Convergence of Method (1.2)

We can show the following result for the semilocal convergence of method (1.2).

Theorem 3.1. *Let F be a nonlinear operator defined on an open set D of a Banach space X with values in a Banach space Y.*

Assume:

the operator F has divided differences of order one and two on $D_0 \subseteq D$;

there exist points x_{-1}, x_0 in D_0 such that $2x_0 - x_{-1} \in D_0$ and $A_0 = [2x_0 - x_{-1}, x_{-1}]$ is invertible on D_0;

Set $A_n = [2x_n - x_{n-1}, x_{n-1}]$ $(n \geq 0)$.

There exist constants α, β such that:

$$\|A_0^{-1}([x, y] - [u, v])\| \leq \alpha(\|x - u\| + \|y - v\|), \tag{3.1}$$

$$\|A_0^{-1}([y, x, y] - [2y - x, x, y])\| \leq \beta\|x - y\| \tag{3.2}$$

for all $x, y, u, v \in D_0$, and condition (2.6) holds;

Define constants γ, δ by

$$\|x_0 - x_{-1}\| \leq \gamma, \tag{3.3}$$

$$\|A_0^{-1}F(x_0)\| \leq \delta, \tag{3.4}$$

$$2\beta\gamma^2 \leq 1; \tag{3.5}$$

Moreover define θ, r, h by

$$\theta = \left\{(\alpha + \beta\gamma)^2 + 3\beta(1 - \beta\gamma^2)\right\}^{1/2}, \tag{3.6}$$

$$r = \frac{1 - \beta\gamma^2}{\alpha + \beta\gamma + \theta}, \tag{3.7}$$

and

$$h(t) = -\beta t^3 - (\alpha + \beta\gamma)t^2 + (1 - \beta\gamma^2)t, \tag{3.8}$$

$$\delta \leq h(r) = \frac{1}{3}\frac{\alpha + \beta\gamma + 2\theta}{1 - 2\beta\gamma^2}r^2; \tag{3.9}$$

(3.10) $$U_0 = U(x_0, r_0) \subseteq D_0,$$

where $r_0 \in (0, r]$ *is the unique solution of equation*

(3.11) $$h(t) = (1 - 2\beta\gamma^2)\delta$$

on interval $(0, r]$.

Then sequence $\{x_n\}$ $(n \geq -1)$ *generated by method* (1.2) *is well defined, remains in* $U(x_0, r_0)$ *for all* $n \geq -1$ *and converges to a solution* x^* *of equation* $F(x) = 0$.

Moreover the following estimates hold for all $n \geq -1$

(3.12) $$\|x_{n+1} - x_n\| \leq t_n - t_{n+1},$$

and

(3.13) $$\|x_n - x^*\| \leq t_n,$$

where

(3.14) $$t_{-1} = r_0 + \gamma, \quad t_0 = r_0,$$

(3.15) $$\gamma_0 = \alpha + 3\beta r_0 + \beta\gamma, \quad \gamma_1 = 3\beta r_0^2 - 2\gamma_0 r_0 - \beta\gamma^2 + 1,$$

and for $n \geq 0$

(3.16) $$t_{n+1} = \frac{\gamma_0 t_n - (t_n - t_{n-1})^2\beta - 2\beta t_n^2}{\gamma_1 + 2\gamma_0 t_n - (t_n - t_{n-1})^2 - 3\beta t_n^2} \cdot t_n.$$

Furthermore if $r_0 \leq r_1$, *and*

(3.17) $$\alpha(2\gamma + r_0 + r_1) < 1,$$

x^* *is the unique solution of equation* (1.1) *in* $\overline{U}(x_0, r_1)$.

Proof. Sequence $\{t_n\}$ $(n \geq -1)$ generated by (3.14) and (3.16) can be obtained if method (1.2) is applied to the scalar polynomial

(3.18) $$f(t) = -\beta t^3 + \gamma_0 t^2 + \gamma_1 t.$$

It is simple calculus to show sequence $\{t_n\}$ $(n \geq -1)$ converges monotonically to zero (decreasingly).

