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ON A DISCUSSION OF NONLINEAR INTEGRAL EQUATION OF TYPE VOLTERRA-HAMMERSTEIN

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ABSTRACT. Here, we consider the existence and uniqueness solution of nonlinear integral equation of the second kind of type Volterra-Hammerstein. Also, the normality and continuity of the integral operator are discussed. A numerical method is used to obtain a system of nonlinear integral equations in position. The solution is obtained, and many applications in one, two and three dimensionals are considered.

1. INTRODUCTION

Many authors have interested in solving the linear and nonlinear integral equation. In [1], Blon and Brunner introduced a class of methods depending on some parameters for the numerical solution of Abel integral equation of the second kind. Abdalkhani, in [2], obtained a numerical solution for the nonlinear Volterra integral equation of the second kind, when the kernel takes Abel's function form. Guoqiang et al., in [3], obtained numerically the solution of two-dimensional nonlinear Volterra integral equation by collocation and iterated collocation methods. In [4], Guoqiang and Jiong analyzed the existence of asymptotic error expansion of the Nyström solution for two-dimensional nonlinear Fredholm integral equation of the second kind. In [5], Abdou obtained, using separation variables method, the solution of the linear Fredholm-Volterra integral equation in one, two and three dimensionals. Many different cases for the linear and nonlinear integral equation with different kernels are discussed and solved by Abdou in [6]. Consider the nonlinear integral equation of the second kind of type Volterra-Hammerstein in n-dimensional

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(t,\tau)K(x,y)\gamma(\tau,y,\phi(y,\tau))dyd\tau, \tag{1.1}$$

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$$(x = \bar{x}(x_1, x_2, \cdots, x_n),\ y = \bar{y}(y_1, y_2, \cdots, y_n);\ \mu, \lambda \text{ are constants}).$$

Here, $f(x,t)$ and $\gamma(t,x,\phi(x,t))$ are two given functions, while the function $\phi(x,t)$ is unknown in Banach space $L_2(\Omega) \times C[0,T]$, where the domain of integration Ω is a closed bounded set depends on the vector of position, while the time $t \in [0,T]$. The two kernels of position $K(x,y)$, and of time $F(t,\tau), t, \tau \in [0,T], t < \infty$ are continuous with their derivatives with respect to position and time respectively. The constant μ defines the kind of the integral equation, while λ is a constant, may be complex, that has a physical meaning. Differentiating Eq.(1.1) with respect to the variable t, we get

$$\mu \frac{\partial \phi(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial t} + \lambda \int_\Omega F(t,t) K(x,y) \gamma(t,y,\phi(y,t)) dy + \lambda \int_0^t \int_\Omega \frac{\partial F(t,\tau)}{\partial t} K(x,y) \gamma(\tau, y, \phi(y,\tau)) dy d\tau. \tag{1.2}$$

The integro-differential equation (1.2) is equivalent to the integral equation (1.1). Therefore, the same solution will verify both of the two equivalent equations after neglecting the constant term. In this work, the existence and uniqueness solution of nonlinear integral equation of the second kind of type Volterra-Hammerstein, under certain conditions are discussed and proved, where the term of Volterra is considered in time, while Hammerstein in position. Moreover, the normality and continuity of the integral operator are considered. A numerical method is used to obtain a system of nonlinear integral equations in position. Also, the degenerate kernel method is considered, for solving the integral system numerically. Finally, many different cases in one, two and three dimensionals with different kernels are solved.

2. THE EXISTENCE AND UNIQUENESS SOLUTION

In this section , Banach fixed point theorem will be used as a source of existence and uniqueness solution of Eq.(1.1). For this, we write it in the integral operator form

$$\bar{W}\phi(x,t) = \frac{1}{\mu} f(x,t) + \frac{1}{\mu} W\phi(x,t), \quad (\mu \neq 0) \tag{2.1}$$

where

$$W\phi(x,t) = \lambda \int_0^t \int_\Omega F(t,\tau) K(x,y) \gamma(\tau, y, \phi(y,\tau)) dy d\tau. \tag{2.2}$$

Also, we assume the following conditions:

(i) The kernel of position $K(x,y), x=\bar{x}(x_1,x_2,\cdots,x_n), y=\bar{y}(y_1,y_2,\cdots,y_n)$, satisfies the discontinuity condition

$$\left\{\int_\Omega\int_\Omega |K(x,y)|^2 dxdy\right\}^{\frac{1}{2}} = c^*, \quad (c^* \text{ is a constant}).$$

(ii) The kernel of time $F(t,\tau)\in C[0,T]$ and satisfies $|F(t,\tau)|\le M$, M is a constant, $\forall t,\tau\in[0,T]$, $0\le\tau\le t\le T<\infty$.

