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THE TRANSFORMATION THEOREM ON ANALOGUE OF WIENER SPACE

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ABSTRACT. In 2002, the author and professor Ryu introduced the concept of analogue of Wiener measure. In this paper, we prove the existence theorem of Fourier-Feynman transform on analogue of Wiener measure in L_2-norm sense.

1. INTRODUCTION

In 1923, Wiener made the reasonable mathematical model of Brownian motions that is called Wiener measure space [16]. In 2002, the author and professor Ryu presented the theories for analogue of Wiener measure that is a generalization of concrete Wiener measure [12]. And they introduced the several papers related to analogue of Wiener space [6, 12, 13, 14, 15]. As for as we know that the dissertation of Brue [2] is the first paper treat the theory of Fourier-Feynman transform on the concrete Wiener space. Since then, this theory was developed by the many mathematicians-Cameron, Storvick, Skoug, Johnson, Chang etc [3, 4, 8]. Here, we will introduce the concept of Fourier-Feynman transform on analogue of Wiener space. Indeed, our following definitions of it is essentially similar to the definition related Fourier-Feynman transform on Wiener space [7].

In this article, we will establish the existence theorem of the Fourier-Feynman transform on analogue of Wiener measure space and find the some properties of our transform.

2. PRELIMINARIES

In this section, we give some definitions, notations and base facts which are need to be understood the next sections.

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Wiener suggested a measure space $(C_0[0,t],\omega)$ where $C_0[0,t]$ is the space of all continuous functions on a closed interval $[0,t]$ which vanish at origin, the so called Wiener space in 1923 [16]. The authors introduced a new measure space, similar to Wiener measure in [12] as following; Let t be a positive real number and let n be a non-negative integer. For $\vec{t}=(t_0,t_1,\cdots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, let $J_{\vec{t}}: C[0,t]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0),x(t_1),\cdots,x(t_n)) .$$

For B_j $(j=0,1,2,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$ where $\mathcal{B}(\mathbb{R})$ is the sets of all Borel subsets of the real number system $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we let

$$\begin{aligned}(2.1)\qquad & m_\varphi\left(J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right)\\ &= \int_{B_0}\left[\int_{\prod_{j=1}^n B_j} W(n+1;\vec{t};u_0,u_1,\cdots,u_n)d\prod_{j=1}^n m_L(u_1,\cdots,u_n)\right] d\varphi(u_0)\end{aligned}$$

where

$$\begin{aligned}& W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\\ &= \left(\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\right)\exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right\}.\end{aligned}$$

By [9], $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure ω_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure ω_φ is called analogue of Wiener measure associated with the probability measure φ. The completion of $(C[0,t],\mathcal{B}(C[0,t]),\omega_\varphi)$ is called the complete analogue of Wiener measure space, denote $(C[0,t],\mathcal{B},\omega_\varphi)$, abbreviation.

By the change of variable formula, we can easily prove the following theorem.

Theorem 2.1 ([12]). (Integration formula for analogue of Wiener measure) *$f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function then the following equality holds.*

$$\begin{aligned}&\int_{C[0,t]} f(x(t_0),x(t_1),\cdots,x(t_n))\,d\omega_\varphi(x)\\ &\overset{*}{=} \int_{\mathbb{R}}\left[\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\right.\end{aligned}$$

$$\times d\prod_{j=1}^{n} m_L((u_1, u_2, \cdots, u_n))\Big] d\varphi(u_0)$$

where $\overset{*}{=}$ *means that if one side exists then both sides exist and the two values are equal.*

Remark. We can found the following theorems for the concrete Wiener measure space in [1].

Theorem 2.2. (Bearman's rotation theorem) $F(x,y)$ *is measurable on* $(C_0[0,t]\times C_0[0,t], \mathcal{B}(C_0[0,t])\times\mathcal{B}(C_0[0,t]))$ *if and only if* $F(x\ cos\ \theta - y\ sin\ \theta,\ x\ sin\ \theta + y\ cos\ \theta)$ *is measurable on* $(C_0[0,t]\times C_0[0,t], \mathcal{B}(C_0[0,t])\times\mathcal{B}(C_0[0,t]))$ *for all real number* θ. *Moreover,*

$$\int_{C_0[0,t]\times C_0[0,t]} F(x,y)\ d\omega\times\omega(x,y)$$
$$= \int_{C_0[0,t]\times C_0[0,t]} F(x\ cos\ \theta - y\ sin\ \theta, x\ sin\ \theta + y\ cos\ \theta)\ d\omega\times\omega(x,y).$$

In the next section, we will treat the Fourier-Feynman transform on analogue of Wiener measure space and we need the concept of scale-invariant measurable subset in analogue of Wiener measure space. Throughout in this paper, without special comments, we assume that φ is a positive Borel measure on $\mathbb{R}$ having a non-zero Radon-Nikodym derivative with respect to $\frac{d\varphi}{dm_L} = f$.

