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NEW ALGORITHM FOR THE DETERMINATION OF AN UNKNOWN PARAMETER IN PARABOLIC EQUATIONS

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ABSTRACT. A new algorithm for the solution of an inverse problem of determining unknown source parameter in a parabolic equation in reproducing kernel space is considered. Numerical experiments are presented to demonstrate the accuracy and the efficiency of the proposed algorithm.

1. INTRODUCTION

Many physical phenomena can be described in terms of parabolic partial differential equations with source control parameters. These type of equations appear for example, in the study of heat condition processes, chemical diffusion, thermoelasticity and control theory [1-6]. In general, these problems are ill-posed. Therefore, a variety of numerical techniques based on regularization, finite difference, finite element and finite volume methods are given to approximate solutions of the equations [7, 8].

In this paper, we present a new algorithm for the following inverse source problem for a parabolic equation

$$(1.1)\qquad \begin{cases} w_t = w_{xx} + p(t)w(t,x) + f(t,x), & 0 \le x \le 1, \quad 0 \le t \le T, \\ w(0,x) = \varphi(x), & 0 \le x \le 1, \\ w(t,0) = w(t,1) = 0, & 0 \le t \le T, \\ w(t,x^*) = E(t), & 0 \le t \le T, \end{cases}$$

where $f(t,x), \varphi(x)$ and $E(t)$ are known functions and x^* is a fixed prescribed interior point in $(0,1)$, while the functions $w(t,x)$ and $p(t)$ are unknown. If $w(t,x)$

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is a temperature, then $Eq.(1.1)$can be regarded as a control problem finding the control function $p(t)$ such that the internal constraint is satisfied. The existence and uniqueness of the equations have been proved ([9]).

Employing a pair of transformations

$$r(t) = \exp\left(-\int_0^t p(s)ds\right), \tag{1.2}$$

$$u(t,x) = r(t)w(t,x) - \varphi(x). \tag{1.3}$$

$Eq.(1.1)$ will be

$$\begin{cases} u_t = u_{xx} + r(t)f(t,x) + \varphi''(x) & 0 \le x \le 1, \quad 0 \le t \le T, \\ u(0,x) = 0, & 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, & 0 \le t \le T, \\ u(t,x^*) = E(t)r(t) - \varphi(x^*), & 0 \le t \le T, \end{cases} \tag{1.4}$$

where $0 \le x \le 1, 0 \le t \le T$ and $u(t,x) \in W_{(2,3)(\Omega)}$.

Using our algorithm, the exact solution $u(t,x)$ can be given in the form of series. And n-term approximation $u_n(t,x)$ for analytic $u(t,x)$ is proved to converge to the exact solution in the large. Furthermore, by additional condition, we can solve source parameter easily. Moreover, the approximate error of $u_n(t,x)$ is monotone decreasing as n becomes larger. And this algorithm can easily be generalized to solve multidimensional problems.

To solve our problem, we give in Section 2 several reproducing spaces needed in this paper. In Section 3, we give the exact solution of $u(t,x)$ under the condition of $r(t)$ being known. Numerical algorithm and convergence results are given in Section 4. Finally, numerical examples are given in Section 5 to illustrate the accuracy of the proposed algorithm.

2. The Reproducing Kernel Spaces

In this section, several reproducing kernel spaces needed are introduced. Throughout this paper, we always consider problems on the domain $\Omega = [0,T] \times [0,1]$.

(1) The reproducing space $W_3[0,1]$

Inner product space $W_3[0,1]$ is defined as $W_3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real value functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. It is endowed with the inner product and the norm of the forms

$$(u(y), v(y))_{W_3[0,1]} = \int_0^1 (36uv + 49u'v' + 14u''v'' + u^{(3)}v^{(3)})dy, \quad u, v \in W_3[0,1] \tag{2.1}$$

$$\|u\|_{W_3[0,1]} = \sqrt{(u,u)_{W_3[0,1]}}.$$

Theorem 2.1. *The space $W_3[0,1]$ is a reproducing kernel space, that is, for any $u(y) \in W_3[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_3[0,1], y \in [0,1]$, such that $(u(y), R_x(y))_{W_3[0,1]} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$(2.2)\; R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y} + c_5e^{3y} + c_6e^{-3y}, & y \le x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y} + d_5e^{3y} + d_6e^{-3y}, & y > x. \end{cases}$$

Proof. Applying to the integrations by parts for (2.1), we have

$$\begin{aligned}(u(y), R_x(y))_{W_3[0,1]} &= \int_0^1 u(y)(36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y))dy \\ &+ u(y)(49R_x'(y) - 14R_x^{(3)}(y) + R_x^{(5)}(y))|_0^1 \\ &+ u'(y)(14R_x^{(2)}(y) - R_x^{(4)}(y))|_0^1 u''(y)R_x^{(3)}(y)|_0^1.\end{aligned} \tag{2.3}$$

Since $R_x(y) \in W_3[0,1]$, it follows that

$$R_x(0) = 0, \qquad R_x(1) = 0. \tag{2.4}$$

For $u \in W_3[0,1]$, thus, $u(0) = u(1) = 0$.

Suppose that $R_x(y)$ satisfies the following generalized differential equations:

$$\begin{cases} 36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y) = \delta(y-x), \\ 14R_x^{(2)}(0) - R_x^{(4)}(0) = 0, \\ 14R_x^{(2)}(1) - R_x^{(4)}(1) = 0, \\ R_x^{(3)}(0) = 0, \\ R_x^{(3)}(1) = 0. \end{cases} \tag{2.5}$$

Then

$$(u(y), R_x(y))_{W_3[0,1]} = \int_0^1 u(y)\delta(y-x)dy = u(x).$$

Hence, $R_x(y)$ is the reproducing kernel of Space $W_3[0,1]$.