(iii) The given function $f(x,t)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_2(\Omega)\times C[0,T]$, and its norm is defined as

$$\|f(x,t)\|_{L_2(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega f^2(x,\tau)dx\right\}^{\frac{1}{2}}d\tau\right| = G, \ (G \text{ is a constant}).$$

(iv) The known continuous function $\gamma(t,x,\phi(x,t))$ for the constants $Q>P$ and $Q>Q_1$ satisfies the following conditions :

(a) $\max\limits_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega |\gamma(\tau,x,\phi(x,\tau))|^2dx\right\}d\tau\right| \le Q_1\|\phi(x,t)\|_{L_2(\Omega)\times C[0,T]}.$

(b) $|\gamma(t,x,\phi_1(x,t)) - \gamma(t,x,\phi_2(x,t)) \le N(t,x)|\phi_1(x,t)-\phi_2(x,t)|$,

where

$$\|N(t,x)\|_{L_2(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega N^2(\tau,x)dx\right\}^{\frac{1}{2}}d\tau\right| = P<\infty.$$

Theorem 1. *If the conditions (i)-(iv) are satisfied, then Eq.(1.1) has a unique solution in Banach space $L_2(\Omega)\times C[0,T]$.*

To prove this theorem, we must consider the following lemmas :

Lemma 1. *Under the condition (i)-(iv − a), the operator $\bar{W}$ maps Banach space $L_2(\Omega)\times C[0,T]$ into itself.*

Proof. In the light of the two formulas (2.1) and (2.2), we have

$$\|\bar{W}\phi(x,t)\| \le \frac{1}{|\mu|}\|f(x,t)\| + \frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega |F(t,\tau)|\,|K(x,y)|\,|\gamma(\tau,y,\phi(y,\tau))|dyd\tau\right\|.$$

Using the conditions (ii) and (iii), then applying Cauchy-Schwarz inequality, we get

$$\|\bar{W}\phi(x,t)\| \le \frac{G}{|\mu|} + \frac{|\lambda|}{|\mu|}M\left\|\left\{\int_\Omega |K(x,y)|^2dy\right\}^{\frac{1}{2}} \cdot \max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega |\gamma(\tau,y,\phi(y,\tau)|^2|dy\right\}^{\frac{1}{2}}d\tau\right|\right\|.$$

In view of conditions (i) and (iv-a), the above inequality can be adapted to,

$$\|\bar{W}\phi(x,t)\| \leq \frac{G}{|\mu|} + \sigma\|\phi(x,t)\|, \; (\sigma = \left|\frac{\lambda}{\mu}\right| Mc^*QT). \tag{2.3}$$

The previous inequality (2.3) shows that, the operator $\bar{W}$ maps the ball S_ρ into itself where,

$$\rho = \frac{G}{[|\mu| - |\lambda| Mc^*QT]}. \tag{2.4}$$

Since $\rho > 0$, $G > 0$, therefore we have $\sigma < 1$. Also, the inequality (2.3) involves the boundedness of the operator W of Eq.(2.2), where

$$\|W\phi(x,y)\| \leq \sigma\|\phi(x,y)\|. \tag{2.5}$$

Moreover , the inequalities (2.3) and (2.5) define the boundedness of the operator $\bar{W}$. □

Lemma 2. *Assume that the conditions (i),(ii) and (iv-b) are verified, then $\bar{W}$ is a contraction operator in Banach space $L_2(\Omega) \times C[0,T]$.*

Proof. For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in Banach space $L_2(\Omega) \times C[0,T]$, and from Eqs.(2.1), (2.2), we find

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\| \leq \frac{|\lambda|}{|\mu|}\Big\| \int_0^t \int_\Omega |F(t,\tau)|\,|K(x,y)|\,|\gamma(\tau,y,\phi_1(y,\tau)) - \gamma(\tau,y,\phi_2(y,\tau))|dyd\tau \Big\|.$$

With the aid of conditions (ii) and (iv-b), the above inequality becomes

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\| \leq M\frac{|\lambda|}{|\mu|}\Big\| \int_0^t \int_\Omega |K(x,y)|\, N(\tau,y)\, |\phi_1(y,\tau) - \phi_2(y,\tau)|dyd\tau \Big\|.$$

Applying Cauchy-Schwarz inequality to Hammerstein integral term then using condition (i), we finally get

$$\|\bar{W}\phi_1(x,t) - \bar{W}\phi_2(x,t)\| \leq \sigma\|\phi_1(x,t) - \phi_2(x,t)\|. \tag{2.6}$$

From inequality (2.6), we see that, the operator $\bar{W}$ is continuous in Banach space $L_2(\Omega) \times C[0,T]$, then $\bar{W}$ is a contraction operator under the condition $\sigma < 1$. □

Proof of Theorem 1. The previous two lemmas (1) and (2) prove that, the operator $\bar{W}$ of Eq.(2.1) is contractive in Banach space $L_2(\Omega) \times C[0,T]$. So, from Banach fixed point theorem, $\bar{W}$ has a unique fixed point which is, of course, the unique solution of Eq.(1.1). □

3. System of Nonlinear Integral Equations in Position

In this section, a numerical method is used in the mixed integral Eq.(1.1), to obtain a system of nonlinear integral equations in position.

If we divide the interval $[0,T]$, $0 \leq \tau \leq t \leq T < \infty$ as $0 = t_0 < t_1 < ... < t_k < ... < t_N = T$, where $t = t_k$, $k = 0,1,2,...,N$, the integral term of Eq.(1.1) becomes

$$(3.1) \qquad \int_0^{t_k} \int_\Omega F(t_k,\tau)\, K(x,y)\, \gamma(\tau,y,\phi(y,\tau))dyd\tau$$
$$= \sum_{j=0}^{k} u^{(j)} F(t_k,t_j) \int_\Omega K(x,y)\, \gamma(t_j,y,\phi(y,t_j))dy + O(\eta_k^{p+1}), \quad (\eta_k \to 0,\, p > 0)$$

where,

$$\eta_k = \max_{0\leq j\leq k} h_j,\ h_j = t_{j+1} - t_j,\ u^{(0)} = \frac{1}{2}h_0,\ u^{(k)} = \frac{1}{2}h_k,\ u^{(j)} = h_j \quad (j \neq 0,k).$$

The values of $u^{(j)}$'s and p; $p \approx k$ are depending on the number of derivatives of $F(t,\tau)$ with respect to time, see [7, 8].