We can have:

(3.19) $$t_{k+1} - t_{k+2} = \frac{2(t_{k+1} - t_k)}{f(2t_{k+1} - t_k) - f(t_k)} f(t_{k+1})$$

$$
\begin{aligned}
&= \frac{\{[\gamma_0 - (2t_k + t_{k+1})\beta](t_k - t_{k+1}) + (t_k - t_{k-1})^2\beta\}(t_k - t_{k+1})}{1 - \beta\gamma^2 - 2(t_0 - t_{k+1})\alpha - [3(t_0 - t_{k+1})(3t_0 + t_{k+1}) - (t_k - t_{k+1})^2]\beta} \\
&\geq \frac{(t_k - t_{k+1})\alpha + (t_{k-1} - t_k)^2\beta}{1 - 2(t_0 - t_{k+1})\alpha - \beta\gamma^2}(t_k - t_{k+1}).
\end{aligned}
\tag{3.20}
$$

We show (3.12) holds for all $k \geq -1$. Using (3.3)–(3.8) and

$$
t_0 - t_1 = \left[1 - \frac{\gamma_0 t_0 - (t_0 - t_{-1})^2\beta - 2\beta t_0^2}{\gamma_1 + 2\gamma_0 t_0 - (t_0 - t_{-1})^2\beta - 3\beta t_0^2}\right] t_0 = \frac{h(r_0)}{1 - 2\beta\gamma^2} = c
\tag{3.21}
$$

we conclude that (3.12) holds for $n = -1, 0$. Assume (3.12) holds for all $n \leq k$ and $x_k \in U(x_0, r_0)$. By (2.6) and (3.12) $x_{k+1} \in U(x_0, r_0)$. By (3.1), (3.2) and (3.12)

(3.22)

$$
\begin{aligned}
\|A_0^{-1}(A_0 - A_{k+1})\| &= \|A_0^{-1}([2x_0 - x_{-1}, x_{-1}] - [x_0, x_{-1}] + [x_0, x_{-1}] - [x_0, x_0] \\
&\quad + [x_0, x_0] - [x_{k+1}, x_0] + [x_{k+1}, x_0] - [x_{k+1}, x_k] \\
&\quad + [x_{k+1}, x_k] - [2x_{k+1} - x_k, x_k])\| \\
&= \|A_0^{-1}(([2x_0 - x_{-1}, x_{-1}, x_0] - [x_0, x_{-1}, x_0])(x_0 - x_{-1}) \\
&\quad + ([x_0, x_0] - [x_{k+1}, x_0]) + ([x_{k+1}, x_0] - [x_{k+1}, x_k]) \\
&\quad + ([x_{k+1}, x_k] - [2x_{k+1} - x_k, x_k]))\| \\
&\leq \beta\gamma^2 + (\|x_0 - x_{k+1}\| + \|x_0 - x_k\| + \|x_k - x_{k+1}\|)\alpha \\
&\leq \beta\gamma^2 + 2(t_0 - t_{k+1})\alpha < 2\beta\gamma^2 + 2\alpha r \leq 1.
\end{aligned}
$$

It follows by the Banach lemma on invertible operators and (3.22) that A_{k+1}^{-1} exists, so that

$$
\|A_{k+1}^{-1} A_0\| \leq [1 - \beta\gamma^2 - (\|x_0 - x_{k+1}\| + \|x_0 - x_k\| + \|x_k - x_{k+1}\|)\alpha]^{-1}.
\tag{3.23}
$$

We can also obtain

$$
\begin{aligned}
&\|A_0^{-1}([x_{k+1}, x_k] - A_k)\| \\
&= \|A_0^{-1}([x_{k+1}, x_k] - [x_k, x_k] \\
&\quad + [x_k, x_k] - [x_k, x_{k-1}] + [x_k, x_{k-1}] - [2x_k - x_{k-1}, x_{k-1}])\| \\
&= \|A_0^{-1}(([x_{k+1}, x_k] - [x_k, x_k]) \\
&\quad + ([x_k, x_{k-1}, x_k] - [2x_k - x_{k-1}, x_{k-1}, x_k])(x_k - x_{k-1}))\| \\
&\leq \alpha\|x_k - x_{k+1}\| + \beta\|x_{k-1} - x_k\|^2.
\end{aligned}
\tag{3.24}
$$