Furthermore, we assume that for any Borel subset B of $C[0,t]\times C[0,t]$ and for any real number θ

$$(2.2)\qquad \int_{C[0,t]\times C[0,t]} \chi_B(x,y)\ d\omega\times\omega(x,y)$$
$$= \int_{C[0,t]\times C[0,t]} \chi_B(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta) d\omega_\varphi\times\omega_\varphi(x,y).$$

Theorem 2.3 ([11]). *If* φ *is a positive measure with* $\frac{d\varphi}{dm_L} = f$ *and* (2.2) *holds then* $f(x)$ *is of the form* Ae^{-ax^2} *where* A, a *are positive constants.*

Theorem 2.4 ([11]). *Let* f *be in* $L_1(\mathbb{R})$ *and let* $\varphi(E) = \int_E f(x)dm_L(x)$ *for* $f > 0$ *and a Borel subset* E *of* $\mathbb{R}$. *If for a measurable function* F,

$$(2.3)\qquad \int_{C[0,t]}\int_{C[0,t]} F(x,y)\ d\omega_\varphi(x)d\omega_\varphi(y)$$

$$= \int_{C[0,t]} \int_{C[0,t]} F(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)\ d\omega_\varphi(x) d\omega_\varphi(y),$$

for all real number θ. Then the function $f(x)$ has the form Ae^{-ax^2} where A, a are positive constants.

Theorem 2.5 ([11]). *For a Borel subset E of $\mathbb{R}$ and positive real numbers A, a, let $\varphi(E) = \int_E Ae^{-ax^2} dm_L(x)$, then for a measurable function F, (2.3) is satisfied.*

3. FOURIER-FEYNMAN TRANSFORM ON ANALOGUE OF WIENER SPACE

In this section, we will develop the theories of fourier-Feynman transform on analogue of Wiener space. First of all, we will establish the existence theorem for our transform. Moreover, we will find the properties of it. In developing our theories, the rotation theorem is key role, so we assume that a measure φ has the Radon-Nikodym derivative with respect to the Lebesgue measure having a form $\frac{d\varphi}{dm_L}(x) = Ae^{-ax^2}$ where A, α are two positive real numbers.

Definition 3.1. Let φ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let $F : C[0,t] \to \mathbb{R}$ be a measurable function. For all $\lambda > 0$, if the integral $\int_{C[0,t]} F(\lambda^{-1}x)\ d\omega_\varphi(x)$ exists, then we denote

$$\int_{C[0,t]} F(\lambda^{-1}x) d\omega_\varphi(x) = J(\lambda)$$

And if there exists a function $J^*(\lambda)$ analytic in the half-plane $\mathbb{C}^+$ such that $J(\lambda) = J^*(\lambda)$ for almost all real $\lambda > 0$, then we write

$$\int_{C[0,t]}^{an\ anw_\lambda} F(x) d\omega_\varphi(x) = J^*(\lambda)$$

and we call that $J^*(\lambda)$ is the *analytic analogue of Wiener integral* of F over $C[0,t]$ with parameter λ. And for non-zero real number q, if the limit $\underset{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}}{lim} J^*(\lambda)$ exists, then we denote

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} J^*(\lambda) = \int_{C[0,t]}^{an\ anf_q} F(x) d\omega_\varphi(x)$$

and we say that the limit is the *analytic analogue of Feynman integral* of F.

Notation. For $\lambda \in \mathbb{C}^+$ and $y \in C[0,t]$ let

$$(T_{an,\lambda}F)(y) = \int_{C[0,t]}^{an\ anw_\lambda} F(x+y) d\omega_\varphi(x).$$

And given a number p such that $1 \leq p \leq \infty$, p and p' will always be related by $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 3.2. Let $\{H_n\}$ and H be analogue of Wiener measurable functions such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C[0,t]} |H_n(\rho y) - H(\rho y))|^2 dy = 0.$$

Then we write

$$\lim_{n\to\infty} (w^2_{\varphi,s}) H_n \overset{an}{\approx} H \tag{3.4}$$

and we call H the *scale invariant limit in the mean of order* 2 of H_n over $C[0,t]$. We define a similar definition for any real number instead of n.

Definition 3.3. Let q be non-zero real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform of F, which we denote by $T^{(p)}_{an,q}F$, by the formula

$$(T^{(p)}_{an,q}F)(y) = \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}^+}} (w^{p'}_{\varphi,s})(T_{an.\lambda}F)(y)$$

whenever this limit exists. Let F be a functional on analogue of Wiener space such that $(T_{an,\lambda}F)(y)$ exists in $\mathbb{C}^+$ for s-almost every y. We define the L_1 analytic analogue of Fourier-Feynman transform of F, which we denote by $T^{(1)}_{an,q}F$, as that functional (if it exists) on analogue of Wiener space such that

$$(T^{(1)}_{an,q}F)(y) = \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}^+}} (T_{an.\lambda}F)(y)$$

for s-almost every y.

Definition 3.4. For each natural number n and a partition $0 = t_0 < t_1 < \cdots < t_n = t$, let $\mathcal{A}_n$ be the collection of functions $F : C[0,t] \to \mathbb{R}$ satisfying (1) and (2) below;

(1) f is a measurable function on $\mathbb{R}^{n+1}$.

(2) $F(x) \overset{an}{\approx} f(x_0), x(t_1), \cdots, x(t_n))$.