Using (3.1) in (1.1), and neglecting $O(\eta_k^{p+1})$, we have

$$(3.2) \qquad \mu\phi^{(k)}(x) = f^{(k)}(x) + \lambda \sum_{j=0}^{k} u^{(j)} F^{(j,k)} \int_\Omega K(x,y)\, \gamma^{(j)}(y,\phi^{(j)}(y))dy.$$

Here, we used the following notations

$$\phi(x,t_k) = \phi^{(k)}(x),\ f(x,t_k) = f^{(k)}(x),\ F(t_k,t_j) = F^{(j,k)}, \ \text{and}$$
$$(3.3) \quad \gamma(t_j,x,\phi(x,t_j)) = \gamma^{(j)}(x,\phi^{(j)}(x)),\ x = \bar{x}(x_1,x_2,...,x_n),\ y = \bar{y}(y_1,y_2,...,y_n).$$

The formula (3.2) represents a system of nonlinear algebraic integral equations in n-dimensional, and its solution depends on the given function $f^{(k)}(x)$, the kind of the kernel $K(x,y)$, and the degree of the known function $\gamma^{(j)}(x,\phi^{(j)}(x))$.

4. On a Method to Solve a System of Nonlinear Integral Equations

The simplicity of finding a solution of the nonlinear algebraic integral system (3.2) with a degenerate method naturally leads one to think of replacing the given kernel $K(x,y)$ approximately by a degenerate kernel $K_n(x,y)$; that is

$$(4.1) \qquad K_n(x,y) = \sum_{i=1}^{n} B_i(x)C_i(y).$$

Here, the set of functions $\{B_i(x)\}$ and $\{C_i(y)\}$ are assumed to be linearly independent, such that

$$\left\{\int_\Omega\int_\Omega |K(x,y)-K_n(x,y)|^2 dxdy\right\}^{\frac{1}{2}} \to 0 \text{ as } n\to\infty. \tag{4.2}$$

Hence, the solution of Eq.(3.2) associated to the kernel $K_n(x,y)$ takes the form

$$\mu\phi_n^{(k)}(x) = f^{(k)}(x) + \lambda\sum_{j=0}^{k} u^{(j)}F^{(j,k)}\int_\Omega K_n(x,y)\,\gamma^{(j)}(y,\phi_n^{(j)}(y))dy. \tag{4.3}$$

Using (4.1) in (4.3), we have

$$\mu\phi_n^{(k)}(x) = f^{(k)}(x) + \lambda\sum_{i=0}^{n}\sum_{j=0}^{k} u^{(j)}F^{(j,k)}A_i^{(j)}B_i(x),\quad (\mu\neq 0),\ k=0,1,2,...,N. \tag{4.4}$$

where

$$A_i^{(j)} = \int_\Omega C_i(y)\gamma^{(j)}(y,\phi_n^{(j)}(y))dy,\quad (j=0,1,2...,k). \tag{4.5}$$

Here, $A_i^{(j)}$'s are constants to be determined from the following formula

$$\begin{aligned} A_m^{(j)} = \int_\Omega C_m(y)\gamma^{(j)}\Big(y,\frac{1}{\mu}f^{(j)}(y) \\ +\frac{\lambda}{\mu}\sum_{i=1}^{n}\sum_{r=0}^{j} u^{(r)}F^{(r,j)}A_i^{(j)}B_i(y)\Big)dy,\quad (m=0,1,2...,n). \end{aligned} \tag{4.6}$$

If we define

$$\begin{aligned} H_m^{(j)}(A_1^{(j)},A_2^{(j)},...,A_n^{(j)}) = \int_\Omega C_m(y)\gamma^{(j)}\Big(y,\frac{1}{\mu}f^{(j)}(y) \\ +\frac{\lambda}{\mu}\sum_{i=1}^{n}\sum_{r=0}^{j} u^{(r)}F^{(r,j)}A_i^{(j)}B_i(y)\Big)dy,\quad (m=0,1,2...,n). \end{aligned} \tag{4.7}$$

then, Eq.(4.6) represents a system of nonlinear algebraic equations which can be written as a matrix equation

$$\begin{bmatrix} A_1^{(j)} \\ A_2^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ A_n^{(j)} \end{bmatrix} = \begin{bmatrix} H_1^{(j)}(A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}) \\ H_2^{(j)}(A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}) \\ \cdot \\ \cdot \\ \cdot \\ H_n^{(j)}(A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}) \end{bmatrix}, \tag{4.8}$$

or in a vector form as

$$\bar{A}^{(j)} = \bar{H}^{(j)}(\bar{A}^{(j)}), \tag{4.9}$$

where,

$$\bar{H}^{(j)} = (H_1^{(j)}, H_2^{(j)}, ..., H_n^{(j)}),$$

and

$$\bar{A}^{(j)} = (A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}).$$

The nonlinear algebraic system (4.8) or (4.9) can be solved numerically.

