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CONCERNING THE RADII OF CONVERGENCE FOR A CERTAIN CLASS OF NEWTON-LIKE METHODS

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ABSTRACT. Local convergence results for three Newton-like methods in Banach space are provided. A comparison is given between the three convergence radii. Then we show that using the largest convergence radius we can pick an initial guess from with we start the corresponding iteration. It turns out that after a finite number of steps we can always use the iterate found as the starting guess for a faster method, since this iterate will be inside the convergence domain of the new method.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where, F is a Fréchet differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. We approximate x^* by generating sequences given by:

Newton-Like method

$$y_{n+1} = y_n - A(y_n)^{-1}F(y_n) \quad (y_0 \in D), \quad (n \geq 0), \tag{2}$$

and Modified Newton-like methods

$$x_{n+1} = x_n - A(x^*)^{-1}F(x_n) \quad (x_0 \in D)\ , (n \geq 0), \tag{3}$$

and

$$z_{n+1} = z_n - A(z_0)^{-1}F(z_n) \quad (z_0 \in D)), \ \ (n \geq 0). \tag{4}$$

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Here $A(x) \in L(X,Y)$ $(x \in D)$ (the space of bounded linear operators from X into Y) is an approximation to the Fréchet-derivative $F'(x)$ of operator F [2], [6]. A survey on local as well as semilocal convergence theorems for Newton-like methods can be found in [1]–[9] and the references there. Here we compare the radii of convergence between method (2)–(4). It turns out that the radius of convergence r_M of method (3) is in general at least as large as the radii r_N and r_{MN} of corresponding methods (2) and (4). Therefore we can use slower (in general) than (2) method (3) and after a finite number of steps N continue with faster method (2) by starting from

$$y_0 = x_N.$$

This may we take advantage of the wider range of starting choices provided by method (3) (or even (4) than the ones given by faster method (2). A numerical example is also provided.

2. Local Convergence Analysis

We can show the main local convergence theorem for method (2) by using Yamanato-type hypotheses [8], [2]:

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator and $A(x) \in L(X,Y)$ be an approximation to $F'(x)$. Assume there exists a solution $x^* \in D$ of equation $F(x) = 0$, and parameters $K \geq 0$, $M \geq 0$, $L \geq 0$, $\mu \in [0,1)$ and $l \in [0,1)$ such that for all $x, y \in D$:*

$$A(x^*)^{-1} \in L(Y,X), \tag{5}$$

$$\left\| A(x^*)^{-1}[F'(x) - F'(y)] \right\| \leq K \|x - y\| \tag{6}$$

$$\left\| A(x^*)^{-1}[A(x) - F'(x)] \right\| \leq M \|x - x\| + \mu, \tag{7}$$

$$\left\| A(x^*)^{-1}[A(x) - A(x^*)] \right\| \leq L \|x - x^*\| + l, \tag{8}$$

$$0 \leq \mu + l < 1, \tag{9}$$

$$L r_N + l < 1, \tag{10}$$

and

$$\bar{U}(x^*, r_N) = \{x \in X \mid \|x - x^*\| \leq r_N\} \subset D, \tag{11}$$

where,

$$r_N = \frac{2(1-l-\mu)}{K+2(M+L)}, \qquad K+2M+L \neq 0 \tag{12}$$

Then sequence $\{y_n\}$ $(n \geq 0)$ generated by Newton-like method (2) is well defined, remains in $\bar{U}$ (x^*, r_N) for all $n \geq 0$ and converges to x^* provided that $y_0 \in U(x^*, r_N)$.

Moreover the following estimates hold for all $n \geq 0$

$$\|y_{n+1} - x^*\| \leq \frac{\bar{a}_n}{\bar{b}_n} \|y_n - x^*\| \leq \frac{a}{b} \|y_n - x^*\|, \tag{13}$$

where,

$$\bar{a}_n = \frac{K}{2} \|y_n - x^*\| + M \|y_n - x^*\| + \mu \tag{14}$$

$$a = \left(\frac{K}{2} + M\right) r_N + \mu, \tag{15}$$

$$\bar{b}_n = 1 - l - L \|y_n - x^*\|, \tag{16}$$

and

$$b = 1 - l - L r_N. \tag{17}$$

Proof. By hypothesis $y_0 \in U(x^*, r_N)$. Let $x \in U(x^*, r_N)$. Using (8) and (10) we get

$$\left\|A(x^*)^{-1}[A(x) - A(x^*)]\right\| \leq L \|x - x^*\| + l < Lr^* + l < 1. \tag{18}$$

It follows by the Banach Lemma on invertible operators [6] and (18) that $A(x)^{-1} \in L(Y, X)$ and

$$\left\|A(x)^{-1}A(x^*)\right\| \leq \frac{1}{1 - l - L\|x - x^*\|} \leq \frac{1}{1 - l - Lr_N} \tag{19}$$

Let us assume $y_k \in U(x^*, r_N), k = 0, 1, ..., n$. It follows y_{n+1} is well defined. In view of (2) we obtain the approximation

(20)

$$\begin{aligned} y_{n+1} - x^* &= A(y_n)^{-1} \int_0^1 [F'(x^* + t(y_n - x^*)) - A(y_n)](y_n - x^*)dt \\ &= [A(y_n)^{-1}A(x^*)]\Big\{A(x^*)^{-1} \int_0^1 [F'(x^* + t(y_n - x^*)) - F'(y_n)](y_n - x^*)dt \\ &\quad + A(x^*)^{-1}[F'(y_n) - A(y_n)](y_n - x^*)\Big\} \end{aligned}$$

Using (6), (7), (12), (19) for $x = y_k$ (14)–(17) and (20) we obtain (13), and

$$\|y_{n+1} - y^*\| < \|y_n - y^*\| < r_N \tag{21}$$

from which it follows

$$y_{n+1} \in U(x^*, r_N) \quad \text{and} \quad \lim_{n\to\infty} y_n = x^*.$$

That completes the proof of the theorem. □

Let us introduce a condition weaker than (6):

(22) $\|A(x^*)^{-1}[F'(x) - F'(x^*)]\| \le K_0 \|x - x^*\|$, for all $x \in D$ and some $K_0 \ge 0$.