In the following, we will get the expression of the reproducing kernel $R_x(y)$.

The characteristic equation of $36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y) = \delta(y-x)$ is given by

$$\lambda^6 - 14\lambda^4 + 49\lambda^2 - 36 = 0,$$

and the characteristic roots are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2, \lambda_4 = -2, \lambda_5 = 3$ and $\lambda_6 = -3$. We denote $R_x(y)$ by

$$R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y} + c_5e^{3y} + c_6e^{-3y}, & y \le x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y} + d_5e^{3y} + d_6e^{-3y}, & y > x. \end{cases}$$

By the definition of Space $W_3[0,1]$, (2.4) and (2.5), coefficients $c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, d_3, d_4, d_5, d_6$ satisfy

$$
(2.6) \qquad \begin{cases} R_x^{(k)}(x+0) = R_x^{(k)}(x-0), & k = 0,1,2,3,4 \\ R_x^{(5)}(x-0) - R_x^{(5)}(x+0) = 1, \\ 14R_x^{(2)}(0) - R_x^{(4)}(0) = 0, \\ 14R_x^{(2)}(1) - R_x^{(4)}(1) = 0, \\ R_x^{(3)}(0) = 0, \\ R_x^{(3)}(1) = 0, \\ R_x(0) = 0, \\ R_x(1) = 0, \end{cases}
$$

from which, the unknown coefficients of $Eq.$(2.2) can be obtained (detail in Appendix A). □

(2) The reproducing kernel space $W_2[0,T]$

Inner product space $W_2[0,T]$ is defined as $W_2[0,T] = \{u(x) \mid u, u'$ are absolutely continuous real value functions, $u, u', u'' \in L^2[0,T], u(0) = 0\}$. The inner product in $W_2[0,T]$is given by

$$
(u(y), v(y))_{W_2[0,T]} = \int_0^T (4uv + 5u'v' + u''v'')dy, \;\; u, v \in W_2[0,T]
$$

$$
\|u\|_{W_2[0,T]} = \sqrt{(u,u)}_{W_2[0,T]}.
$$

Similarly, $W_2[0,T]$is a reproducing kernel space and the corresponding reproducing kernel is

$$
(2.7) \qquad R_x^{\{2\}}(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y}, & y \le x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y}, & y > x, \end{cases}
$$

where

$$
\nabla = 6(-7 - 9e^{2T} + 9e^{4T} + 7e^{6T}),
$$

$$
c_1 = \frac{1}{\nabla}(e^{-2x}(4e^{3T} - 7e^{3x} - 9e^{2T+x} + 7e^{6T+x} + 9e^{4T+3x} - 4e^{3T+4x})),
$$

$$
c_2 = -c_1,
$$

$$
c_3 = \frac{1}{2\nabla}(e^{-2x}(-9e^{4T} - 7e^{6T} + 7e^{4x} - 8e^{3(T+x)} + 8e^{3T+x} + 9e^{2T+4x})),
$$

$$
c_4 = -c_3,
$$

$$
d_1 = \frac{1}{\nabla}(-(e^{-2x}(-1 + e^{2x})(4e^{3T} - 7e^{x} - 9e^{4T+x} + 4e^{3T+2x}))),
$$

$$
d_2 = \frac{1}{\nabla}(e^{2T-2x}(-1 + e^{2x})(4e^{T} - 9e^{x} + 7e^{4T+x} + 4e^{T+2x})),
$$

$$d_3 = \frac{1}{2\nabla}(e^{-2x}(-1+e^{2x})(7+9e^{2T}+7e^{2x}+9e^{2(T+x)}-8e^{3T+x})),$$
$$d_4 = \frac{1}{2\nabla}(-(e^{3T-2x}(-1+e^{2x})(9e^{T}+7e^{3T}-8e^{x}+9e^{T+2x}+7e^{3T+2x}))).$$

(3) The reproducing kernel space $W_1[0,1]$

The inner product space $W_1[0,1]$ is defined by $W_1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_1[0,1]$ are given respectively by

$$(u(x), v(x))_{W_1[0,1]} = \int_0^1 (uv + u'v')dx, \quad \| u \|_{W_1[0,1]} = \sqrt{(u,u)_{W_1[0,1]}},$$

where $u(x), v(x) \in W_1[0,1]$. In Ref[10], the author has proved that $W_1[0,1]$ is a reproducing kernel space and its reproducing kernel is

$$R_x^{\{1\}}(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh(|x-y|-1)].$$

(4) The reproducing kernel space $W_{(2,3)}(\Omega)$

$W_{(2,3)}(\Omega)$ is defined by

$$W_{(2,3)}(\Omega) = \left\{ \sum_{i,j=1}^{\infty} \alpha_{i,j} g_i(t) h_j(x) \middle| \sum_{i,j=1}^{\infty} |\alpha_{i,j}|^2 < \infty \right\} \tag{2.8}$$

where $g_i(t), h_j(x)$ are respectively the complete orthonormal systems of $W_2[0,T]$ and $W_3[0,1]$.

The inner product of $W_{(2,3)}(\Omega)$ is defined by

$$(u(t,x), v(t,x)) = \sum_{i,j=1}^{\infty} \alpha_{i,j}\beta_{i,j} \tag{2.9}$$

where $u(t,x) = \sum\limits_{i,j=1}^{\infty} \alpha_{i,j} g_i(t) h_j(x)$ and $v(t,x) = \sum\limits_{i,j=1}^{\infty} \beta_{i,j} g_i(t) h_j(x)$. The norm is denoted by $\|u\|_{W_{(2,3)}} = \sqrt{(u,u)_{W_{(2,3)}}}$.