Now, we shall show that the unique solution of the nonlinear algebraic system (4.4) corresponds to the unique solution of (4.8). Also, under the condition (4.2),we shall prove that, for each positive integer $n > n_0$, the unique solution of Eq.(4.3) converges to the unique solution of Eq.(1.1).

Theorem 2. *Suppose that $K_n(x,y) \in C([\Omega] \times [\Omega])$ and satisfy condition (4.2), then there exists a positive integer n_0, such that for each $n > n_0$, the integral equation*

$$\mu\phi_n(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(t,\tau)K_n(x,y)\gamma(\tau,y,\phi_n(y,\tau))dyd\tau, \tag{4.10}$$

has a unique solution $\phi_n(x,t) \in L_2(\Omega) \times C[0,T]$.

Proof. In view of condition (4.2) and condition (i) of Theorem (1), there exists a positive integer n_0, such that

$$\left\{ \int_\Omega \int_\Omega |K_n(x,y)|^2 dxdy \right\}^{\frac{1}{2}} \leq c^*, \quad \forall n > n_0. \tag{4.11}$$

Define the operator

$$\bar{V}\phi_n(x,t) = \frac{1}{\mu}f(x,t) + \frac{1}{\mu}V\phi_n(x,t), \quad (\mu \neq 0), \tag{4.12}$$

where,

$$V\phi_n(x,t) = \lambda \int_0^t \int_\Omega F(t,\tau)K_n(x,y)\gamma(\tau,y,\phi_n(y,\tau))dyd\tau. \tag{4.13}$$

Taking in account the conditions of Theorem 1 with the condition (4.11), and proceeding the same proof of Lemma 1 and 2, we see that $\bar{V}$ is a bounded and continuous operator which maps Banach space $L_2(\Omega) \times C[0,T]$ into itself, where

$$\|\bar{V}\phi_n(x,t)\| \leq \frac{G}{|\mu|} + \sigma\|\phi_n(x,t)\|;\ \phi_n \in L_2(\Omega) \times C[0,T],\ \forall n > n_0, \tag{4.14}$$

and

$$\|\bar{V}\phi_n^{(1)}(x,t) - \bar{V}\phi_n^{(2)}(x,t)\| \leq \sigma\|\phi_n^{(1)}(x,t) - \phi_n^{(2)}(x,t)\|; \tag{4.15}$$
$$\phi_n^{(1)}, \phi_n^{(2)} \in L_2(\Omega) \times C[0,T],\ \forall n > n_0.$$

Hence, $\bar{V}$ is a contraction operator under the condition $\sigma < 1$. Moreover, by virtue of Banach fixed point theorem, $\bar{V}$ has a unique fixed point which is, the unique solution of Eq.(4.10). □

Theorem 3. *Under the same assumptions of Theorem 1 and 2, the sequence of solution $\{\phi_n(x,t)\}$ of Eq.(4.10) converges to the unique solution $\phi(x,t)$ of Eq.(1.1) in Banach space $L_2(\Omega) \times C[0,T]$.*

Proof. From the two formulas (1.1) and (4.10), we get

$$\begin{aligned}&\|\phi(x,t) - \phi_n(x,t)\| \\ &\leq \frac{|\lambda|}{|\mu|}\Big\{\Big\|\int_0^t\int_\Omega |K(x,y) - K_n(x,y)|\,|\gamma(\tau,y,\phi(y,\tau))dyd\tau\Big\| \\ &+ \Big\|\int_0^t\int_\Omega |F(t,\tau)|\,|K_n(x,y)|\,|\gamma(\tau,y,\phi(y,\tau)) - \gamma(\tau,y,\phi_n(y,\tau))|dyd\tau\Big\|\Big\}\end{aligned}$$

Introducing conditions (ii) and (iv-b), then applying Cauchy-Schwartz inequality the previous inequality becomes,

$$\begin{aligned}\|\phi(x,t) - \phi_n(x,t)\| \leq \frac{|\lambda|}{|\mu|}M\Big\{\Big\|&\Big(\int_\Omega |F(t,\tau)|\,|K(x,y) - K_n(x,y)|dy\Big)^{\frac{1}{2}} \\ &\cdot \max_{0\leq t\leq T}\Big|\int_0^t\Big(\int_\Omega |\gamma(\tau,y,\phi(y,\tau))|^2dy\Big)^{\frac{1}{2}}d\tau\Big\| \\ &+ \Big\|\Big(\int_\Omega |K_n(x,y)|^2dy\Big)^{\frac{1}{2}} \cdot \max_{0\leq t\leq T}\Big|\int_0^t\Big(\int_\Omega N^2(\tau,y)\,|\phi(y,\tau) \\ &- \phi_n(y,\tau)|^2dy\Big)^{\frac{1}{2}}d\tau\Big|\Big\|\Big\}.\end{aligned}$$

In view of conditions (4.11) and (iv-a), the above inequality takes the form

$$\begin{aligned}\|\phi(x,t) - \phi_n(x,t)\| \leq \frac{|\lambda|}{|\mu|}MTQ\{&\|K(x,y) - K_n(x,y)\|\,\|\phi(x,t)\| \\ &+ c^*\|\phi(x,t) - \phi_n(x,t)\|\}.\end{aligned}$$