Lemma 3.5. *For a non-zero complex number λ with $Re\lambda \geq 0$ and for $f \in L_2(\mathbb{R}^{n+1})$, define*

$$g(v_0, v_1, \cdots, v_n) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(u_j - v_j)^2} dm_L(u_0)\cdots dm_L(u_n),$$

then $g \in L_2(\mathbb{R}^{n+1})$ *and* $\| g \|_2 \leq \| f \|_2$.

Proof. (I) Case $\lambda = ip$, $(p > 0)$;
Let $v_j = \frac{z_j}{p}$, $j = 0, 1, \cdots, n$, then

$$g\left(\frac{z_0}{p}, \frac{z_1}{p}, \cdots, \frac{z_n}{p}\right)(ip)^{-\frac{n+1}{2}} e^{\frac{ip}{2}\sum_{j=0}^{n}\frac{z_j^2}{p^2}}$$
$$= \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(u_0, u_1, \cdots, u_n) e^{-\frac{ip}{2}\sum_{j=0}^{n} u_j^2} e^{i\sum_{j=0}^{n}(u_j z_j)}$$
$$dm_L(u_0) \cdots dm_L(u_n).$$

Since $f \in L_2(\mathbb{R}^{n+1})$, $e^{-\frac{ip}{2}\sum_{j=0}^{n} u_j^2} f(u_0, u_1, \cdots, u_n) \in L_2(\mathbb{R}^{n+1})$. Then as the Fourier transform of $e^{-\frac{ip}{2}\sum_{j=0}^{n} u_j^2} f(u_0, u_1, \cdots, u_n)$,

$$g\left(\frac{z_0}{p}, \frac{z_1}{p}, \cdots, \frac{z_n}{p}\right)(ip)^{-\frac{n+1}{2}} e^{\frac{ip}{2}\sum_{j=0}^{n}\frac{z_j^2}{p^2}} \in L_2(\mathbb{R}^{n+1}).$$

Hence $g \in L_2(\mathbb{R}^{n+1})$.

(II) Case $\lambda = p$, $(p > 0)$;
From

$$|g(v_0, v_1, \cdots, v_n)|$$
$$\leq \left(\frac{p}{2\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |f(u_0, u_1, \cdots, u_n)| e^{-\frac{p}{2}\sum_{j=0}^{n}(u_j - v_j)^2}$$
$$dm_L(u_0) \cdots dm_L(u_n),$$
$$\| g \|_2^2$$
$$\leq \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\frac{p}{2\pi}\right)^{n+1} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |f(u_0, u_1, \cdots, u_n)| e^{-\frac{p}{2}\sum_{j=0}^{n}(u_j - v_j)^2}$$
$$dm_L(u_0) \cdots dm_L(u_n) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |f(x_0, x_1, \cdots, x_n)| e^{-\frac{p}{2}\sum_{j=0}^{n}(x_j - v_j)^2}$$
$$dm_L(x_0) \cdots dm_L(x_n) dm_L(v_0) \cdots dm_L(v_n).$$

By the change of variables, $u_j = u_j$, $v_j - u_j = z_j$ and $v_j - x_j = y_j$ for $j = 0, 1, \cdots, n$, and by the Cauchy-Schwarz inequality,

$$\| g \|_2^2$$
$$\leq \left(\frac{p}{2\pi}\right)^{n+1} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |f(u_0, u_1, \cdots, u_n)|$$
$$|f(z_0 + u_0 - y_0, \cdots, z_n + u_n - y_n)| e^{-\frac{p}{2}\sum_{j=0}^{n}(z_j^2 + y_j^2)}$$

$$dm_L(u_0)\cdots dm_L(u_n)\ dm_L(z_0)\cdots dm_L(z_n)\ dm_L(y_0)\cdots dm_L(y_n)$$
$$\leq \left(\frac{p}{2\pi}\right)^{n+1}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \| f \|_2^2\, e^{-\frac{p}{2}\sum_{j=0}^{n}(z_j^2+y_j^2)}$$
$$dm_L(z_0)\cdots dm_L(z_n)\ dm_L(y_0)\cdots dm_L(y_n)$$
$$= \| f \|_2^2,$$

as desired.

(III) Case $\lambda = p+iq$, $(p>0)$;

Since

$$|g(v_0,v_1,\cdots,v_n)|$$
$$\leq \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} |f(u_0,u_1,\cdots,u_n)|e^{-\frac{|\lambda|}{2}\sum_{j=0}^{n}(u_j-v_j)^2}$$
$$dm_L(u_0)\cdots dm_L(u_n),$$

let

$$h(v_0,v_1,\cdots,v_n)$$
$$= \left(\frac{p}{2\pi}\right)^{\frac{n+1}{2}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(u_0,u_1,\cdots,u_n)e^{-\frac{p}{2}\sum_{j=0}^{n}(u_j-v_j)^2}$$
$$dm_L(u_0)\cdots dm_L(u_n),$$

then by (II), $h\in L_2(\mathbb{R}^{n+1})$ and $\| h \|_2 \leq \| f \|_2$. So,

$$|g(v_0,v_1,\cdots,v_n)| \leq \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n+1}{2}} |h(v_0,v_1,\cdots,v_n)|.$$

