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ON 4-PERMUTING 4-DERIVATIONS IN PRIME AND SEMIPRIME RINGS

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ABSTRACT. Let R be a 2-torsion free semiprime ring. Suppose that there exists a 4-permuting 4-derivation $\Delta : R \times R \times R \times R \to R$ such that the trace is centralizing on R. Then the trace of Δ is commuting on R. In particular, if R is a 3!-torsion free prime ring and Δ is nonzero under the same condition, then R is commutative.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, R will represent an associative ring, and Z will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We will also use the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Then a map $f : R \to R$ is said to be *commuting* (resp. *centralizing*) on R if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z$) for all $x \in R$. A map $\Delta : R \times R \times R \times \cdots \times R \to R$ will be said to be *n-permuting* ($n \geq 3$) if the equation $\Delta(x_1, x_2, , \cdots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_1, x_2, \cdots, x_n \in R$ and for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$. Recall that R is *semiprime* if $xRx = \{0\}$ implies $x = 0$ and R is *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$.

An additive map $d : R \to R$ is called a *derivation* if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

By a *bi-derivation* we mean a bi-additive map $B : R \times R \to R$ (i.e., B is additive in both arguments) which satisfies the relations

$$B(xy, z) = B(x, z)y + xB(y, z),$$
$$B(x, yz) = B(x, y)z + yB(x, z)$$

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for all $x, y \in R$. Let B be symmetric, that is, $B(x,y) = B(y,x)$ for all $x, y \in R$. The map $\beta : R \to R$ defined by $\beta(x) = B(x,x)$ for all $x, y \in R$ is called the trace of B. If R is a noncommutative 2-torsion free prime ring and $B : R \times R \to R$ is a symmetric bi-derivation, then it follows from [1, Theorem 3.5] that $B = 0$.

A 3-additive map $D : R \times R \times R \to R$ (i.e., additive in each argument) will be called a 3-*derivation* if the relations

$$D(x_1x_2, y, z) = D(x_1, y, z)x_2 + x_1D(x_2, y, z),$$
$$D(x, y_1y_2, z) = D(x, y_1, z)y_2 + y_1D(x, y_2, z)$$

and

$$D(x, y, z_1z_2) = D(x, y, z_1)z_2 + z_1D(x, y, z_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i \in R$, $i = 1, 2$. We obtained some results concerning 3-permuting 3-derivations of prime and semiprime rings in [2].

Here we introduce the following map:

A 4-additive map $\Delta : R \times R \times R \times R \to R$ (i.e., additive in each argument) will be called a 4-*derivation* if the relations

$$\Delta(x_1x_2, y, z, w) = \Delta(x_1, y, z, w)x_2 + x_1\Delta(x_2, y, z, w),$$
$$\Delta(x, y_1y_2, z, w) = \Delta(x, y_1, z, w)y_2 + y_1\Delta(x, y_2, z, w),$$
$$\Delta(x, y, z_1z_2, w) = \Delta(x, y, z_1, w)z_2 + z_1\Delta(x, y, z_2, w)$$

and

$$\Delta(x, y, z, w_1w_2) = \Delta(x, y, z, w_1)w_2 + w_1\Delta(x, y, z, w_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i, w_i \in R$, $i = 1, 2$. If Δ is 4-permuting, then the above four relations are equivalent to each other.

For example, let R be commutative. A map $\Delta : R \times R \times R \times R \to R$ defined by $(x, y, z, w) \mapsto d(x)d(y)d(z)d(w)$ for all $x, y, z, w \in R$ is a 4-permuting 4-derivation, where d is a derivation on R.

On the other hand, let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \ a, b \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is a complex field. It is clear that R is a noncommutative ring under matrix addition and matrix multiplication. We define a map $\Delta : R \times R \times R \times R \to R$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_4 & b_4 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1a_2a_3a_4 \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that Δ is a 4-permuting 4-derivation.