Thus, we have

(4.16)
$$\|\phi(x,t)-\phi_n(x,t)\| \leq \frac{|\lambda|MTQ}{[|\mu|-|\lambda|Mc^*QT]}\|\phi(x,t)\|\,\|K(x,y)-K_n(x,y)\|, (\sigma<1).$$

Since $\|K(x,y)-K_n(x,y)\| \to 0$ as $n\to\infty$, so $\|\phi(x,t)-\phi_n(x,t)\|\to 0$ as $n\to\infty$. This completes the proof. □

Theorem 4. *Assume that, the known continuous functions $\gamma^{(j)}(y,\Psi(y,A_i^{(j)}))$ in Eq.*(4.7) *satisfy the following conditions*

$$\left\{\int_\Omega |\gamma^{(j)}(y,\Psi(y,A_i^{(j)}))|^2 dy\right\}^{\frac{1}{2}} \leq L\left(\sum_{i=1}^{n}|A_i^{(j)}|^2\right)^{\frac{1}{2}}, \quad (L \text{ is a constant}) \tag{4.17}$$

and

$$|\gamma^{(j)}(y,\Psi(y,A_i^{(j)}))-\gamma^{(j)}(y,\Psi(y,D_i^{(j)}))| \leq M_1|\Psi(y,A_i^{(j)})-\Psi(y,D_i^{(j)})|, \tag{4.18}$$
$(M_1$ *is a constant*).

Then, the algebraic system (4.8) *has a unique solution $\bar{A}^{(j)}$, and $\phi_n^{(k)}(x)$ is the unique solution of Eq.*(4.4) *in the Banach space ℓ_2.*

To prove this theorem, we must consider the following two lemmas:

Lemma 3. *Under condition* (4.17), *the operator $\bar{H}^{(j)}$ of Eq.*(4.9) *maps Banach space ℓ_2 into itself.*

Proof. Let U be the set of functions $\Xi=\{\xi_i\}$ in Banach space ℓ_2 such that

$$\|\Xi\|_{\ell_2}=\left(\sum_{i=1}^{\infty}|\xi_i|^2\right)^{\frac{1}{2}} \leq \beta,\quad \beta \text{ is a constant.}$$

From Eq.(4.7), we have

$$|H_m^{(j)}(A_1^{(j)},A_2^{(j)},\dots,A_n^{(j)})| \leq \int_\Omega |C_m(y)|\left|\gamma^{(j)}\left(y,\frac{1}{\mu}f^{(j)}(y) + \frac{\lambda}{\mu}\sum_{i=1}^{n}\sum_{r=0}^{j}u^{(r)}F^{(r,j)}A_i^{(j)}B_i(y)\right)\right|dy.$$

Applying Cauchy-Schwarz inequality, and using condition, the above inequality becomes,

$$\left(\sum_{m=1}^{n}|H_m^{(j)}(A_1^{(j)},A_2^{(j)},\dots,A_n^{(j)})|^2\right)^{\frac{1}{2}} \leq M_2\left(\sum_{i=1}^{n}|A_i^{(j)}|^2\right)^{\frac{1}{2}},$$

where, $M_2 = L\left(\sum_{m=1}^{n}\int_\Omega |C_m(y)|^2 dy\right)^{\frac{1}{2}}$. As $n \to \infty$, the last inequality can be adapted to,

$$\|\bar{H}^{(j)}(\bar{A}^{(j)})\|_{\ell_2} \le M_2 \|\bar{A}^{(j)}\|_{\ell_2}. \tag{4.19}$$

Hence, $\bar{H}^{(j)}$ is a bounded operator which maps the set U into itself where,

$$\beta = M_2 \|\bar{A}^{(j)}\|_{\ell_2}. \tag{4.20}$$

□

Lemma 4. *Under condition* (4.18), $\bar{H}^{(j)}$ *is a contraction operator in the Banach space* ℓ_2.

Proof. Let $\bar{A}^{(j)} = (A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)})$ and $\bar{D}^{(j)} = (D_1^{(j)}, D_2^{(j)}, ..., D_n^{(j)})$ be any two vectors. In view of Eq.(4.7), we get

$$\begin{aligned} &|H_m^{(j)}(A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}) - H_m^{(j)}(D_1^{(j)}, D_2^{(j)}, ..., D_n^{(j)})| \\ &\le \int_\Omega |C_m(y)| \left|\gamma^{(j)}\left(y, \frac{1}{\mu} f^{(j)}(y) + \frac{\lambda}{\mu}\sum_{i=1}^{n}\sum_{r=0}^{j} u^{(r)} F^{(r,j)} A_i^{(j)} B_i(y)\right)\right. \\ &\left. - \gamma^{(j)}\left(y, \frac{1}{\mu} f^{(j)}(y) + \frac{\lambda}{\mu}\sum_{i=1}^{n}\sum_{r=0}^{j} u^{(r)} F^{(r,j)} D_i^{(j)} B_i(y)\right)\right| dy \end{aligned}$$