Hence $g\in L_2(\mathbb{R}^{n+1})$. Now

$$\| g \|_2^2$$
$$= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} |g(v_0,v_1,\cdots,v_n)|^2 dm_L(v_0)\cdots dm_L(v_n)$$
$$= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} g(v_0,v_1,\cdots,v_n)\overline{g(v_0,v_1,\cdots,v_n)}dm_L(v_0)\cdots dm_L(v_n)$$
$$= \left(\frac{|\lambda|}{2\pi}\right)^{n+1}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(u_0,u_1,\cdots,u_n)\overline{f(x_0,x_1,\cdots,x_n)}$$
$$\left[\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(u_j-y_j)^2}e^{-\frac{\bar{\lambda}}{2}\sum_{j=0}^{n}(x_j-y_j)^2}dm_L(u_0)\cdots dm_L(u_n)\right]$$
$$dm_L(x_0)\cdots dm_L(x_n)\ dm_L(v_0)\cdots dm_L(v_n).$$

Since

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(u_j-y_j)^2} e^{-\frac{\bar{\lambda}}{2}\sum_{j=0}^{n}(x_j-y_j)^2}\, dm_L(v_0)\cdots dm_L(v_n)$$
$$= \left(\frac{\pi}{4p}\right)^{\frac{n+1}{2}} e^{-\frac{|\lambda|^2}{4p}\sum_{j=0}^{n}(u_j-x_j)^2}$$

and let $u_j - x_j = s_j$, $x_j = x_j$ for $j = 0, 1, \cdots, n$, then by the Cauchy-Schwarz inequality

$$\| g \|_2^2$$
$$\leq \left(\frac{|\lambda|^2}{4p\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \| f \|_2^2\, e^{-\frac{|\lambda|^2}{4p}\sum_{j=0}^{n} s_j^2}\, dm_L(s_0)\cdots dm_L(s_n)$$
$$= \| f \|_2^2,$$

as desired. □

Lemma 3.6. *For $1 \geq p \geq 2$ and all non-zero real number q, if $F_1 \overset{an}{\approx} F_2$, then the existence of $T_q^{(p)}(F_1)$ assure the existence of $T_q^{(p)}(F_2)$, and $T_q^{(p)}(F_1) \overset{an}{\approx} T_q^{(p)}(F_2)$.*

Proof. For all positive real numbers α, β, let $F = F_1 - F_2$. Then $F(\sqrt{\alpha^2+\beta^2}\ x) = 0$ for almost all x in $(C[0,t], \omega_\varphi)$ and

$$\int_{C[0,t]\times C[0,t]} |F(\alpha x + \beta y)| d\omega_\varphi \times \omega_\varphi(x,y)$$
$$= \int_{C[0,t]} |F(\sqrt{\alpha^2+\beta^2}\ z)| d\omega_\varphi(z)$$
$$= 0.$$

Hence for almost all (x, y) in $(C[0,t] \times C[0,t], \omega_\varphi \times \omega_\varphi)$, $|F(\alpha x + \beta y)| = 0$. By Fubini theorem, for all positive real numbers α, β and for ω_φ-a.e. y, $\int_{C[0,t]} |F(\alpha x + \beta y)| d\omega_\varphi(x) = 0$, that is, $\int_{C[0,t]} [F_1(\alpha x + \beta y) - F_2(\alpha x + \beta y)] d\omega_\varphi(x) = 0$. Then we can obtain that for all positive real number β

$$\int_{C[0,t]}^{an\ an\lambda} F_1(x + \beta y) d\omega_\varphi(x) = \int_{C[0,t]}^{an\ an\lambda} F_2(x + \beta y) d\omega_\varphi(x) \tag{3.5}$$

and so if one side exists then both sides exist in (3.5). Thus if $T_q^{(p)}(F_1)$ exists then $T_q^{(p)}(F_2)$ also exists, and $T_q^{(p)}(F_1) \overset{an}{\approx} T_q^{(p)}(F_2)$. □

Theorem 3.7. *For a partition $0 = t_0 < t_1 < \cdots < t_n = t$, let $F(x) = f(x(t_0), x(t_1), \cdots, x(t_n))$ be in $\mathcal{A}_n$. Then for each y in $C[0,t]$ and for all complex numbers λ with*

$Re\lambda > 0$, $\int_{C[0,t]}^{an\ anw\lambda} F(x+y)d\omega_\varphi(x)$ exists. Moreover,

(3.6)
$$\begin{aligned}&\int_{C[0,t]}^{an\ anw\lambda} F(x+y)d\omega_\varphi(x)\\ &= A\lambda^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j-t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(v_0,v_1,\cdots,v_n)\\ &\quad e^{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(y(t_j)-y(t_{j-1}))]^2}{t_j-t_{j-1}}} e^{-a\lambda(v_0-y(t_0))^2}\\ &\quad dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0).\end{aligned}$$

Moreover, let $h(y(t_0),y(t_1),\cdots,y(t_n);\lambda)$ be the right side in (3.6), then

$$\| h(y(t_0),y(t_1),\cdots,y(t_n);\lambda)\|_2 \le A\sqrt{\frac{\pi}{a}}\| f\|_2$$

and $h(w_0,w_1,\cdots,w_n;\lambda)$ is analytic of λ.