Using (1.2), (3.23) and (2.24) we get

$$
\|x_{k+2} - x_{k+1}\| = \|A_{k+1}^{-1} F(x_{k+1})\| = \|A_{k+1}^{-1}(F(x_{k+1}) - F(x_k) - A_k(x_{k+1} - x_k))\|
$$

(3.25)

$$\leq \|A_{k+1}^{-1}A_0\| \, \|A_0^{-1}([x_{k+1},x_k]-A_k)\| \cdot \|x_k - x_{k+1}\|$$
$$\leq \frac{\alpha\|x_k - x_{k+1}\| + \beta\|x_{k-1}-x_k\|^2}{1-\beta\gamma^2 - \alpha(\|x_0 - x_{k+1}\| + \|x_0 - x_k\| + \|x_k - x_{k+1}\|)}\|x_k - x_{k+1}\|$$
$$\leq \frac{[\alpha(t_k - t_{k+1}) + \beta(t_{k-1}-t_k)^2](t_k - t_{k+1})}{1-\beta\gamma^2 - 2(t_0 - t_{k+1})\alpha} \leq t_{k+1} - t_{k+2},$$

which together with (3.11) completes the induction.

It follows from (3.12) that sequence $\{x_n\}$ $(n \geq -1)$ is Cauchy in a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, r_0)$ (since $\overline{U}(x_0, r_0)$ is a closed set). By letting $k \to \infty$ in (3.25) we obtain $F(x^*) = 0$.

Finally to show uniqueness, define operator

$$B_0 = [y^*, x^*] \tag{3.26}$$

where y^* is a solution of equation (1.1) in $\overline{U}(x_0, r_1)$. We can have

(3.27)

$$\begin{aligned}\|A_0^{-1}(A_0 - B_0)\| &\leq \alpha[\|y^* - (2x_0 - x_1)\| + \|x^* - x_{-1}\|] \\ &\leq \alpha[\|(y^* - x_0) - (x_0 - x_{-1}\| + \|(x^* - x_0) + (x_0 - x_{-1})\|] \\ &\leq \alpha[\|y^* - x_0\| + 2\|x_0 - x_{-1}\| + \|x^* - x_0\|] \\ &\leq \alpha(2\gamma + r_0 + r_1) < 1.\end{aligned}$$

It follows from the Banach lemma on invertible operators and (3.27) that linear operator B is invertible.

We deduce from (3.26) and the identity

$$F(x^*) - F(y^*) = B_0(x^* - y^*) \tag{3.28}$$

that

$$x^* = y^*. \tag{3.29}$$

The proof of Theorem 3.1 is now complete. □

Remark 3.2. (a) It follows from (3.12), (3.13), (3.20) and (3.25) that the order of convergence of scalar sequence $\{t_n\}$ and iteration $\{x_n\}$ is quadratic.

(b) The conclusions of Theorem 3.1 hold in a weaker setting. Indeed assume:

$$\|A_0^{-1}([x_0, x_0] - [x, x_0])\| \leq \alpha_0\|x - x_0\|, \tag{3.30}$$

$$\|A_0^{-1}([x, x_0] - [x, y])\| \leq \alpha_1\|y - x_0\|, \tag{3.31}$$

$$\|A_0^{-1}([y, x] - [2y - x, x])\| \leq \alpha_2\|y - x\|, \tag{3.32}$$

(3.33) $$\|A_0^{-1}([y,x]-[x,x])\| \leq \alpha_3\|y-x\|,$$

(3.34) $$\|A_0^{-1}([2x_0-x_{-1},x_0]-[x,y])\| \leq \alpha_4(\|2x_0-x_{-1}-x\|+\|x_0-y\|)$$

and

(3.35) $$\|A_0^{-1}([2x_0-x_{-1},x_{-1},x_0]-[x_0,x_{-1},x_0])\| \leq \beta_0\|x_0-x_{-1}\|$$

for all $x, y \in D_0$.

It follows from (3.1), (3.2) and (3.30)–(3.35) that

(3.36) $$\alpha_i \leq 2\alpha, \quad i=1,2,3,4$$

and

(3.37) $$\beta_0 \leq \beta.$$

For the derivation of: (3.23), we can use (3.30)–(3.32) and (3.32) instead of (3.1) and (3.2), respectively; (3.24), we can use (3.33) instead of (3.1); (3.27), we can use (3.25) instead of (3.1). The resulting majorizing sequence call it $\{s_n\}$ is also converging to zero and is finer than $\{t_n\}$ because of (3.36) and (3.37).