Introducing condition (4.18), then applying Cauchy-Schwarz inequality three times, respectively, the above inequality takes the form,

$$\begin{aligned} &\left(\sum_{m=1}^{n} |H_m^{(j)}(A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}) - H_m^{(j)}(D_1^{(j)}, D_2^{(j)}, ..., D_n^{(j)})|^2\right)^{\frac{1}{2}} \\ &\le M_3 \left(\sum_{i=1}^{n} |A_i^{(j)} - D_i^{(j)}|^2\right)^{\frac{1}{2}}, \quad (M_3 \text{ is a constant}), \end{aligned}$$

where,

$$\begin{aligned} M_3 = \frac{|\lambda|}{|\mu|} M_1 &\left(\sum_{r=0}^{n} |u^{(r)}|^2\right)^{\frac{1}{2}} \left(\sum_{r=0}^{j} |F^{(r,j)}|^2\right)^{\frac{1}{2}} \\ &\cdot \left(\sum_{m=1}^{n}\int_\Omega |C_m(y)|^2 dy\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}\int_\Omega |B_i(y)|^2 dy\right)^{\frac{1}{2}}. \end{aligned}$$

As $n \to \infty$, the previous inequality can be reduced to

$$\|\bar{H}^{(j)}(\bar{A}^{(j)}) - \bar{H}^{(j)}(\bar{D}^{(j)})\|_{\ell_2} \le M_3 \|\bar{A}^{(j)} - \bar{D}^{(j)}\|_{\ell_2}. \tag{4.21}$$

Thus, $\bar{H}^{(j)}$ is a continuous operator in the Banach space ℓ_2. If $M_3 < 1$, then $\bar{H}^{(j)}$ is a contraction operator. □

Proof of Theorem 4. By virtue of lemmas (3) and (4), we see that $\bar{H}^{(j)}$ is a contraction operator which maps Banach space ℓ_2 into itself. Hence, by Banach fixed point theorem, $\bar{H}^{(j)}$ has a unique fixed point $\bar{A}^{(j)}$ which is the unique solution of the algebraic system (4.8) . For this solution, it is obvious that $\phi_n^{(k)}(x)$ is the unique of Eq.(4.4) by Theorem 2. □

5. Examples

Here, two examples will be introduced as an illustration for solving nonlinear integral equation of type Volterra-Hammerstein with degenerate kernel.

Example 1. Consider the nonlinear Volterra-Hammerstein integral equation of the second kind

$$(5.1)\quad \phi(x,t) - \int_0^t \int_0^1 \tau^3(1+xy)\phi^2(y,\tau)dyd\tau = xt - \frac{1}{24}xt^6 - \frac{1}{18}t^6, \ (\phi(x,t) = xt).$$

If we divide the interval $[0,T]$ as $0 = t_0 < t_1 < t_2 < t_3 = t$, $t = t_k$; $k = 0,1,2,3$, the integral Eq.(5.1) takes the form

$$(5.2)\quad \phi_k(x) - \sum_{j=0}^{k} u_j t_j^3 \int_0^1 (1+xy)\phi_j^2(y)dy = xt_k - \frac{1}{24}xt_k^6 - \frac{1}{18}t_k^6, \ (k = 0,1,2,3).$$

Using the degenerate method, we have the following nonlinear algebraic system

$$(5.3)\qquad \phi_k(x) - \sum_{j=0}^{k} u_j t_j^3[\alpha_j + x\gamma_j] = xt_k - \frac{1}{24}xt_k^6 - \frac{1}{18}t_k^6, \ (k = 0,1,2,3),$$

where,

$$(5.4)\qquad \alpha_k = \int_0^1 \left[yt_k - \frac{1}{24}yt_k^6 - \frac{1}{18}t_k^6 + \sum_{j=0}^{k} u_j t_j^3(\alpha_j + y\gamma_j)\right]^2 dy, \ (k = 0,1,2,3),$$

and

$$(5.5)\qquad \gamma_k = \int_0^1 y\left[yt_k - \frac{1}{24}yt_k^6 - \frac{1}{18}t_k^6 + \sum_{j=0}^{k} u_j t_j^3(\alpha_j + y\gamma_j)\right]^2 dy, \ (k = 0,1,2,3),$$

Thus, the solution of the nonlinear algebraic system (5.3) leads us to

$$\phi_0(x) = 0, \ \phi_1(x) = \frac{xt}{3}, \ \phi_2(x) = \frac{2xt}{3}, \ \phi_3(x) = xt.$$

It is obvious that $\phi_3(x)$ is the exact solution of Eq.(5.1).

Example 2. Consider the nonlinear Volterra -Hammerstein integral equation of the second kind

$$\phi(x,t) - \int_0^t \int_0^1 \tau^3 e^{xy} \phi^2(y,\tau)\,dy\,d\tau = xt - \left[\frac{e^x}{x} - \frac{2}{x^2}e^x + \frac{2}{x^3}e^x - \frac{2}{x^3}\right]\frac{t^6}{6}, \tag{5.6}$$

$$(\phi(x,t) = xt).$$

Dividing the interval $[0,T]$ as $0 = t_0 < t_1 < t_2 < t_3 = t,\ t = t_k;\ k = 0,1,2,3,$ the integral Eq.(5.6) can be written in the form

$$\phi_k(x) - \sum_{j=0}^{k} u_j t_j^3 \int_0^1 e^{xy} \phi_j^2(y)\,dy = xt_k - \left[\frac{e^x}{x} - \frac{2}{x^2}e^x + \frac{2}{x^3}e^x - \frac{2}{x^3}\right]\frac{t_k^6}{6}, \tag{5.7}$$

$$(k = 0,1,2,3).$$

The kernel $K(x,y) = e^{xy}$ can be approximated by a degenerate one

$$K_n(x,y) = \sum_{m=0}^{n} \frac{(xy)^m}{m!}, \tag{5.8}$$

which is a Taylor polynomial of degree n at $x = 0$, and satisfies

$$\left\{\int_0^1 \int_0^1 |K(x,y) - K_n(x,y)|^2\,dx\,dy\right\}^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty. \tag{5.9}$$