Proof. For all positive real number λ and for all y in $C[0,t]$, we can find the equality (3.6) by the elementary calculus from the definition of $\mathcal{A}_n$, the analogue of Wiener integration formula, the Radon-Nikodym derivative $\frac{d\varphi}{dm_L}(x) = Ae^{-ax^2}$ and the change of variable with $\lambda^{-\frac{1}{2}}u_j + y(t_j) = v_j$ $(j=0,1,\cdots,n)$. Now, since

$$\begin{aligned}&h(w_0,w_1,\cdots,w_n;\lambda)\\ &= A\lambda^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j-t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(v_0,v_1,\cdots,v_n)\\ &\quad e^{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(w_j-w_{j-1})]^2}{t_j-t_{j-1}}} e^{-a\lambda(v_0-w_0)^2}\\ &\quad dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0),\end{aligned}$$

by the change of variable, $v_j' = \frac{v_j-v_{j-1}}{\sqrt{t_j-t_{j-1}}}$, $w_j' = \frac{w_j-w_{j-1}}{\sqrt{t_j-t_{j-1}}}$ $(j=1,2,\cdots,n)$, $v_0'=v_0$ and $w_0'=w_0$,

$$\begin{aligned}&h\left(w_0', w_0'+\sqrt{t_1}w_1',\cdots,w_0'+\sum_{j=1}^{n}\sqrt{t_j-t_{j-1}}w_j';\lambda\right)\\ &= A\lambda^{\frac{n+1}{2}}(2\pi)^{-\frac{n}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f\left(v_0', v_0'+\sqrt{t_1}v_1',\cdots,v_0'+\sum_{j=1}^{n}\sqrt{t_j-t_{j-1}}v_j'\right)\\ &\quad e^{-\frac{\lambda}{2}\sum_{j=1}^{n}(v_j'-w_j')^2} e^{-a\lambda(v_0'-w_0')^2}\, dm_L(v_n')\cdots dm_L(v_1')dm_L(v_0').\end{aligned}$$

And again using the change of variable, $w'_j = w''_j$, $v'_j = v''_j$ $(j = 1, 2, \cdots, n)$, $\sqrt{2a}w'_0 = w''_0$ and $\sqrt{2a}v'_0 = v''_0$, and let

$$\begin{aligned} &h^*(w''_0, w''_1, \cdots, w''_n; \lambda) \\ &= h\Big(\frac{1}{\sqrt{2a}}w''_0, \frac{1}{\sqrt{2a}}w''_0 + \sqrt{t_1}w''_1, \cdots, \frac{1}{\sqrt{2a}}w''_0 + \sum_{j=1}^{n}\sqrt{t_j - t_{j-1}}w''_j; \lambda\Big) \end{aligned}$$

and

$$\begin{aligned} &f^*(v''_0, v''_1, \cdots, v''_n) \\ &= f\Big(\frac{1}{\sqrt{2a}}v''_0, \frac{1}{\sqrt{2a}}v''_0 + \sqrt{t_1}v''_1, \cdots, \frac{1}{\sqrt{2a}}v''_0 + \sum_{j=1}^{n}\sqrt{t_j - t_{j-1}}v''_j\Big), \end{aligned}$$

then

$$\begin{aligned} &h^*(w''_0, w''_1, \cdots, w''_n; \lambda) \\ &= A\sqrt{\frac{\pi}{a}}\Big(\frac{\lambda}{2\pi}\Big)^{\frac{n+1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(v''_0, v''_1, \cdots, v''_n) e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(v''_j - w''_j)^2} \\ &dm_L(v''_n) \cdots dm_L(v''_1) dm_L(v''_0). \end{aligned}$$

Hence by Lemma 3.5

$$\| h^*(w''_0, w''_1, \cdots, w''_n; \lambda) \|_2 \leq A\sqrt{\frac{\pi}{a}} \| f^*(v''_0, v''_1, \cdots, v''_n) \|_2 .$$

Now, we show that $h^*(w_0, w_1, \cdots, w_n; \lambda)$ is analytic of λ. From the dominated convergence theorem, if $\lim_{m \to \infty} \lambda_m = \lambda$ in $\mathbb{C}^+$,

$$\lim_{m \to \infty} h^*(w_0, w_1, \cdots, w_n; \lambda_m) = h^*(w_0, w_1, \cdots, w_n; \lambda),$$

and so, h^* is a continuous function of λ. And let Δ be a simple closed interval on $\mathbb{C}^+$, then by the Fubini theorem,

$$\begin{aligned} &\int_{\Delta} h^*(w_0, w_1, \cdots, w_n; \lambda)\, d\lambda \\ &= A\sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(v_0, v_1, \cdots, v_n) \Big(\int_{\Delta} \Big(\frac{\lambda}{2\pi}\Big)^{\frac{n+1}{2}} e^{-\frac{\lambda}{2}\sum_{j=0}^{n}(v_j - w_j)^2}\, d\lambda \Big) \\ &dm_L(v_n) \cdots dm_L(v_1) dm_L(v_0) \\ &= 0. \end{aligned}$$

Thus, from the Morera theorem, h^* is an analytic function of λ, that is, h is analytic.