By [11], it is easy to prove that the following properties hold.

Property 2.1. *If $u(t) \in W_2[0,T]$, $v(x) \in W_3[0,1]$, then $u(t)\cdot v(x) \in W_{(2,3)}(\Omega)$ hold.*

Property 2.2. *If $u(t,x) = u_1(t)\cdot u_2(x)$, $v(t,x) = v_1(t)\cdot v_2(x) \in W_{(2,3)}(\Omega)$, then*

$$(u(t,x), v(t,x))_{W_{(2,3)}} = (u_1(t), v_1(t))_{W_2}(u_2(x), v_2(x))_{W_3}.$$

Property 2.3. *Space $W_{(2,3)}(\Omega)$ is a reproducing space, and its reproducing kernel is $K_{(\xi,\eta)}(t,x) = R_\xi^{\{2\}}(t)R_\eta(x)$ where $R_\xi^{\{2\}}(t)$, $R_\eta(x)$ are given by (2.7) and (2.2) respectively.*

Similarly, we can define $W_{(1,1)}(\Omega)$ and$W_{(1,1)}(\Omega)$ is also a reproducing kernel space.

3. The Exact Solution of Eq.(1.4)

If $r(t)$ is known, then we can give the exact solution of *Eq.*(1.4). Note that $Lu = u_t - u_{xx}$ in *Eq.*(1.4), it is clear that $L : W_{(2,3)}(\Omega) \to W_{(1,1)}(\Omega)$ is bounded linear operator. Put $M = (t,x), M_i = (t_i, x_i), \varphi_i(M) = K_{M_i}(M)$ and $\psi_i(M) = L^*\varphi_i(M)$ where K is the reproducing kernel of $W_{(1,1)}(\Omega)$ and L^* is the conjugate operator of L. $\{\bar{\psi}_i(M)\}_{i=1}^{\infty}$ derives from Gram-Schmidt orthonormalization of $\{\psi_i(M)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(M) = \sum_{k=1}^{i} \beta_{ik}\psi_k(M), (\beta_{ii} > 0, i = 1, 2, \cdots). \tag{3.1}$$

Theorem 3.1. *For Eq.(1.4),if $\{M_i\}_{i=1}^{\infty}$ is dense on Ω, then $\{\psi_i(M)\}_{i=1}^{\infty}$ is the complete system of $W_{(2,3)}(\Omega)$.*

Proof. For each fixed $u(M) \in W_{(2,3)}(\Omega)$, let $(u(M), \psi_i(M)) = 0, \quad (i = 1, 2, \cdots)$, that is,

$$(u(M), (L^*\varphi_i)(M)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(M_i) = 0 \tag{3.2}$$

Note that $\{M_i\}_{i=1}^{\infty}$ is dense on Ω, therefore $(Lu)(M) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If $\{M_i\}_{i=1}^{\infty}$ is dense on Ω and the solution of Eq.(1.4) is unique, then the solution of Eq.(1.4) satisfies the form*

$$u(M) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(M_k, r(t_k))\bar{\psi}_i(M), \tag{3.3}$$

where $F(M, r(t)) = r(t)f(t,x) + \varphi''(x)$.

Proof. By Theorem 3.1, it is clear that $\{\bar{\psi}_i(M)\}_{i=1}^{\infty}$ is the complete orthonormal system of $W_{(2,3)}(\Omega)$. Note that $(v(M), \varphi_i(M)) = v(M_i)$ for each $v(M) \in W_{(1,1)}(\Omega)$, then

$$u(M) = \sum_{i=1}^{\infty}(u(M), \bar{\psi}_i(M))\bar{\psi}_i(M)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(M), L^*\varphi_k(M))\bar{\psi}_i(M) \tag{3.4}$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(F(M,r(t)),\varphi_k(M))\bar{\psi}_i(M)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(M_k,r(t_k))\bar{\psi}_i(M),$$

which proves the theorem. □

4. An Iteration Procedure

In this section, a new algorithm of obtaining the solution (3.3) is presented. (3.3) can be denoted by $u(M) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i(M)$, where $A_i = \sum_{k=1}^{i}\beta_{ik}F(M_k, r(t_k))$. Let $M_1 = (0,0)$, it follows that $r(0) = 1$, then $F(M_1, r(t_1))$ is known.Considering the numerical computation, we put $r_0(t_1) = r(t_1)$.The iteration algorithm is performed through the following two equations

$$u_n(M) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(M_k, r_{k-1}(t_k))\bar{\psi}_i(M), \tag{4.1}$$

$$r_n(t) = \frac{u_n(t,x^*) + \varphi(x^*)}{E(t)}. \tag{4.2}$$

Let

$$B_i = \sum_{k=1}^{i}\beta_{ik}F(M_k, r_{k-1}(t_k)), \tag{4.3}$$

then

$$u_n(M) = \sum_{i=1}^{n} B_i\bar{\psi}_i(M). \tag{4.4}$$

Due to (4.2), the convergence of $u_n(M)$ can lead to that of $r_n(t)$. So we only need to show the convergence of $u_n(M)$. Now, two lemmas are needed.

Lemma 4.1. *If $r(t) \in W_1[0,1]$, then there exists $M > 0$, such that $|r(t)| \le M\|r\|_{W_1[0,1]}$.*

It is easy to obtain from the definition of reproducing kernel and Schwarz inequality.