Introducing the degenerate kernel $K_n(x,y)$ instead of the kernel $K(x,y)$ in (5.7), we get

$$\phi_k(x) - \sum_{j=0}^{k} u_j t_j^3 \int_0^1 \sum_{m=0}^{n} \frac{x^m y^m}{m!} \phi_j^2(y)\,dy = F_k(x), \quad (k = 0,1,2,3) \tag{5.10}$$

where,

$$F_k(x) = F_k(x,t_k) = xt_k - \left[\frac{e^x}{x} - \frac{2}{x^2}e^x + \frac{2}{x^3}e^x - \frac{2}{x^3}\right]\frac{t_k^6}{6}, \quad (k = 0,1,2,3). \tag{5.11}$$

Therefore, we have

$$\phi_k(x) - \sum_{j=0}^{k} \sum_{m=0}^{n} u_j t_j^3 \frac{x^m}{m!} A_j^{(m)} = F_k(x), \quad (k = 0,1,2,3), \tag{5.12}$$

where,

$$A_k^{(m)} = \int_0^1 y^m \left[F_k(y) + \sum_{j=0}^{k} \sum_{m=0}^{n} u_j t_j^3 \frac{x^m}{m!} A_j^{(m)}\right]^2 dy, \quad (k = 0,1,2,3). \tag{5.13}$$

The solution of (5.12), with the aid of (5.13), gives the following results

$$\phi_0(x) = 0, \quad \phi_1(x) = \frac{xt}{3}, \quad \phi_2(x) = \frac{2xt}{3}, \quad \phi_3(x) = xt.$$

Here, $\phi_3(x)$ is the exact solution of Eq.(5.6).

6. Applications and Special Cases

In this section, we shall discuss different formulas that can be established from the type of nonlinear Volterra-Hammerstein integral equation.

(1) Hammerstein integral equation:

If in Eq.(3.2), $x = \bar{x}(x_1)$, $\Omega = [0,1]$ and $k = 0$, we have the integral equation

$$\Psi(x) + \lambda_0 \int_0^1 K(x,y) Z(y, \Psi(y)) dy = g(x), \tag{6.1}$$

where, $\Psi(x) = \phi^{(0)}(x) - \dfrac{f^{(0)}(x)}{\mu}$ and $\lambda_0 = u^{(0)} F^{(0,0)} \dfrac{\lambda}{\mu}$.

The formula (6.1) represents a Hammerstein integral equation of the second kind. Lardy, in [9], used a variation of Nyström method to solve the integral Eq.(6.1), when the kernel $K(x,y)$ is in a continuous form. In [10], Kumar and Sloan, and in [11], Kumar used and developed a new collocation type method to solve the formula (6.1) in $L_2[0,1]$, numerically. In [12], the solution of Eq.(6.1) using degenerate kernel method is obtained by Kaneko and Xu. In [13], Hacia obtained, using numerical method, approximately the solution of the system of Hammerstein integral equations in a Banach space. In Hilbert space Hacia, in [14], using projection iteration methods, obtained the solution of nonlinear operator equations of Hammerstein type of (6.1). In [15] and [16,17], Bannas and Emmanuele respectively have studied the type of problem of Eq.(6.1) in $L_1[0,1]$ when $\bar{x} = x(x_1)$ their analysis depend on the technique of noncompactness.

(2) One dimensional Fredholm integral equation with discontiuous kernel:

If in Eq.(3.2), $k = 0$, $\gamma^{(0)}(y, \phi^{(0)}(y)) = \phi^{(0)}(y) = \phi(y)$, $\Omega = [-1,1]$ and $\bar{x} = x(x_1)$, we have the following integral equation

$$\mu\phi(x) = f(x) + \lambda' \int_{-1}^1 K(x,y)\phi(y) dy, \quad (\lambda' = \lambda u^{(0)} F^{(0,0)}). \tag{6.2}$$

The formula (6.2) represents a Fredholm integral equation of the second kind. When the kernel takes the two forms

$$K(x,y) = \begin{cases} (\ln|x-y|)^q, & q = 1,2,\dots,N, \\ |x-y|^{-\nu}, & 0 \le \nu < 1 \end{cases}. \tag{6.3}$$

Abdou et al., in [18], obtained numerically the solution of (6.2) with the kernel (6.3) using Toeplitz matrix and product Nyström method. Using a numerical method and orthogonal polynomials of Legendre series Abdou and Nasr in [19], obtained the solution of (6.2) when the kernel takes a Cauchy form, $K(x,y) = \frac{1}{x-y}$. If we let in (6.3), $q = 1$ and differentiate (6.2) with respect to x, we obtain the following integro-differential equation with Cauchy kernel

$$\mu\frac{d\phi(x)}{dx} - \lambda' \int_{-1}^{1} \frac{\phi(x)}{x-y}dy = h(x). \tag{6.4}$$

Using in (6.4), the substitution $y = 2u - 1$, $x = 2v - 1$, we have

$$\frac{d\Theta}{dv} - \lambda_1 \int_0^1 \frac{\Theta(u)}{v-u}du = g(u),\ g(u) = \frac{2}{\mu}h(2v-1),\ \lambda_1 = \frac{2\lambda'}{\mu},\ \phi(2v-1) = \Theta(v), \tag{6.5}$$

Frankel, in his work [20], obtained the solution of (6.5) under the condition $\Theta(0) = \Theta(1) = 0$, using Chebyshev polynomials. Eq.(6.5) has appeared in both combined infrared gaseous radiation and molecular conduction.