Therefore if (2.7), (3.30)–(3.35) are used in Theorem 3.1 instead of (3.1) we draw the same conclusions but with weaker conditions, and corresponding error bounds are such that:

(3.38) $$\|x_{n+1}-x_n\| \leq s_n - s_{n+1} \leq t_{n+1}-t_n$$

and

(3.39) $$\|x_n - x^*\| \leq s_n \leq t_n$$

for all $n \geq 0$.

(c) Condition (3.2) can be replaced by the stronger (not really needed in the proof) but more popular,

(3.40) $$\|A_0^{-1}([v,x,y]-[u,x,y])\| \leq \beta_1\|u-v\|$$

for all $v, u, x, y \in D_0$.

(d) As already noted at the end of Remark 2.2, conditions (2.7) and (3.10) can be replaced by

(3.41) $$U_2 = U(x_0, R_0) \subseteq D_0 \quad \text{with} \quad R_0 = 3r_0$$

provided that $x_{-1} \in U_2$.

Indeed if $x_{n-1}, x_n \in U_0$ $(n \geq 0)$ then

$$\|2x_n - x_{n-1}\| \leq 2\|x_n - x_0\| + \|x_{n-1} - x_0\| < 3r_0.$$

That is $2x_n - x_{n-1} \in U_2$ $(n \geq 0)$.

We can also provide a posteriori estimates for method (1.2):

Proposition 3.3. *Assume hypotheses of Theorem 3.1 hold. Define scalar sequences* $\{p_n\}$ *and* $\{q_n\}$ *for all* $n \geq 1$ *by:*

$$p_n = \alpha\|x_{n-1} - x_n\|^2 + \beta\|x_{n-1} - x_{n-2}\|^2\|x_{n-1} - x_n\| \tag{3.42}$$

and

$$q_n = 1 - 2\alpha\|x_n - x_0\| + \beta\gamma^2. \tag{3.43}$$

Then the following error bounds hold for all $n \geq 1$:

$$\|x_n - x^*\| \leq \varepsilon_n \leq t_n, \tag{3.44}$$

where,

$$\varepsilon_n = 2\{q_n + (q_n^2 - 4\alpha p_n)^{1/2}\}^{-1} p_n. \tag{3.45}$$

Proof. As in (3.22) we can have in turn:

$$\begin{aligned} \|A_0^{-1}(A_0 - [x_n, x^*])\| &= \|A_0^{-1}(A_0 - [x_0, x_0] + [x_0, x_0] - [x_n, x^*])\| \\ &\leq \beta\gamma^2 + \alpha(\|x_0 - x_n\| + \|x_0 - x^*\|) \\ &\leq \beta\gamma^2 + \alpha(2t_0 - t_n) \\ &< \beta\gamma^2 + 2\alpha r_0 \leq 1. \end{aligned} \tag{3.46}$$

It follows from (3.42) and the Banach lemma on invertible operators that linear operator $[x_n, x^*]$ is invertible, with

$$\|[x_n, x^*]^{-1}A_0\| \leq (q_n - \alpha\|x_n - x^*\|)^{-1}. \tag{3.47}$$

Using (1.2) we obtain the approximation:

$$x_n - x^* = ([x_n, x^*]^{-1}A_0)(A_0^{-1}F(x_n)). \tag{3.48}$$

By (3.24), (3.47) and (3.48) we obtain the left-hand side estimate of (3.44).

Moreover we can have in turn:

(3.49)

$$\begin{aligned}\|x_n - x^*\| &\le \frac{p_n}{q_n - \alpha\|x_n - x^*\|} \le \frac{\alpha(t_{n-1}-t_n)^2 + \beta(t_{n-1}-t_n)(t_{n-1}-t_{n-2})^2}{1-\beta\gamma^2 - 2\alpha(t_0 - t_n) - \alpha t_n} \\ &\le \frac{\{[\gamma_0 - \beta(2t_{n-1}+t_n)](t_{n-1}-t_n) + \beta(t_{n-2}-t_{n-1})^2\}(t_{n-1}-t_n)}{1-\beta\gamma^2 - 2\alpha(t_0-t_n) - \alpha t_n - (3r_0+\gamma)(t_0-t_n)\beta - \beta t_n^2 - \beta\gamma r_0} \\ &\le \frac{-\beta t_n^3 + \alpha t_n^2 + \gamma_1}{-\beta r_n^2 + \alpha t_n + \gamma_1} = t_n.\end{aligned}$$