Lemma 4.2. *If $r_n(t) \overset{\|\cdot\|_{W_1}}{\longrightarrow} \bar{r}(t)$, $(n \to \infty)$, $M_n = (t_n, x_n) \to M = (t,x)$, $(n \to \infty)$, $\|r_n\|$ is bounded and $F(M, r(t))$ is continuous in M, then $F(M_n, r_{n-1}(t_n)) \to F(M, \bar{r}(t))$ as $n \to \infty$.*

Proof. From the given condition $r_n(t) \to \bar{r}(t)$, $(n \to \infty)$, and Lemma4.1, it follows that, for any $M \in \Omega$,$r_n(t)$ converges uniformly to $\bar{r}(t)$. So we can easily prove the conclusion of Lemma 4.2. □

Theorem 4.1. *Suppose that* $\| u_n \|$ *is bounded in* (4.1), *if* $\{M_i\}_{i=1}^{\infty}$ *is dense in* Ω, *then the n-term approximate solution* $u_n(M)$ *derived from the above method converges to the exact solution* $u(M)$ *of Eq.*(1.4) *and* $u(M) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(M)$, *where* B_i *is given by* (4.3).

Proof. (1) First, we will prove the convergence of $u_n(M)$. By (4.4), we infer that

$$u_{n+1}(M) = u_n(M) + B_{n+1}\overline{\psi}_{n+1}(M). \tag{4.5}$$

From the orthonormality of $\{\overline{\psi}_i(M)\}_{i=1}^{\infty}$and(4.1), it follows that

$$\| u_{n+1} \|^2 = \| u_n \|^2 + (B_{n+1})^2. \tag{4.6}$$

From (4.6), it holds that $\|u_{n+1}\| \geq \|u_n\|$. Due to the condition that $\|u_n\|$ is bounded, $\|u_n\|$ is convergent and there exists constant c such that

$$\sum_{i=1}^{\infty}(B_i)^2 = c$$

This implies that $B_i \in l^2, i = 1, 2, \cdots$ If $m > n$, then

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots u_{n+1} - u_n\|^2.$$

From $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1}\|^2 + \cdots + \|u_{n+1} - u_n\|^2.$$

Furthermore,

$$\|u_m - u_{m-1}\|^2 = (B_m)^2.$$

Consequently,

$$\|u_m - u_n\|^2 = \sum_{i=n+1}^{m} (B_i)^2 \to 0, (n \to \infty).$$

For(4.1), let $n \to \infty$, then

$$\bar{u}(M) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(M).$$

Due to the orthogonality of $\{\overline{\psi}_i(M)\}_{i=1}^{\infty}$,

$$(\bar{u}(M), \bar{\psi}_i(M)) = B_i.$$

So

$$\sum_{k=1}^{i} \beta_{ik}(L\bar{u})(M_k) = \sum_{k=1}^{i} \beta_{ik}(\bar{u}(M), (L^*\varphi_k)(M))$$
$$= \left(\bar{u}(M), \sum_{k=1}^{i} \beta_{ik}\psi_k(M)\right)$$
$$= (\bar{u}(M), \bar{\psi}_i(M)) = B_i.$$

If $i = 1$, then

$$(L\bar{u})M_1 = F(M_1, r_0(t_1)).$$

If $i = 2$, then

$$\beta_{21}(L\bar{u}(M_1) + \beta_{22}(L\bar{u}(M_2) = \beta_{21}F(M_1, r_0(t_1)) + \beta_{21}F(M_2, r_1(t_2)).$$

It is clear that

$$(L\bar{u}(M_2) = F(M_2, r_1(t_2)).$$

In the same way,

$$(L\bar{u}(M_j) = F(M_j, r_{j-1}(t_j)),\ j = 1, 2, \cdots \tag{4.7}$$

Since $\{M_i\}_{i=1}^{\infty}$ is dense in Ω, for $\forall Y = (t, x) \in \Omega$, there exists subsequence $\{M_{nj}\}_{j=1}^{\infty}$ such that

$$M_{nj} \to Y = (t, x),\ (j \to \infty).$$

For (4.7), let $j \to \infty$, by the convergence of u_n and Lemma 4.2, we have

$$(L\bar{u})(Y) = F(Y, r(t)) \tag{4.8}$$

That is, $\bar{u}(M)$ is the solution of *Eq.*(1.4) and

$$u(M) = \sum_{i=1}^{\infty} B_i\bar{\psi}_i(M), \tag{4.9}$$

then the prove is complete. □

From the proof of Theorem 4.1 and *Eq.*(4.8), we can see that for arbitrary initial value, the iteration sequence is convergent and the limiting function is the solution of *Eq.*(1.4). Furthermore, this immediately leads to the following conclusion.

Corollary. *The n-term approximate solution $u_n(M)$ in* (4.4) *can converge to exact solution in the large.*

Table 1

(t,x)	True solution $u(t,x)$	Approximate solution u_{81}	Absolute error	Relative error
(0.06,0.06)	0.198968	0.198968	5.57763E-07	2.80328E-06
(0.12,0.12)	0.415059	0.415057	2.43845E-06	5.87495E-06
(0.18,0.18)	0.641501	0.641495	6.4493E-06	1.00535E-05
(0.24,0.24)	0.87023	0.870219	1.07722E-05	1.23786E-05
(0.30,0.30)	1.09206	1.09205	1.32012E-05	1.20883E-05
(0.36.0.36)	1.29692	1.29091	8.76693E-06	6.75983E-06
(0.42,0.42)	1.47415	1.47415	2.37183E-06	1.60895E-06
(0.48,0.48)	1.61289	1.61292	3.44639E-05	2.13678E-05
(0.54,0.54)	1.70248	1.70254	6.76741E-05	3.97504E-05
(0.60,0.60)	1.73294	1.73308	1.41218E-04	8.14904E-05
(0.66,0.66)	1.69547	1.69566	1.90033E-04	1.12083E-04
(0.72.0.72)	1.58297	1.58326	2.93295E-04	1.85281E-04
(0.78,0.78)	1.39052	1.39085	3.23009E-04	2.32293E-04
(0.84,0.84)	1.11592	1.11627	3.49175E-04	3.12904E-04
(0.90,0.90)	0.760059	0.760378	3.19067E-04	4.19792E-04