□

Theorem 3.8. *Let $f \in L_2(\mathbb{R}^{n+1})$ and let q be a non-zero real number. For a partition $0 = t_0 < t_1 < \cdots < t_n = t$, define*

$$(3.7) \qquad \begin{aligned} &g(v_0, v_1, \cdots, v_n) \\ &= \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_0, u_1, \cdots, u_n) \\ &\quad e^{\frac{iq}{2}\sum_{j=1}^{n}\frac{[(u_j-u_{j-1})-(v_j-v_{j-1})]^2}{t_j-t_{j-1}}} e^{iqa(u_0-v_0)^2}\, dm_L(u_n)\cdots dm_L(u_1)dm_L(u_0). \end{aligned}$$

Then g is in $L_2(\mathbb{R}^{n+1})$, and for all $u_0, u_1, \cdots, u_n$

$$(3.8) \qquad \begin{aligned} &f(u_0, u_1, \cdots, u_n) \\ &= \frac{\sqrt{2}a}{A\sqrt{\pi}}(iq)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(v_0, v_1, \cdots, v_n) \\ &\quad e^{-\frac{iq}{2}\sum_{j=1}^{n}\frac{[(u_j-u_{j-1})-(v_j-v_{j-1})]^2}{t_j-t_{j-1}}} e^{-iqa(u_0-v_0)^2} dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0). \end{aligned}$$

And

$$(3.9) \qquad \| f \|_2 = \frac{4a^2}{A} \| g \|_2 .$$

Proof. Let $p = |q|$ ($q = \pm p$) and define

$$(3.10) \qquad \begin{aligned} &f^*(z_0, z_1, \cdots, z_n) \\ &= e^{\pm\frac{i}{2}\sum_{j=1}^{n} z_j^2} A e^{\pm iaz_0^2} f\left(\frac{1}{\sqrt{p}}z_0, \frac{1}{\sqrt{p}}z_0 + \sqrt{\frac{t_1}{p}}z_1, \cdots, \frac{1}{\sqrt{p}}z_0 \right. \\ &\quad \left. + \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}z_j\right). \end{aligned}$$

Then since $f \in L_2(\mathbb{R}^{n+1})$, $f^* \in L_2(\mathbb{R}^{n+1})$. Now, let

$$(3.11) \qquad \tilde{g}(w_0, w_1, \cdots, w_n) = (\mathcal{F}(f^*))(w_0, w_1, \cdots, w_n)$$

where $\mathcal{F}$ is the Fourier transform, and let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$T(u_0, u_1, \cdots, u_n) = (z_0, z_1, \cdots, z_n)$$

with $z_0 = \sqrt{p}u_0$, $z_j = \sqrt{\frac{p}{t_j - t_{j-1}}}(u_j - u_{j-1})$ for $j = 1, 2, \cdots, n$. For a positive real number B, let $D_B = T(D_B')$ where $D_B' = [-B, B]^{n+1}$, then by the change of variable, $w_0 = \mp 2a\sqrt{p}v_0$ and $w_j = \mp\sqrt{\frac{p}{t_j-t_{j-1}}}(v_j - v_{j-1})$,

(3.12)
$$\begin{aligned}
&\tilde{g}(w_0, w_1, \cdots, w_n) \\
&= \lim_{B\to\infty}(2\pi)^{-\frac{n+1}{2}} A \int \underset{D'_B}{\cdots} \int f\Bigg(\frac{1}{\sqrt{p}}z_0, \frac{1}{\sqrt{p}}z_0 + \sqrt{\frac{t_1}{p}}z_1, \cdots, \\
&\quad \frac{1}{\sqrt{p}}z_0 + \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}z_j\Bigg)\, e^{\pm\frac{i}{2}\sum_{j=1}^{n}(z_j \pm w_j)^2} e^{\pm ia(z_0 \pm \frac{w_0}{2a})^2} \\
&\quad e^{\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\mp ia(\frac{w_0}{2a})^2}\, dm_L(z_n)\cdots dm_L(z_1)dm_L(z_0) \\
&= \lim_{B\to\infty} A(2\pi)^{-\frac{n+1}{2}} \sqrt{\frac{p^{n+1}}{\prod_{j=1}^{n}(t_j - t_{j-1})}} \int \underset{D_B}{\cdots} \int f(u_0, u_1, \cdots, u_n) \\
&\quad e^{\pm\frac{ip}{2}\sum_{j=1}^{n}\frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}} e^{\pm iap(u_0 - v_0)^2} e^{\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\mp i\frac{w_0^2}{4a}} \\
&\quad dm_L(u_n)\cdots dm_L(u_1)dm_L(u_0) \\
&= (\pm i)^{\frac{n+1}{2}} e^{\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\mp i\frac{w_0^2}{4a}} \\
&\quad \lim_{B\to\infty}\frac{A}{\sqrt{2\pi}}(\mp ip)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int \underset{D_B}{\cdots} \int f(u_0, u_1, \cdots, u_n) \\
&\quad e^{\pm\frac{ip}{2}\sum_{j=1}^{n}\frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}} e^{\pm iap(u_0 - v_0)^2} \\
&\quad dm_L(u_n)\cdots dm_L(u_1)dm_L(u_0).
\end{aligned}$$

