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AN EXTENSION OF THE CONTRACTION MAPPING THEOREM

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ABSTRACT. An extension of the contraction mapping theorem is provided in a Banach space setting to approximate fixed points of operator equations. Our approach is justified by numerical examples where our results apply whereas the classical contraction mapping principle cannot.

1. INTRODUCTION

Many problems in applied sciences are reduced to finding fixed points of operators on appropriate spaces. One of the most important results in fixed point theory is the so-called contraction mapping theorem or principle [1] and [3, Theorem 1 (1.XVI)].

It asserts that if a contraction operator F is mapping a closed subset D of a Banach space X into itself then F has a unique fixed point x^* in D.

In this study we show how to extend this result in cases not covered before.

Our approach is justified by a numerical example appearing in many discretization studies in connection with the solution of two point boundary value problems [1] and [2].

2. APPROXIMATION OF FIXED POINTS

We provide the following fixed point theorem:

Theorem 1. *Let $Q\colon D \subseteq X \to X$ be an operator, p, q scalars with $p < 1$, $q \geq 0$ and x_0 a point in D such that:*

(1) $$\|Q(x) - Q(y)\| \leq q\|x - y\|^p \quad \text{for all } x, y \in D.$$

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Set $a = q^{\frac{1}{1-p}}$ *and* $b = q^{\frac{1}{p}}\|Q(x_0) - x_0\|$. *Moreover assume:*

$$0 \leq b < 1 \tag{2}$$

and

$$U(x_0, r) = \{x \in X \mid \|x - x_0\| \leq r\} \subseteq D,$$

where,

$$r \geq \|Q(x_0) - x_0\| + a\big[b^p + \cdots + b^{p^n}\big]. \tag{3}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ *generated by successive approximations*

$$x_{n+1} = Q(x_n) \quad (n \geq 0) \tag{4}$$

remains in $U(x_0, r)$ *for all* $n \geq 0$ *and converges to a fixed point* $x^* \in U(x_0, r)$, *so that for all* $n \geq 1$:

$$\|x_{n+1} - x_n\| \leq ab^{p^n} \leq ab^{np} \tag{5}$$

and

$$\begin{aligned}\|x_n - x^*\| &\leq a \lim_{m\to\infty}\big[b^{p^n} + \cdots + b^{p^{n+m-1}}\big] = c_n\\ &\leq a \lim_{m\to\infty}\big[b^{np} + \cdots + b^{(n+m-1)p}\big]\\ &\leq \frac{ab^{np}}{1-b^p}.\end{aligned} \tag{6}$$

Moreover if

$$r < q^{-1/p} \tag{7}$$

then x^* *is the unique fixed point of* Q *on* $U(x_0, r)$.

Proof. By hypothesis (3) $Q(x_0) = x_1 \in U(x_0, r)$. Let us assume $x_k \in U(x_0, r)$ for $k = 0, 1, \ldots, n$. Then by (1), (2) and (4) we obtain in turn

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|Q(x_n) - Q(x_{n-1})\| \leq q\|x_n - x_{n-1}\|^p\\ &\leq q(q\|x_{n-1} - x_{n-2}\|^p)^p\\ &\leq \cdots \leq q^{1+p+p^2+\cdots+p^{n-1}}\|x_1 - x_0\|^{p^n}\\ &\leq q^{\frac{p^n-1}{p-1}}\|x_1 - x_0\|^p \leq ab^{p^n} \leq ab^{np},\end{aligned}$$

which shows (5).

Using (3) and (5) we get

$$\begin{aligned} \|x_{n+1} - x_0\| &\le \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\le a\big[b^{p^n} + \cdots + b^p\big] + \|x_1 - x_0\| \le r, \end{aligned} \tag{8}$$

which shows $x_{n+1} \in U(x_0, r)$. Moreover for all $m \ge 0$ we can have

$$\begin{aligned} \|x_{n+m} - x_n\| &\le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le a\big[b^{p^{n+m-1}} + \cdots + b^{p^n}\big] \\ &\le a\big[b^{(n+m-1)p} + \cdots + b^{np}\big] \\ &= ab^{np}\frac{1 - b^{mp}}{1 - b^p}, \end{aligned} \tag{9}$$

which shows that $\{x_n\}$ is a Cauchy sequence in a Banach space X and as such it converges to some $x^* \in U(x_0, r)$ (since $U(x_0, r)$ is a closed set). By letting $m \to \infty$ in (9) we obtain (6). Furthermore by letting $n \to \infty$ in (4) we obtain $x^* = Q(x^*)$ (since Q is continuous by (1)).