(3) Two and three dimensional integral equation:

Let in Eq.(3.2), $k = 0$, $\gamma^{(0)}(y, \phi^{(0)}(y)) = \phi^{(0)}(y) = \phi(y)$ and $x = \bar{x}(x_1, x_2, x_3)$, then we have

$$\mu\phi(x,y) = f(x,y) + \lambda \iint_{\Omega} K(x-\xi, y-\eta)\phi(\xi,\eta)d\xi d\eta. \tag{6.6}$$

(i) If,

$$K(x-\xi, y-\eta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}},$$

and

$$\Omega = \{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a,\ z = 0\},$$

we have an integral equation of the second kind with potential kernel. This case was discussed and solved by Abdou, in [21]. Also, Abdou in [22], discussed the structure resolvent of the integral equation with potential kernel.

(ii) If,

$$K(x-\xi, y-\eta) = [(x-\xi)^2 + (y-\eta)^2]^{-\nu},\ \Omega = \{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a,\ z = 0\}.$$

The integral equation was investigated from the semi-symmetric Hertz problem for two different elastic materials in three-dimensional when the modules of elasticity is changing according to the power law $\sigma_i = K_0\varepsilon_i^{\nu}$ $(0 \le \nu < 1)$, where σ_i and ε_i, $i = 1, 2, 3$, are the stress and strain rate intensities respectively, while K_0, ν, are physical constants. Abdou and Salma, in [23], discussed the integral equation of the second kind with generalized potential kernel in the contact problems in the theory of elasticity and established its solution.

(iii) Let in (1.1),

$$\gamma(\tau, y, \phi(y,\tau)) = \phi(y,\tau),\ F(t,\tau) = 1,\ K(x-\xi, y-\eta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}},$$

$$\Omega = \{(x,y,z) \in \Omega : |x| < \infty, |y| < \infty,\ z = 0\},$$

we have the following integral equation

$$\mu\phi(x,y,t) = f(x,y,t) + \lambda \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\phi(\xi,\eta,\tau)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\, d\xi d\eta d\tau. \tag{6.7}$$

The integral Eq.(6.7) is established from the three-dimensional half-space of contact problem. To construct the general solution of (6.7), we use the following Fourier integral transformations

$$\phi_\alpha(x,t) = \int_{-\infty}^{\infty} \phi(x,y,t) e^{i\alpha y} dy,\quad f_\alpha(x,t) = \int_{-\infty}^{\infty} f(x,y,t) e^{i\alpha y} dy. \tag{6.8}$$

Hence, we obtain the Volterra-Wiener-Hopf integral equation as

$$\mu\phi_\alpha(x,t) = f_\alpha(x,t) + \lambda \int_0^t \int_0^\infty K_0(|\alpha(x-\xi)|)\phi_\alpha(\xi,\tau) d\xi d\tau, \tag{6.9}$$

where α is the Fourier parameter, and K_0 is the Macdonald kernel (see [24])

$$K_0(|\alpha(x-\xi)|) = \int_0^\infty \frac{\cos\alpha y}{\sqrt{(x-\xi)^2 + y^2}}\, dy. \tag{6.10}$$

Using the following notations

$$|\alpha|x = u,\ |\alpha|\xi = s,\ \alpha = \frac{-\lambda}{\mu\alpha},\ \Psi(u,t) = \phi_\alpha\Big(\frac{u}{|\alpha|}, t\Big),\ g(u,t) = \frac{1}{\mu} f_\alpha\Big(\frac{u}{|\alpha|}, t\Big). \tag{6.11}$$

The integral Eq.(6.9) takes the form

$$\Psi(u,t) - a \int_0^t \int_0^\infty K_0(|u-s|\Psi(s,\tau) ds d\tau = g(u,t), \tag{6.12}$$

$$(0 < u < \infty,\ 0 \le t \le T < \infty,\ a < 0)$$

The formula (6.12) represents a mixed type of Volterra-Wiener-Hopf integral equation of the second kind with Macdonald kernel. Abdou and Badr, in [25], obtained the general solution of (6.12), using separation of variables method.

(iv) If we let in (1.1), $\mu = 0$, $\gamma(\tau, y, \phi(y,\tau)) = \phi(y,\tau)$, we have the following integral equation

$$\int_0^t \int_\Omega F(t,\tau) K(x,y) \phi(y,\tau) dy d\tau = g(x,t), \tag{6.13}$$

under the condition

$$\int_{\Omega} \phi(x,t)dx = P(t). \tag{6.14}$$

Abdou and Salma, in [26], obtained many spectral relationships for the Volterra-Fredholm integral equation of the first kind, when the kernel takes different forms.

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