Let a map $\delta : R \to R$ defined by $\delta(x) = \Delta(x,x,x,x)$ for all $x \in R$, where $\Delta : R \times R \times R \times R \to R$ is a 4-permuting map, be the *trace* of Δ. It is obvious that, in case when $\Delta : R \times R \times R \times R \to R$ is a 4-permuting map which is also 4-additive, the trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + 4\Delta(x,x,x,y) + 6\Delta(x,x,y,y) + 4\Delta(x,y,y,y)$$

for all $x, y \in R$. Since we have

$$\Delta(0,y,z,w) = \Delta(0+0,y,z,w) = \Delta(0,y,z,w) + \Delta(0,y,z,w)$$

for all $y, z, w \in R$, we obtain $\Delta(0,y,z,w) = 0$ for all $y, z, w \in R$. Hence we get

$$0 = \Delta(0,y,z,w) = \Delta(x-x,y,z,w) = \Delta(x,y,z,w) + \Delta(-x,y,z,w)$$

and so we see that $\Delta(-x,y,z,w) = -\Delta(x,y,z,w)$ for all $x, y, z \in R$. This tells us that δ is an even function.

A study concerning the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of E. C. Posner [4] which states that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Since then, a great deal of work in this context has been done by a number of authors (see, e.g., [1] and references therein). For example, as a study concerning centralizing (commuting) maps, J. Vukman [5, 6] investigated symmetric bi-derivations on prime and semiprime rings.

In this paper, we apply the results due to E. C. Posner [4] and J. Vukman [5] to 4-permuting 4-derivations, respectively.

2. The Main Results

We first need the following well-known lemma [3].

Lemma 1. *Let R be a prime ring. Let $d : R \to R$ be a derivation and $a \in R$. If $ad(x) = 0$ holds for all $x \in R$, then we have either $a = 0$ or $d = 0$.*

We begin our investigation of 4-permuting 4-derivations with the next result.

Lemma 2. *Let R be a noncommutative 3!-torsion free prime ring. Suppose that there exists a 4-permuting 4-derivation $\Delta : R \times R \times R \times R \to R$ such that δ is commuting on R, where δ is the trace of Δ. Then we have $\Delta = 0$.*

Proof. Suppose that

$$[\delta(x), x] = 0 \quad \text{for all } \; x \in R. \tag{1}$$

The substitution $x = x + y$ to linearize (1) leads to

$$\begin{aligned} 0 = {} & [\delta(y), x] + 4[\Delta(x,x,x,y), x] + 6[\Delta(x,x,y,y), x] + 4[\Delta(x,y,y,y), x] \\ & + [\delta(x), y] + 4[\Delta(x,x,x,y), y] + 6[\Delta(x,x,y,y), y] + 4[\Delta(x,y,y,y), y] \end{aligned} \tag{2}$$

for all $x, y \in R$. Putting $-x$ instead of x in (2) and comparing (2) with the result, we arrive at

$$[\delta(x), y] + 4[\Delta(x,x,x,y), x] + 6[\Delta(x,x,y,y), y] + 4[\Delta(x,y,y,y), x] = 0 \tag{3}$$

for all $x, y \in R$ since δ is even. We set $y = x + y$ in (3) and then use (1) and (3) to obtain

$$[\delta(x), y] + 4[\Delta(x,x,x,y), x] + 3[\Delta(x,x,y,y), x] + 2[\Delta(x,x,x,y), y] = 0 \tag{4}$$

for all $x, y \in R$. Replacing x by $-x$ in (4), we have

$$3[\Delta(x,x,y,y), x] + 2[\Delta(x,x,x,y), y] = 0 \quad \text{for all } \; x \in R. \tag{5}$$

We let $y = x + y$ in (5) and then employ (1) and (5) to get

$$\begin{aligned} 0 &= 2[\delta(x), y] + 8[\delta(x,x,x,y), x] + 3[\Delta(x,x,y,y), x] + 2[\Delta(x,x,x,y), y] \\ &= 2[\delta(x), y] + 8[\delta(x,x,x,y), x] \end{aligned}$$

which reduces to the equation

$$0 = [\delta(x), y] + 4[\delta(x,x,x,y), x] \quad \text{for all } \; x, y \in R. \tag{6}$$

Let us write in (6) xy instead of y. Then we get

$$\begin{aligned} 0 &= [\delta(x), xy] + 4[\Delta(x,x,x,xy), x] \\ &= x[\delta(x), y] + 4\delta(x)[y, x] + 4x[\Delta(x,x,x,y), x] \\ &= x\{[\delta(x), y] + 4[\Delta(x,x,x,y), x]\} + 4\delta(x)[y, x] \end{aligned}$$

which implies that

$$\delta(x)[y, x] = 0 \quad \text{for all } \; x, y \in R \tag{7}$$

on account of (6). From (7) and Lemma 2.1, we have $\delta(x) = 0$ for all $x \in R$ $(x \notin Z)$ since for every fixed $x \in R$, a map $y \mapsto [y, x]$ is a derivation on R.