That completes the proof of Proposition 3.3. □

A simple numerical example follows to show:

(a) how to choose divided difference in method (1.2);

(b) method (1.2) is faster than the Secant method

$$x_{n+1} = x_n - [x_n, x_{n-1}]^{-1} F(x_n) \quad (n \ge 0) \tag{3.50}$$

(c) method (1.2) can be as fast as Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \ge 0). \tag{3.51}$$

Note that the analytical representation of $F'(x_n)$ may be complicated which makes the use of method (1.2) very attractive.

Example 3.4. Let $X = Y = \mathbf{R}$, and define function F on $D_0 = D = (.4, 1.5)$ by

$$F() = x^2 - 6x + 5. \tag{3.52}$$

Moreover define divided difference of order one appearing in method (1.2) by

$$[2y - x, x] = \frac{F(2y-x) - F(x)}{2(y-x)}. \tag{3.53}$$

In this case method (1.2) becomes

$$x_{n+1} = \frac{x_n^2 - 5}{2(x_n - 3)}, \tag{3.54}$$

and coincides with Newton's method (3.51) applied to F. Furthermore Secant method (3.50) becomes:

$$x_{n+1} = \frac{x_{n-1}x_n - 5}{x_{n-1} + x_n - 6}. \tag{3.55}$$

Choose $x_{-1} = .6$ and $x_0 = .7$. Then we obtain:

n	Method (1.2)	Secant method (3.55)
1	.980434783	.96875
2	.999905228	.997835498
3	.999999998	.99998323
4	$1 = x^*$	.999999991
5	—	1

We conclude this section with an example involving a nonlinear integral equation:

Example 3.5. Let $H(x, t, x(t))$ be a continuous function of its arguments which is sufficiently many times differentiable with respect to x. It can easily be seen that if operator F in (1.2) is given by

$$F(x(s)) = x(s) - \int_0^1 H(s, t, x(t))dt, \tag{3.56}$$

then divided difference of order one appearing in (1.2) can be defined as

$$h_n(s,t) = \frac{H(s,t,2x_n(t) - x_{n-1}(t)) - H(s,t,x_{n-1}(t))}{2(x_n(t) - x_{n-1}(t))}, \tag{3.57}$$

provided that if for $t = t_m$ we get $x_n(t) = x_{n-1}(t)$, then the above function equals $H'_x(s, t_m, x_n(t_m))$. Note that this way $h_n(s,t)$ is continuous for all $t \in [0,1]$.

4. Monotone Convergence of Method (1.2)

We refer the reader to [8], [14], [16], [21] for the concepts concerning partially ordered topological spaces (POTL-spaces).

The monotone convergence of method (1.2) is examined in the next result.

Theorem 4.1. *Let F be a nonlinear operator defined on an open subset of a regular POTL-space X with values in a POTL-space Y. Let x_0, y_0, y_{-1} be points of D such that:*

$$x_0 \le y_0 \le y_{-1}, \quad D_0 = \langle x_0, y_{-1}\rangle \subseteq D, \quad F(x_0) \le 0 \le F(y_0). \tag{4.1}$$

Moreover assume: there exists a divided difference $[\cdot,\cdot]\colon D \to L(X,Y)$ such that for all $(x,y) \in D_0^2$ with $x \le y$

$$2y - x \in D_0, \tag{4.2}$$

and

$$F(y) - F(x) \le [x, 2y - x](y - x). \tag{4.3}$$

Furthermore, assume that for any $(x, y) \in D_0^2$ *with* $x \leq y$*, and* $(x, 2y - x) \in D_0^2$ *the linear operator* $[x, 2y - x]$ *has a continuous non-singular, non-negative left subinverse.*