From the definition of g, g exists and $g \in L_2(\mathbb{R}^{n+1})$. Hence

$$\begin{aligned}
&e^{\pm\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\pm i\frac{w_0^2}{4a}} \tilde{g}(w_0, w_1, \cdots, w_n) \\
&= (\pm i)^{\frac{n+1}{2}} g(v_0, v_1, \cdots, v_n) \\
&= (\pm i)^{\frac{n+1}{2}} g\Bigg(\mp\frac{1}{2a\sqrt{p}}w_0, \mp\frac{1}{2a\sqrt{p}}w_0 \mp \sqrt{\frac{t_1}{p}}w_1, \cdots, \\
&\qquad \mp\frac{1}{2a\sqrt{p}}w_0 \mp \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}w_j\Bigg),
\end{aligned}$$

that is,

$$\tilde{g}(w_0, w_1, \cdots, w_n) = (\pm i)^{\frac{n+1}{2}} g^*(w_0, w_1, \cdots, w_n)$$

where

(3.13) $$g^*(w_0, w_1, \cdots, w_n)$$

$$= e^{\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\mp i\frac{w_0^2}{4a}} g\Bigg(\mp \frac{1}{2a\sqrt{p}}w_0, \mp\frac{1}{2a\sqrt{p}}w_0 \mp \sqrt{\frac{t_1}{p}}w_1, \cdots ,$$
$$\mp\frac{1}{2a\sqrt{p}}w_0 \mp \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}w_j\Bigg).$$

From (3.11),

$$g^*(w_0, w_1, \cdots , w_n) = (\pm i)^{-\frac{n+1}{2}}(\mathcal{F}(f^*))(w_0, w_1, \cdots , w_n).$$

By the Plancherel theorem, $\| g^* \|_2 = \| f^* \|_2$ and

$$f^*(z_0, z_1, \cdots , z_n)$$
$$= (\pm i)^{\frac{n+1}{2}}(2\pi)^{-\frac{n+1}{2}} \lim_{B\to\infty}\int \underset{D_B}{\cdots}\int g^*(w_0, w_1, \cdots , w_n)e^{-i\sum_{j=1}^{n} w_j z_j}$$
$$dm_L(w_n)\cdots dm_L(w_1)dm_L(w_0).$$

This means that

$$e^{\pm\frac{i}{2}\sum_{j=1}^{n} z_j^2} A e^{\pm iaz_0^2} f\Bigg(\frac{1}{\sqrt{p}}z_0, \frac{1}{\sqrt{p}}z_0 + \sqrt{\frac{t_1}{p}}z_1, \cdots , \frac{1}{\sqrt{p}}z_0 + \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}z_j\Bigg)$$
$$= (\pm i)^{\frac{n+1}{2}}(2\pi)^{-\frac{n+1}{2}} \lim_{B\to\infty}\int \underset{D_B}{\cdots}\int e^{-i\sum_{j=1}^{n} w_j z_j} e^{\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2} e^{\mp i\frac{w_0^2}{4a}}$$
$$g\Bigg(\mp \frac{1}{2a\sqrt{p}}w_0, \mp\frac{1}{2a\sqrt{p}}w_0 \mp \sqrt{\frac{t_1}{p}}w_1, \cdots , \mp\frac{1}{2a\sqrt{p}}w_0 \mp \sum_{j=1}^{n}\sqrt{\frac{t_j - t_{j-1}}{p}}w_j\Bigg)$$
$$dm_L(w_n)\cdots dm_L(w_1)dm_L(w_0),$$

by the change of variable,

$$e^{\pm\frac{ip}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}} A e^{\pm ipau_0^2} f(u_0, u_1, \cdots , u_n)$$
$$= (\pm i)^{\frac{n+1}{2}}(2\pi)^{-\frac{n+1}{2}} \lim_{B\to\infty}\int \underset{D_B}{\cdots}\int 2ap^{\frac{n+1}{2}} \prod_{j=1}^{n}(t_j - t_{j-1})^{-\frac{1}{2}}$$
$$e^{\mp\frac{ip}{2}\sum_{j=1}^{n}\frac{[(u_j-u_{j-1})-(v_j-v_{j-1})]^2}{t_j-t_{j-1}}} e^{\pm\frac{ip}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}} e^{\mp ipav_0^2} e^{\pm 2ipau_0v_0}$$
$$g(v_0, v_1, \cdots , v_n)\; dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0).$$