Table 2

t	True solution $p(t)$	Approximate solution p_{81}	Absolute error	Relative error
0.06	1.0036	1.00321	3.86715E-04	3.85328E-04
0.12	1.0144	1.01284	1.56012E-03	1.53798E-03
0.18	1.0324	1.03006	2.34023E-03	2.26679E-03
0.24	1.0576	1.05373	3.87024E-03	3.65946E-03
0.30	1.09	1.08429	5.7076E-03	5.23633E-03
0.36	1.1296	1.12177	7.82698E-03	6.92898E-03
0.42	1.1764	1.16501	1.13855E-02	9.67827E-03
0.48	1.2304	1.21605	1.43484E-02	1.16616E-02
0.54	1.2916	1.27135	2.02475E-02	1.56763E-02
0.60	1.36	1.33567	2.4331E-02	1.78905E-02
0.66	1.4356	1.40241	3.3189E-02	2.31186E-02
0.72	1.5184	1.47951	3.88941E-02	2.56152E-02
0.78	1.6084	1.55938	4.90215E-02	3.04784E-02
0.84	1.7056	1.64771	5.78872E-02	3.39395E-02
0.90	1.81	1.73684	7.31555E-02	4.04174E-02

Theorem 4.2. *Assume $u(M)$ is the solution of Eq.(1.4) and $\gamma_n(M)$ is the approximate error of $u_n(M)$, where $u_n(M)$ is given by (4.4), then the error $\gamma_n(M)$ is monotone decreasing in the sense of $\|\cdot\|_{W_{(2,3)}(\Omega)}$.*

Proof. From (4.4), (4.9), it follows that

$$\|\gamma_n\|^2_{W_{(2,3)}} = \left\| \sum_{i=n+1}^{\infty} B_i \bar{\psi}_i(x) \right\|^2_{W_{(2,3)}} = \sum_{i=n+1}^{\infty} (B_i)^2,$$

Table 3

(t,x)	True solution $u(t,x)$	Approximate solution u_{144}	Absolute error	Relative error
(0.06,0.06)	0.198968	0.198968	2.88968E-07	1.45233E-06
(0.12,0.12)	0.415059	0.415058	1.0469E-06	2.52229E-06
(0.18,0.18)	0.641501	0.641499	2.48782E-06	3.87813E-06
(0.24,0.24)	0.87023	0.870226	4.34356E-06	4.99128E-06
(0.30,0.30)	1.09206	1.09205	4.64225E-06	4.25091E-06
(0.36.0.36)	1.29692	1.29691	1.85604E-06	1.43112E-06
(0.42,0.42)	1.47415	1.47415	3.63194E-06	2.46376E-06
(0.48,0.48)	1.61289	1.6129	1.30126E-05	8.0679E-06
(0.54,0.54)	1.70248	1.70251	3.14264E-05	1.84592E-05
(0.60,0.60)	1.73294	1.73299	5.45239E-05	3.14633E-05
(0.66,0.66)	1.69547	1.69554	7.32428E-05	4.31991E-05
(0.72.0.72)	1.58297	1.58305	8.44003E-05	5.33178E-05
(0.78,0.78)	1.39052	1.39061	9.10671E-05	6.54913E-05
(0.84,0.84)	1.11592	1.11597	5.27523E-05	4.72726E-05
(0.90,0.90)	0.760059	0.760046	1.339E-05	1.76171E-05

Table 4

t	True solution $p(t)$	Approximate solution p_{144}	Absolute error	Relative error
0.06	1.0036	1.00329	3.07732E-04	3.06629E-04
0.12	1.0144	1.01372	6.83165E-04	6.73467E-04
0.18	1.0324	1.03107	1.32526E-03	1.28367E-03
0.24	1.0576	1.05506	2.53737E-03	2.39918E-03
0.30	1.09	1.0868	3.19584E-03	2.93197E-03
0.36	1.1296	1.12532	4.27506E-03	3.78458E-03
0.42	1.1764	1.17032	6.08278E-03	5.17067E-03
0.48	1.2304	1.22143	8.96667E-03	7.28761E-03
0.54	1.2916	1.28076	1.0843E-02	8.39503E-03
0.60	1.36	1.34643	1.35727E-02	9.97992E-03
0.66	1.4356	1.41814	1.74614E-02	1.21631E-02
0.72	1.5184	1.49537	2.3028E-02	1.51659E-02
0.78	1.6084	1.58134	2.70618E-02	1.68253E-02
0.84	1.7056	1.6743	3.13025E-02	1.83528E-02
0.90	1.81	1.76897	4.10324E-02	2.26699E-02

which shows that the error γ_n is monotone decreasing in the sense of $\|\cdot\|_{W_{(2,3)}(\Omega)}$. □

5. Numerical Test

To illustrate the description above and to test iteration algorithm developed in this article for solving the one-dimensional diffusion with a control function, we include a numerical example for which the exact solution is known. Consider $Eq.(1.1)$

with

$$\varphi(x) = \sin(\pi x),$$
$$f(t,x) = e^t(\pi^2 - t^2)\sin(\pi x),$$
$$E(t) = e^t \sin(\pi x^*),$$

for which the exact solution is $w(t,x) = e^t \sin(\pi x)$, and $p(t) = 1 + t^2$. In the process of computation, all the symbolic and numerical computations performed by using Mathematica 5.0. Taking $T = 1.0$, $x^* = 0.4$ and choosing 81 and 144 points on $\Omega = [0,T] \times [0,1]$, respectively, then we obtain the approximate solution u_{81} and u_{144} on Ω. The numerical results are given in Table 1, 2 and 3, 4. Results obtained by the method have been compared with the exact solution of each example and are found to be in good agreement with each other.