Now, let $x \in R$ $(x \in Z)$ and $y \in R$ $(y \notin Z)$. Then $y + x \notin Z$ and $-y \notin Z$. Thus we have

$$\begin{aligned} 0 = \delta(y+x) &= \delta(y) + \delta(x) + 4\Delta(y,y,y,x) + 6\Delta(y,y,x,x) + 4\Delta(y,x,x,x) \\ &= \delta(x) + 4\Delta(y,y,y,x) + 6\Delta(y,y,x,x) + 4\Delta(y,x,x,x) \end{aligned}$$

and

$$\begin{aligned} 0 = \delta(y-x) &= \delta(y) + \delta(x) - 4\Delta(y,y,y,x) + 6\Delta(y,y,x,x) - 4\Delta(y,x,x,x) \\ &= \delta(x) - 4\Delta(y,y,y,x) + 6\Delta(y,y,x,x) - 4\Delta(y,x,x,x) \end{aligned}$$

which shows that

$$\delta(x) + 6\Delta(x,x,y,y) = 0. \tag{8}$$

Replacing $y \in R\,(y \notin Z)$ by $2y$ in (8) and using (8), we obtain that

$$18\Delta(x,x,y,y) = 0 = \Delta(x,x,y,y)$$

and so the relation (8) gives $\delta(x) = 0$ for all $x \in R$ $(x \in Z)$. Therefore we conclude that $\delta(x) = 0$ for all $x \in R$.

On the other hand, since the relation

$$\delta(x+y) = \delta(x) + \delta(y) + 4\Delta(x,x,x,y) + 6\Delta(x,x,y,y) + 4\Delta(x,y,y,y)$$

is fulfilled for all $x, y \in R$, it follows that

$$2\Delta(x,x,x,y) + 3\Delta(x,x,y,y) + 2\Delta(x,y,y,y) = 0 \quad \text{for all} \quad x, y \in R \tag{9}$$

and putting $x = -x$ in (9) and utilizing (9) yield

$$3\Delta(x,x,y,y) = 0 = \Delta(x,x,y,y) \quad \text{for all} \quad x, y \in R. \tag{10}$$

Let us substitute $y + w$ for y in (10) and then use (10). Then we obtain that

$$2\Delta(x,x,y,w) = 0 = \Delta(x,x,y,w) \quad \text{for all} \quad x, y, w \in R. \tag{11}$$

Finally, replacing x by $x + z$ in (11) and applying (11), we get

$$2\Delta(x,y,z,w) = 0 = \Delta(x,y,z,w) \quad \text{for all} \quad x, y, z, w \in R,$$

that is, $\Delta(x,y,z,w) = 0$ for all $x, y, z, w \in R$ which completes the proof of the theorem. □

We continue with the following result for 4-permuting 4-derivations on semiprime rings.

Theorem 1. *Let R be a noncommutative 2-torsion free semiprime ring. Suppose that there exists a 4-permuting 4-derivation $\Delta : R \times R \times R \times R \to R$ such that δ is centralizing on R, where δ is the trace of Δ. Then δ is commuting on R.*

Proof. Assume that

$$(12) \qquad [\delta(x), x] \in Z \quad \text{for all } x \in R.$$

By linearizing (12) and again using (12), we obtain

$$(13) \qquad \begin{aligned} Z \ni\ & [\delta(y), x] + 4[\Delta(x,x,x,y), x] + 6[\Delta(x,x,y,y), x] + 4[\Delta(x,y,y,y), x] \\ & + [\delta(x), y] + 4[\Delta(x,x,x,y), y] + 6[\Delta(x,x,y,y), y] + 4[\Delta(x,y,y,y), y] \end{aligned}$$

for all $x, y \in R$. We substitute $-x$ for x in (13) and compare (13) with the result to get

$$(14) \qquad [\delta(x), y] + 4[\Delta(x,x,x,y), x] + 6[\Delta(x,x,y,y), y] + 4[\Delta(x,y,y,y), x] \in Z$$

for all $x, y \in R$ since R is 2-torsion free.