Then there exist two sequences $\{x_n\}$ $(n \geq 1)$*,* $\{y_n\}$ $(n \geq 1)$*, and two points* x^**,* y^* *of* X *such that for all* $n \geq 0$:

$$F(y_n) + [y_{n-1}, 2y_n - y_{n-1}](y_{n+1} - y_n) = 0, \tag{4.4}$$

$$F(x_n) + [y_{n-1}, 2y_n - y_{n-1}](x_{n+1} - x_n) = 0, \tag{4.5}$$

$$F(x_n) \leq 0 \leq F(y_n), \tag{4.6}$$

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{4.7}$$

$$\lim_{n \to \infty} x_n = x^*, \quad \lim_{n \to \infty} y_n = y^*. \tag{4.8}$$

Finally, if linear operators $A_n = [y_{n-1}, 2y_n - y_{n-1}]$ *are inverse non-negative, then any solution of the equation* $F(x) = 0$ *from the interval* D_0 *belongs to the interval* $\langle x^*, y^* \rangle$ *(i.e.,* $x_0 \leq v \leq y_0$ *and* $F(v) = 0$ *imply* $x^* \leq v \leq y^*$ *).*

Proof. Let $\overline{A}_0$ be a continuous non-singular, non-negative left subinverse of A_0. Define the operator $Q\colon \langle 0, y_0 - x_0 \rangle \to X$ by

$$Q(x) = x - \overline{A}_0[F(x_0) + A_0(x)].$$

It is easy to see that Q is isotone and continuous. We also have:

$$\begin{aligned} Q(0) &= -\overline{A}_0 F(x_0) \geq 0, \\ Q(y_0 - x_0) &= y_0 - x_0 - \overline{A}_0(F(y_0)) + \overline{A}_0(F(y_0) - F(x_0) - A_0(y_0 - x_0)) \\ &\leq y_0 - x_0 - \overline{A}_0(F(y_0)) \leq y_0 - x_0. \end{aligned}$$

According to Kantorovich's theorem on POTL-spaces [4], [12], [20] for fixed points, operator Q has a fixed point $w \in \langle 0, y_0 - x_0 \rangle$. Set $x_1 = x_0 + w$. Then we get

$$F(x_0) + A_0(x_1 - x_0) = 0, \quad x_0 \leq x_1 \leq y_0. \tag{4.9}$$

By (4.3) and (4.9) we deduce:

$$F(x_1) = F(x_1) - F(x_0) + A_0(x_0 - x_1) \leq 0.$$

Consider the operator $H\colon \langle 0, y_0 - x_1 \rangle \to X$ given by

$$H(x) = x + \overline{A}_0(F(y_0) - A_0(x)).$$

Operator H is clearly continuous, isotone and we have:

$$\begin{aligned} H(0) &= \overline{A}_0 F(y_0) \geq 0, \\ H(y_0 - x_1) &= y_0 - x_1 + \overline{A}_0 F(x_1) + \overline{A}_0[F(y_0) - F(x_1) - A_0(y_0 - x_1)] \\ &\leq y_0 - x_1 + \overline{A}_0 F(x_1) \leq y_0 - x_1. \end{aligned}$$

By Kantorovich's theorem there exists a point $z \in \langle 0, y_0 - x_1 \rangle$ such that $H(z) = z$. Set $y_1 = y_0 - z$ to obtain

$$F(y_0) + A_0(y_1 - y_0) = 0, \quad x_1 \leq y_1 \leq y_0. \tag{4.10}$$

Using (4.3), (4.10) we get:

$$F(y_1) = F(y_1) - F(y_0) - A_0(y_1 - y_0) \geq 0.$$

Proceeding by induction we can show that there exist two sequences $\{x_n\}$ $(n \geq 1)$, $\{y_n\}$ $(n \geq 1)$ satisfying (4.4)–(4.7) in a regular space X, and as such they converge to points $x^*, y^* \in X$ respectively. We obviously have $x^* \leq y^*$. If $x_0 \leq u \leq y_0$ and $F(u) = 0$, then we can write

$$A_0(y_1 - u) = A_0(y_0) - F(y_0) - A_0(u) = A_0(y_0 - u) - (F(y_0) - F(u)) \geq 0$$

and

$$A_0(x_1 - u) = A_0(x_0) - F(x_0) - A_0(u) = A_0(x_0 - u) - (F(x_0) - F(u)) \leq 0.$$

If the operator A_0 is inverse non-negative then it follows that $x_1 \leq u \leq y_1$. Proceeding by induction we deduce that $x_n \leq u \leq y_n$ holds for all $n \geq 0$. Hence we conclude

$$x^* \leq u \leq y^*.$$

That completes the proof of Theorem 4.1. □

In what follows we give some natural conditions under which the points x^* and y^* are solutions of equation $F(x) = 0$.