6. APPENDIX

APPENDIX A. THE COEFFICIENTS OF THE REPRODUCING KERNEL $R_x(y)$

$$\Delta_1 = 48(-1+e)e^{3x}(57121 + 171363e + 287970e^2 + 409502e^3 + 283644e^4 + 283644e^5 + 409502e^6 + 287970e^7 + 171363e^8 + 57121e^9)$$

$$\Delta_2 = 60\,(-1+e)\,e^{3x}\,(57121 + 114242\,e + 173728\,e^2 + 235774\,e^3 + 47870\,e^4 + 235774\,e^5 + 173728\,e^6 + 114242\,e^7 + 57121\,e^8)$$

$$\Delta_3 = 5\Delta_1$$

$$c_1 = \frac{1}{\Delta_1}(-6318\,e^4 - 19548\,e^5 - 19548\,e^6 - 19548\,e^7 - 6318\,e^8 - 55926\,e^{4x} - 7648\,e^{5x} + 6453\,e^{6x} + 7488\,e^{3+x} + 30816\,e^{4+x} + 30816\,e^{5+x} + 57121\,e^{2\,(5+x)} + 30816\,e^{6+x} + 30816\,e^{7+x} + 7488\,e^{8+x} + 54756\,e^{2+2x} + 108232\,e^{3+2x} + 165353\,e^{4+2x} + 222474\,e^{5+2x} + 39495\,e^{6+2x} + 229764\,e^{7+2x} + 171363\,e^{8+2x} + 114242\,e^{9+2x} - 111852\,e^{1+4x} - 167778\,e^{2+4x} - 223704\,e^{3+4x} - 37080\,e^{4+4x} - 223704\,e^{5+4x} - 167778\,e^{6+4x} - 111852\,e^{7+4x} - 55926\,e^{8+4x} - 15296\,e^{1+5x} - 22944\,e^{2+5x} - 46272\,e^{3+5x} - 46272\,e^{4+5x} - 22944\,e^{5+5x} - 15296\,e^{6+5x} - 7648\,e^{7+5x} + 12906\,e^{1+6x} + 19359\,e^{2+6x} + 32724\,e^{3+6x} + 19359\,e^{4+6x} + 12906\,e^{5+6x} + 6453\,e^{6+6x})$$

$$c_2 = \frac{1}{\Delta_1}(6453\,e^4 + 12906\,e^5 + 19359\,e^6 + 32724\,e^7 + 19359\,e^8 + 12906\,e^9$$

$$+ 6453\,e^{10} + 57121\,e^{4x} - 7648\,e^{3+x} - 15296\,e^{4+x} - 22944\,e^{5+x} - 55926\,e^{2\,(5+x)}$$
$$- 46272\,e^{6+x} - 46272\,e^{7+x} - 22944\,e^{8+x} - 15296\,e^{9+x} - 7648\,e^{10+x}$$
$$- 55926\,e^{2+2\,x} - 111852\,e^{3+2\,x} - 167778\,e^{4+2\,x} - 223704\,e^{5+2\,x} - 37080\,e^{6+2\,x}$$
$$- 223704\,e^{7+2\,x} - 167778\,e^{8+2\,x} - 111852\,e^{9+2\,x} + 114242\,e^{1+4\,x}$$
$$+ 171363\,e^{2+4\,x} + 229764\,e^{3+4\,x} + 39495\,e^{4+4\,x} + 222474\,e^{5+4\,x} + 165353\,e^{6+4\,x}$$
$$+ 108232\,e^{7+4\,x} + 54756\,e^{8+4\,x} + 7488\,e^{2+5\,x} + 30816\,e^{3+5\,x} + 30816\,e^{4+5\,x}$$
$$+ 30816\,e^{5+5\,x} + 30816\,e^{6+5\,x} + 7488\,e^{7+5\,x} - 6318\,e^{2+6\,x} - 19548\,e^{3+6\,x}$$
$$- 19548\,e^{4+6\,x} - 19548\,e^{5+6\,x} - 6318\,e^{6+6\,x})$$

$$c_3 = -\frac{2}{\Delta_1}(1080\,e^4 - 105840\,e^5 + 1080\,e^6 + 9560\,e^{4\,x} - 118305\,e^{5\,x} + 51624\,e^{6\,x}$$
$$- 1280\,e^{3+x} + 243745\,e^{4+x} + 55841\,e^{5+x} + 60766\,e^{6+x} + 59486\,e^{7+x}$$
$$+ 57121\,e^{8+x} + 57121\,e^{9+x} - 9360\,e^{2+2\,x} - 29160\,e^{3+2\,x} - 9360\,e^{4+2\,x}$$
$$- 29160\,e^{5+2\,x} - 9360\,e^{6+2\,x} + 9560\,e^{1+4\,x} + 19120\,e^{2+4\,x} + 38720\,e^{3+4\,x}$$
$$+ 19120\,e^{4+4\,x} + 9560\,e^{5+4\,x} + 9560\,e^{6+4\,x}$$
$$- 118305\,e^{1+5\,x} - 121950\,e^{2+5\,x} - 121950\,e^{3+5\,x} - 118305\,e^{4+5\,x}$$
$$- 118305\,e^{5+5\,x} + 51624\,e^{1+6\,x} + 52704\,e^{2+6\,x} + 51624\,e^{3+6\,x} + 51624\,e^{4+6\,x})$$