Finally to show uniqueness, let $y^* \in U(x_0, r)$ be a fixed point of Q. Then we can get as above

$$\begin{aligned} \|x_{n+1} - y^*\| = \|Q(x_n) - Q(y^*)\| &\le q\|x_n - y^*\|^p \\ &\le q(q\|x_{n-1} - y^*\|^p)^p \\ &\le \cdots \le ab_1^{p^n}, \end{aligned} \tag{10}$$

where,

$$b_1 = q^{\frac{1}{p}r}.$$

It follows by (7) and (10) that $\lim_{n\to\infty} x_n = y^*$.

Hence we deduce

$$x^* = y^*.$$

That completes the proof of Theorem 1.

Remark 1. (a) Case $p > 1$. Under hypotheses (1), (2) and (3) the proof of the theorem goes through since $\big[b^{p^n} + \cdots + b^{p^{n+m-1}}\big]$ is still a Cauchy sequence but estimate (6) only holds as

$$\|x_n - x^*\| \le c_n. \tag{11}$$

(b) Case $p = 1$ [3, Theorem 1 (1.XV)]. We simply have the contraction mapping principle with hypotheses

$$0 \leq q < 1 \tag{12}$$

$$r \geq \frac{\|Q(x_0) - x_0\|}{1-q} \tag{13}$$

corresponding to (2) and (3) respectively whereas estimates (5) and (6) are

$$\|x_{n+1} - x_n\| \leq q^n \|Q(x_0) - x_0\| \tag{14}$$

and

$$\|x_n - x^*\| \leq \frac{\|Q(x_0) - x_0\|}{1-q} q^n. \tag{15}$$

3. Applications

Next we provide a numerical example to show that our Theorem can apply in cases where $p \neq 1$. That is in cases where the contraction mapping principle cannot apply.

Example 1. Let $X = \mathbf{R}^{m-1}$ with $m \geq 2$ an integer. Define operator Q on a subset D of X by

$$Q(w) = H + h^2(p+1)M \tag{16}$$

where, h is a real parameter, $p_1 \geq 0$,

$$H = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & -1 \\ 0 & \cdots & -1 & & 2 \end{bmatrix},$$

$$M = \begin{bmatrix} w_1^p & & \cdots & 0 \\ 0 & w_2^p & \cdots & 0 \\ 0 & \cdots & \cdots & w_{m-1}^p \end{bmatrix},$$

and $w = (w_1, w_2, \ldots, w_{m-1})$.

Finding fixed points of operator Q is very important in many discretization studies since e.g. many two point boundary value problems can be reduced to finding such points [1] and [3].

Let $w \in \mathbf{R}^{m-1}$, $M \in \mathbf{R}^{m-1} \times \mathbf{R}^{m-1}$, and define the norms of w and M by

$$\|w\| = \max_{1 \leq j \leq m-1} |w_j|$$

and

$$\|M\| = \max_{1\leq j\leq m-1} \sum_{k=1}^{m-1} |m_{jk}|.$$

For all $w, z \in \mathbf{R}^{m-1}$ for which $|w_j| > 0$, $|z_j| > 0$, $j = 1, 2, \ldots, m-1$ we obtain, for say $p = \frac{1}{2}$

$$\begin{aligned}\|Q(w) - Q(z)\| &= \left\|\operatorname{diag}\left\{\frac{3}{2}h^2\left(w_j^{1/2} - z_j^{1/2}\right)\right\}\right\| \\ &= \frac{3}{2}h^2 \max_{1\leq j\leq m-1} \left|w_j^{1/2} - z_j^{1/2}\right| \\ &\leq \frac{3}{2}h^2[\max|w_j - z_j|]^{1/2} \\ &= \frac{3}{2}h^2\|w - z\|^{1/2}. \qquad (17)\end{aligned}$$

Set $q = \frac{3}{2}h^2$. Then $a = q^2$ and $b = q^2\|Q(x_0) - x_0\|$. It follows by (17) that the contraction mapping principle cannot be used to obtain fixed points of operator Q since $p = \frac{1}{2} \neq 1$. However our theorem can apply provided that $x_0 \in D$ is such that

$$\|Q(x_0) - x_0\| \leq q^{-2}. \qquad (18)$$

That is (2) is satisfied. Furthermore r should be chosen to satisfy (3) and we can certainly set $U(x_0, r) = D$.

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