Proposition 4.2. *Under the hypotheses of Theorem 4.1, assume that F is continuous at x^* and y^* if one of the following conditions is satisfied:*

(a) $x^* = y^*$;

(b) *X is normal, and there exists an operator $T\colon X \to Y$ $(T(0) = 0)$ which has an isotone inverse continuous at the origin and such that $A_n \leq T$ for sufficiently large n;*

(c) *Y is normal and there exists an operator $Q\colon X \to Y$ $(Q(0) = 0)$ continuous at the origin and such that $A_n \le Q$ for sufficiently large n;*

(d) *operators A_n $(n \ge 0)$ are equicontinuous.*

Then we deduce

$$F(x^*) = F(y^*) = 0. \tag{4.11}$$

Proof. (a) Using the continuity of F and (4.6) we get

$$F(x^*) \le 0 \le F(y^*).$$

Hence, we conclude

$$F(x^*) = 0.$$

(b) Using (4.4)–(4.7) we get

$$0 \ge F(x_n) = A_n(x_n - x_{n+1}) \ge T(x_n - x_{n+1}),$$
$$0 \le F(y_n) = A_n(y_n - y_{n+1}) \le T(y_n - y_{n+1}).$$

Therefore, it follows:

$$0 \ge T^{-1}F(x_n) \ge x_n - x_{n+1}, \quad 0 \le T^{-1}F(y_n) \le y_n - y_{n+1}.$$

By the normality of X, and

$$\lim_{n\to\infty}(x_n - x_{n+1}) = \lim_{n\to\infty}(y_n - y_{n+1}) = 0,$$

we get $\lim_{n\to\infty} T^{-1}F(x_n)) = \lim\limits_{n\to\infty} T^{-1}(F(y_n)) = 0$. Using the continuity of F we obtain (4.11).

(c) As before for sufficiently large n

$$0 \ge F(x_n) \ge Q(x_n - x_{n+1}), \quad 0 \le F(y_n) \le Q(y_n - y_{n+1}).$$

By the normality of Y and the continuity of F and Q we obtain (4.11).

(d) It follows from the equicontinuity of operator A_n that $\lim\limits_{n\to\infty} A_n v_n = 0$ whenever $\lim\limits_{n\to\infty} v_n = 0$. Therefore, we get $\lim\limits_{n\to\infty} A_n(x_n - x_{n+1}) = \lim\limits_{n\to\infty} A_n(y_n - y_{n+1}) = 0$. By (4.4), (4.5), and the continuity of F at x^* and y^* we obtain (4.11).

That completes the proof of Proposition 4.2. □

Remark 4.3. Hypotheses of Theorem 4.1 can be weakened along the lines of Remarks 2.2, and 3.2 above and the works in [18, pp. 102–105], [8], [16] on the monotone convergence of Newton-like methods. However, we leave the details to the motivated reader.

Remark 4.4. We finally note that (1.2) is a special case of the class of methods of the form:

(4.12) $$x_{n+1} = x_n - [(1+\lambda_n)x_n - \lambda_n x_{n-1}, x_{n-1}]^{-1} F(x_n) \quad (n \geq 0)$$

where λ_n are real numbers depending on x_{n-1} and x_n, i.e.,

(4.13) $$\lambda_n = \lambda(x_{n-1}, x_n) \quad (n \geq 0), \quad \lambda\colon X^2 \to R,$$

and are chosen so that in practice, e.g.,

(4.14) $$\text{for all } x, y \in D \Rightarrow (1 + \lambda(x,y))y - \lambda(x,y)x \in D.$$

Note that setting $\lambda(x, y) = 1$ for all $x, y \in D$ in (4.12) we obtain (1.2).

Using (4.12) instead of (1.2) all the results obtained here can immediately be reproduced in this more general setting.

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