Thus

$$f(u_0, u_1, \cdots , u_n)$$

$$= \frac{2a}{A\sqrt{2\pi}}(\pm ip)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \lim_{B\to\infty}\int \underset{D_B}{\cdots} \int g(v_0, v_1, \cdots, v_n)$$

$$e^{\mp\frac{ip}{2}\sum_{j=1}^{n}\frac{[(u_j-u_{j-1})-(v_j-v_{j-1})]^2}{t_j-t_{j-1}}} e^{\mp ipa(u_0-v_0)^2}\, dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0),$$

and so, we obtain (3.8). And from (3.10) and (3.13), we have (3.9). □

Theorem 3.9. *For a nonzero real number q and a partition $0 = t_0 < t_1 < \cdots < t_n = t$, if $F(x) = f(x(t_0), \cdots, x(t_n))$ is in $\mathcal{A}_n$. Then $G \equiv T_q(F)$ exists and $G(y) \approx g(y(t_0), \cdots, y(t_n)) \in \mathcal{A}_n$ where*

$$(3.14)\qquad g(w_0, w_1, \cdots, w_n)$$

$$= \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \lim_{B\to\infty}\int_{D_B}\cdots\int_{D_B} f(v_0, v_1, \cdots, v_n)$$

$$e^{\frac{iq}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(w_j-w_{j-1})]^2}{t_j-t_{j-1}}} e^{iqa(v_0-w_0)^2}\, dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0).$$

Moreover,

$$(3.15)\qquad \| T_q(F) \|_2 \le 4\sqrt{a^3} \| g \|_2 .$$

Proof. Let

$$(3.16)\qquad \int_{C[0,t]}^{an\ anw\lambda} F(x + y)d\omega_\varphi(x)$$

$$= A\lambda^{\frac{n+1}{2}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(v_0, v_1, \cdots, v_n)$$

$$e^{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(y(t_j)-y(t_{j-1}))]^2}{t_j-t_{j-1}}} e^{-a\lambda(v_0-y(t_0))^2}$$

$$dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0)$$

$$= h(y(t_0), y(t_1), \cdots, y(t_n); \lambda).$$

Then from Theorem 3.7 h is analytic for λ.
Define for positive real number B,

$$f_B(v_0, v_1, \cdots, v_n) = \begin{cases} 0, |v_j| \le B & \text{for all } j = 0, \cdots, n \\ f(v_0, v_1, \cdots, v_n), |v_j| > B & \text{for some } j = 0, \cdots, n. \end{cases}$$

Then for two positive real numbers ρ and B,

$$\|g(\rho y(t_0), \rho y(t_1), \cdots, \rho y(t_n)) - h(\rho y(t_0), \rho y(t_1), \cdots, \rho y(t_n); \lambda)\|_2$$

$$= \rho^{-(n+1)} \Bigg\| \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}}$$

$$\lim_{B\to\infty} \int_{D_B} \cdots \int_{D_B} f(v_0, v_1, \cdots, v_n) e^{\frac{iq}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}}$$

$$e^{iqa(v_0-w_0)^2}\, dm_L(v_n)\cdots dm_L(v_1) dm_L(v_0)$$

$$-A\lambda^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$

$$f(v_0, v_1, \cdots, v_n) e^{-\frac{\lambda}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}} e^{-a\lambda(v_0-w_0)^2}$$

$$dm_L(v_n)\cdots dm_L(v_1) dm_L(v_0) \Bigg\|_2$$

$$\leq \rho^{-(n+1)} \Bigg\| \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}}$$

$$\int_{-B}^{B} \cdots \int_{-B}^{B} f(v_0, v_1, \cdots, v_n) e^{\frac{iq}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}}$$

$$e^{iqa(v_0-w_0)^2}\, dm_L(v_n)\cdots dm_L(v_1) dm_L(v_0)$$

$$-A\lambda^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-B}^{B} \cdots \int_{-B}^{B}$$

$$f(v_0, v_1, \cdots, v_n) e^{-\frac{\lambda}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}} e^{-a\lambda(v_0-w_0)^2}$$

$$dm_L(v_n)\cdots dm_L(v_1) dm_L(v_0) \Bigg\|_2$$

$$+\rho^{-(n+1)} \Bigg\| \frac{A}{\sqrt{2\pi}}(-iq)^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_B(v_0, v_1, \cdots, v_n) e^{\frac{iq}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}}$$

$$e^{iqa(v_0-w_0)^2}\, dm_L(v_n)\cdots dm_L(v_1) dm_L(v_0) \Bigg\|_2$$

$$+\rho^{-(n+1)} \Bigg\| - A\lambda^{\frac{n+1}{2}} \left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$

$$f_B(v_0, v_1, \cdots, v_n) e^{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j-v_{j-1})-(w_j-w_{j-1})]^2}{t_j-t_{j-1}}} e^{-a\lambda(v_0-w_0)^2}$$

$$dm_L(v_n)\cdots dm_L(v_1)dm_L(v_0)\Bigg\|_2$$

$$= I_1 + I_2 + I_3.$$

From Theorem 3.8 and Theorem 3.7 applied by f_B and $\|f_B\|_2$ converges to zero as $B \to \infty$, for each positive number ϵ, we can say $I_2 + I_3 \le \frac{\epsilon}{2}$. And the integral of in the norm $\|\cdot\|_2$ in I_1 is dominated by the function

$$\frac{A}{\sqrt{2\pi}}\left[\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right]^{-\frac{1}{2}} |f(v_0, v_1, \cdots, v_n)|\, 2^{\frac{n+3}{2}}(1+|q|)^{\frac{n+1}{2}}.$$

with $|\lambda + iq| < 1$ and $Re\ \lambda > 0$. Hence by dominated convergence theorem, I_1 converges to zero as $\lambda \to -iq$ ($Re\ \lambda > 0$). Thus the theorem is proved. □

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