$$c_4 = -\frac{1}{\Delta_2}(51624\,e^5 + 51624\,e^6 + 52704\,e^7 + 51624\,e^8 + 51624\,e^9 + 57121\,e^{5\,x}$$
$$- 118305\,e^{4+x} - 118305\,e^{5+x} - 121950\,e^{6+x} - 121950\,e^{7+x} - 118305\,e^{8+x}$$
$$- 118305\,e^{9+x} + 9560\,e^{3+2\,x} + 9560\,e^{4+2\,x} + 19120\,e^{5+2\,x} + 38720\,e^{6+2\,x}$$
$$+ 19120\,e^{7+2\,x} + 9560\,e^{8+2\,x} + 9560\,e^{9+2\,x} - 9360\,e^{3+4\,x} - 29160\,e^{4+4\,x}$$
$$- 9360\,e^{5+4\,x} - 29160\,e^{6+4\,x} - 9360\,e^{7+4\,x} + 57121\,e^{1+5\,x} + 59486\,e^{2+5\,x}$$
$$+ 60766\,e^{3+5\,x} + 55841\,e^{4+5\,x} + 243745\,e^{5+5\,x} - 1280\,e^{6+5\,x} + 1080\,e^{3+6\,x}$$
$$- 105840\,e^{4+6\,x} + 1080\,e^{5+6\,x})$$

$$c_5 = \frac{1}{\Delta_3}e^{-3x}(3645e^4 - 179334e^5 - 122213e^6 + 121532e^7 + 116607e^8$$
$$+ 114242e^9 + 57121e^{10} + 32265e^{4x} - 206496e^{5x} + 117110e^{6x}$$
$$- 4320e^{3+x} + 419040e^{5+x} - 4320e^{6+x} - 31590e^{2+2x} - 97740e^{3+2x}$$
$$- 97740e^{4+2x} - 97740e^{5+2x} - 31590e^{6+2x} + 64530e^{1+4x} + 96795e^{2+4x}$$
$$+ 163620e^{3+4x} + 96795e^{4+4x} + 64530e^{5+4x} + 32265e^{6+4x} - 412992e^{1+5x}$$
$$- 417312e^{2+5x} - 417312e^{3+5x} - 412992e^{4+5x} - 206496e^{5+5x}$$

$$+ 234220e^{1+6x} + 235500e^{2+6x} + 234220e^{3+6x} + 117110e^{4+6x})$$

$$\begin{aligned} c_6 = \frac{1}{\Delta_3} e^{-3x} (& 117110e^6 + 234220e^7 + 235550e^8 + 234220e^9 + 117110e^{10} \\ & + 57121e^{6x} - 206496e^{5+x} + 32265e^{10+2x} - 412992e^{6+x} - 417312e^{8+x} \\ & - 412992e^{9+x} - 206496e^{10+x} + 32265e^{4+2x} + 64530e^{5+2x} + 96795e^{6+2x} \\ & + 163620e^{7+2x} + 96795e^{8+2x} + 64530e^{9+2x} - 31590e^{4+4x} - 97740e^{7+4x} \\ & - 31590e^{8+4x} - 4320e^{4+5x} + 419040e^{5+5x} + 419040e^{6+5x} - 4320e^{7+5x} \\ & + 114242e^{1+6x} + 116607e^{2+6x} + 121532e^{3+6x} - 122213e^{4+6x} \\ & - 179334e^{5+6x} + 3645e^{6+6x}) \end{aligned}$$

$$\begin{aligned} d_1 = \frac{1}{\Delta_1} (& -6318\, e^4 - 19548\, e^5 - 19548\, e^6 - 19548\, e^7 - 6318\, e^8 + 57121\, e^{2x} \\ & - 55926\, e^{4x} - 7648\, e^{5x} + 6453\, e^{6x} + 7488\, e^{3+x} + 30816\, e^{4+x} \\ & + 30816\, e^{5+x} + 30816\, e^{6+x} + 30816\, e^{7+x} + 7488\, e^{8+x} + 114242\, e^{1+2x} \\ & + 171363\, e^{2+2x} + 229764\, e^{3+2x} + 39495\, e^{4+2x} + 222474\, e^{5+2x} \\ & + 165353\, e^{6+2x} + 108232\, e^{7+2x} + 54756\, e^{8+2x} - 111852\, e^{1+4x} \\ & - 167778\, e^{2+4x} - 223704\, e^{3+4x} - 37080\, e^{4+4x} - 223704\, e^{5+4x} \\ & - 167778\, e^{6+4x} - 111852\, e^{7+4x} - 55926\, e^{8+4x} - 15296\, e^{1+5x} \\ & - 22944\, e^{2+5x} - 46272\, e^{3+5x} - 46272\, e^{4+5x} - 22944\, e^{5+5x} \\ & - 15296\, e^{6+5x} - 7648\, e^{7+5x} + 12906\, e^{1+6x} + 19359\, e^{2+6x} \\ & + 32724\, e^{3+6x} + 19359\, e^{4+6x} + 12906\, e^{5+6x} + 6453\, e^{6+6x}) \end{aligned}$$