Letting $y = x + y$ in (14) and using (14) give

$$(15) \qquad [\delta(x), y] + 4[\Delta(x,x,x,y), x] + 3[\Delta(x,x,y,y), x] + 2[\Delta(x,x,x,y), y] \in Z$$

for all $x, y \in R$. We set $x = -x$ in (15) and compare (15) with the result to obtain

$$(16) \qquad 3[\Delta(x,x,y,y), x] + 2[\Delta(x,x,x,y), y] \in Z$$

for all $x, y \in R$ since R is 2-torsion free.

Replacing x by $x + y$ in (16) and using (16), we have

$$(17) \qquad [\delta(x), y] + 4[\Delta(x,x,x,y), x] \in Z \quad \text{for all } x, y \in R.$$

Taking $y = x^2$ in (17) and invoking (12) show that

$$(18) \qquad Z \ni [\delta(x), x^2] + 4[\Delta(x,x,x,x^2), x] = 10[\delta(x), x]x \quad \text{for all } y \in R$$

and commuting with $\delta(x)$ in (18) gives

$$(19) \qquad 10[\delta(x), x]^2 = 0 \quad \text{for all } y \in R.$$

On the other hand, substituting y by xy in (17), we obtain

$$(20) \qquad \begin{aligned} Z \ni\ & [\delta(x), xy] + 4[\Delta(x,x,x,xy), x] \\ & = x\{[\delta(x), y] + 4[\Delta(x,x,x,y), x]\} + 4\delta(x)[y, x] + 5[\delta(x), x]y \end{aligned}$$

for all $x, y \in R$ and hence we have, for all $x, y \in R$,

$$[x\{[\delta(x), y] + 4[\Delta(x,x,x,y), x]\},\ x] + [4\delta(y)[y, x] + 5[\delta(x), x]y,\ x] = 0.$$

So we get

(21) $$4\delta(x)[[y,x],x] + 9[\delta(x),x][y,x] = 0 \quad \text{for all } x, y \in R$$

according to (17).

Substituting $\delta(x)y$ for y in (21), it follows that

$$0 = \delta(x)\{4\delta(x)[[y,x],x] + 9[\delta(x),x][y,x]\} + 4[[\delta(x),x],x]$$
$$+8\delta(x)[\delta(x),x][y,x] + 9[\delta(x),x]^2 y \quad \text{for all } x, y \in R$$

which, by (1) and (21), implies that

(22) $$8\delta(x)[\delta(x),x][y,x] + 9[\delta(x),x]^2 y = 0 \quad \text{for all } x, y \in R.$$

Letting $y = [\delta(x), x]$ in (22), we arrive at $9[\delta(x),x]^3 = 0$ and so we have

$$9[\delta(x),x]^2 R 9[\delta(x),x]^2 = 0 \quad \text{for all } x \in R.$$

Since R is semiprime, we deduce that

(23) $$9[\delta(x),x]^2 = 0 \quad \text{for all } x \in R.$$

Thus, the relations (19) and (23) yield $[\delta(x),x]^2 = 0$ for all $x \in R$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $[\delta(x),x] = 0$ for all $x \in R$. This completes the proof of the theorem. □

The following result is an analogue of Posner's theorem [4].

Theorem 2. *Let R be a 3!-torsion free prime ring. Suppose that there exists a nonzero 4-permuting 4-derivation $\Delta : R \times R \times R \times R \to R$ such that δ is centralizing on R, where δ be the trace of Δ. Then R is commutative.*

Proof. Suppose that R is noncommutative. Then it follows from Theorem 2.3 that δ is commuting on R. Hence Lemma 2.2 gives $\Delta = 0$ which guarantees the conclusion of the theorem. □

References

1. M. Brešar: Commuting maps: a survey. *Taiwanese J. Math.* **8** (2004), no. 3, 361-397.
2. Y.-S. Jung & K.-H. Park: On prime and semiprime rings with permuting 3-derivations. *Submitted.*
3. J. Mayne: Centralizing mappings of prime rings. *Canad. Math. Bull.* **27** (1984), 122-126.
4. E.C. Posner: Derivations in prime rings. *Proc. Amer. Math. Soc.* **8** (1957), 1093-1100.

5. J. Vukman : Symmetric bi-derivations on prime and semi-prime rings. *Aequationes Math.* **38** (1989), 245-254.
6. J. Vukman : Two results concerning symmetric bi-derivations on prime rings. *Aequationes Math.* **40** (1990), 181-189.

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