$$\begin{aligned} d_2 = \frac{1}{\Delta_1} e^{2-3x} (& 6453\, e^2 + 12906\, e^3 + 19359\, e^4 + 32724\, e^5 + 19359\, e^6 \\ & + 12906\, e^7 + 6453\, e^8 - 55926\, e^{2x} + 54756\, e^{4x} + 7488\, e^{5x} - 6318\, e^{6x} \\ & - 7648\, e^{1+x} - 15296\, e^{2+x} - 22944\, e^{3+x} - 46272\, e^{4+x} - 46272\, e^{5+x} \\ & - 22944\, e^{6+x} - 15296\, e^{7+x} - 7648\, e^{8+x} - 111852\, e^{1+2x} \\ & - 167778\, e^{2+2x} - 223704\, e^{3+2x} - 37080\, e^{4+2x} - 223704\, e^{5+2x} \\ & - 167778\, e^{6+2x} - 111852\, e^{7+2x} - 55926\, e^{8+2x} + 108232\, e^{1+4x} \\ & + 165353\, e^{2+4x} + 222474\, e^{3+4x} + 39495\, e^{4+4x} + 229764\, e^{5+4x} \\ & + 171363\, e^{6+4x} + 114242\, e^{7+4x} + 57121\, e^{8+4x} + 30816\, e^{1+5x} \\ & + 30816\, e^{2+5x} + 30816\, e^{3+5x} + 30816\, e^{4+5x} + 7488\, e^{5+5x} \\ & - 19548\, e^{1+6x} - 19548\, e^{2+6x} - 19548\, e^{3+6x} - 6318\, e^{4+6x}) \end{aligned}$$

$$
\begin{aligned}
d_3 = -\frac{1}{\Delta_2}(&1080\,e^4 - 105840\,e^5 + 1080\,e^6 + 57121\,e^x + 9560\,e^{4x} \\
&- 118305\,e^{5x} + 51624\,e^{6x} + 57121\,e^{1+x} + 59486\,e^{2+x} + 60766\,e^{3+x} \\
&+ 55841\,e^{4+x} + 243745\,e^{5+x} - 1280\,e^{6+x} - 9360\,e^{2+2x} - 29160\,e^{3+2x} \\
&- 9360\,e^{4+2x} - 29160\,e^{5+2x} - 9360\,e^{6+2x} + 9560\,e^{1+4x} \\
&+ 19120\,e^{2+4x} + 38720\,e^{3+4x} + 19120\,e^{4+4x} + 9560\,e^{5+4x} \\
&+ 9560\,e^{6+4x} - 118305\,e^{1+5x} - 121950\,e^{2+5x} - 121950\,e^{3+5x} \\
&- 118305\,e^{4+5x} - 118305\,e^{5+5x} + 51624\,e^{1+6x} + 52704\,e^{2+6x} \\
&+ 51624\,e^{3+6x} + 51624\,e^{4+6x})
\end{aligned}
$$

$$
\begin{aligned}
d_4 = -\frac{1}{\Delta_2}e^{3-3x}(&51624\,e^2 + 51624\,e^3 + 52704\,e^4 + 51624\,e^5 + 51624\,e^6 + 9560\,e^{2x} \\
&- 9360\,e^{4x} - 1280\,e^{5x} + 1080\,e^{6x} - 118305\,e^{1+x} - 118305\,e^{2+x} - 121950\,e^{3+x} \\
&- 121950\,e^{4+x} - 118305\,e^{5+x} - 118305\,e^{6+x} + 9560\,e^{1+2x} + 19120\,e^{2+2x} \\
&+ 38720\,e^{3+2x} + 19120\,e^{4+2x} + 9560\,e^{5+2x} + 9560\,e^{6+2x} - 29160\,e^{1+4x} \\
&- 9360\,e^{2+4x} - 29160\,e^{3+4x} - 9360\,e^{4+4x} + 243745\,e^{1+5x} + 55841\,e^{2+5x} \\
&+ 60766\,e^{3+5x} + 59486\,e^{4+5x} + 57121\,e^{5+5x} + 57121\,e^{6+5x} - 105840\,e^{1+6x} \\
&+ 1080\,e^{2+6x})
\end{aligned}
$$

$$
\begin{aligned}
d_5 = \frac{1}{\Delta_3}e^{-3x}(&57121 + 114242e + 116607e^2 + 121532e^3 - 122213e^4 - 179334e^5 \\
&+ 3645e^6 + 32265e^{4x} - 206496e^{5x} + 117110e^{6x} - 4320e^{3+x} + 419040e^{4+x} \\
&+ 419040e^{5+x} - 4320e^{6+x} - 31590e^{2+2x} - 97740e^{3+2x} - 97740e^{4+2x} \\
&- 97740e^{5+4x} - 31590e^{6+2x} + 64530e^{1+4x} + 96795e^{2+4x} + 163620e^{3+4x} \\
&+ 96795e^{4+4x} + 64530e^{5+4x} + 32265e^{6+4x} - 412992e^{1+5x} - 417312e^{2+5x} \\
&- 417312e^{3+5x} - 412992e^{4+5x} - 206496e^{5+5x} + 234220e^{1+6x} + 235550e^{2+6x} \\
&+ 234220e^{3+6x} + 117110e^{4+6x})
\end{aligned}
$$

$$
\begin{aligned}
d_6 = \frac{1}{\Delta_3}e^{4-3x}(&117110e^2 + 234220e^3 + 235500e^4 + 234220e^5 + 117110e^6 + 32265e^{2x} \\
&- 31590e^{4x} - 4320e^{5x} + 3645e^{6x} - 206496e^{1+x} - 412992e^{2+x} - 417312e^{3+x} \\
&- 417312e^{4+x} - 412992e^{5+x} - 206496e^{6+x} + 64530e^{1+2x} + 96795e^{2+2x} \\
&+ 163620e^{3+2x} + 96795e^{4+2x} + 64530e^{5+2x} + 32265e^{6+2x} - 97740e^{1+4x} \\
&- 97740e^{2+4x} - 97740e^{3+4x} - 31590e^{4+4x} + 419040e^{1+5x} + 419040e^{2+5x} \\
&- 4320e^{3+5x} - 179334e^{1+6x} - 122213e^{2+6x} + 121532e^{3+6x} + 116607e^{4+6x}
\end{aligned}
$$

$$+ 114242e^{5+6x} + 57121e^{6+6x})$$

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