Clearly,

$$K_0 \le K \tag{23}$$

holds in general and $\frac{K}{K_0}$ can be arbitrarily large [1], [2]. Then using the approximation

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - A(x^*)^{-1}F(x_n) \\ &= -A(x^*)^{-1} \int_0^1 \left[F'x^* + t(x_n - x^*)) - F'(x^*)\right](x_n - x^*)dt \\ &\quad + \left[F'(x^*) - A(x^*)\right](x_n - x^*) \end{aligned} \tag{24}$$

instead of (20) exactly as in Theorem 2.1 we can show the corresponding local convergence theorem for modified method (3):

Theorem 2.2. *Let D, F, x^* be as in Theorem 2.1. Assume together with (22) that*

$$\|A(x^*)^{-1}[A(x^*) - F'(x^*)]\| \le \mu < 1 \tag{25}$$

holds true.

Moreover, assume:

$$\bar{U}(x^*, r_M) \subseteq D, \tag{26a}$$

where,

$$r_M = \frac{2(1-\mu)}{K_0}, \quad K_0 \neq 0. \tag{27}$$

Then sequence $\{x_n\}$ generated by Modified Newton-like method (3) remains in $\bar{U}(x^*, r_M)$ and converges to x^* provided that $x_0 \in U(x^*, r_M)$. Moreover the following estimates are true for all $n \ge 0$:

$$\|x_{n+1} - x^*\| \le \alpha_n \|x_n - x^*\| \le \alpha \|x_n - x\|, \tag{28}$$

where,

$$\alpha_n = \frac{K_0}{2}\|x_n - x^*\| + \mu, \tag{29}$$

and

$$\alpha = \frac{K_0}{2} r_M + \mu. \tag{30}$$

Let us introduce conditions

$$\left\| A(x^*)^{-1}[F'(x) - F'(z_0)] \right\| \le K_1 \|x - z_0\|, \quad K_1 \ge 0, \tag{31}$$

$$\left\| A(x^*)^{-1}[A(z_0) - F'(z_0)] \right\| \le M_0 \|z_0 - x^*\| + \mu, \quad \mu_0 \ge 0, M_0 \ge 0, \tag{32}$$

and

$$\left\| A(x^*)^{-1}[A(z_0) - A(x^*)] \right\| \le L_0 \|z_0 - x^*\| + l_0, \tag{33}$$

for all $z_0, x \in D$. Then using the approximation

(34)

$$z_{n+1} - x^* = [A(z_0)^{-1} A(x^*)] A(x^*)^{-1} \left\{ \int_0^1 [F'(x^* + t(z_n - x^*)) - F'(z_0](z_n - x^*) dt + [F'(z_0) - A(z_0)](z_n - x^*) \right\}$$

obtained via (4) instead of (20) exactly as in Theorem 2.1 we can show the corresponding local convergence theorem for method (4):

Theorem 2.3. *Let D, F, x^*be as in Theorem 2.1. Moreover, assume:*

$$\begin{aligned} L_0 r_{MN} + l_0 &< 1, \\ 0 \le l_0 + \mu_0 &< 1, \end{aligned} \tag{35}$$

and

$$\bar{U}(x^*, r_{MN}) \subseteq D, \tag{36}$$

where,

$$r_{MN} = \frac{2(1 - l_0 - \mu_0)}{K_1 + 2(M_0 + L_0)}. \tag{37}$$

Then sequence $\{z_n\}$ remains in $\bar{U}(x^*, r_{MN})$ for all $n \ge 0$ and converges to x^* provided that $z_0 \in \bar{U}(x^*, r_{MN})$. Moreover the following estimates hold for all $n \ge 0$

$$\|z_{n+1} - x^*\| \le \frac{\bar{\beta}_n}{\bar{\gamma}} \|z_n - x^*\| \le \frac{\beta}{\gamma} \|z_n - x^*\|, \tag{38}$$

where,

$$\bar{\beta}_n = \int_0^1 K_1 \|x^* + t(z_n - x^*) - z_0\| + M_0 \|x^* - z_0\| + \mu_0, \tag{39}$$

(40) $$\bar{\gamma} = 1 - l_0 - L_0 \|x^* - z_0\|,$$

(41) $$\beta = (K_1 + M_0) r_{MN} + \mu_0,$$

and

(42) $$\gamma = 1 - l_0 - L_0 r_{MN}.$$

Clearly for a fixed $z_0 \in D$

(43) $$K_1 \le K,$$

(44) $$M_0 \le M,$$

(45) $$\mu_0 \le \mu,$$

(46) $$L_0 \le L,$$

and

(47) $$l_0 \le l$$

hold in general.

By comparing (12), (27) and (37) we deduce

(48) $$r_N \le r_M$$

and

(49) $$r_N \le r_{MN}.$$

Moreover if

(50) $$\frac{1 - l_0 - \mu_0}{K_1 + 2(M_0 + L_0)} \le \frac{1 - \mu}{K_0},$$

then we conclude

(51) $$r_{MN} \le r_M.$$

In the special case when $A(x) = F'(x) (x \in D)$ (Newton's method) (12),(27) and (37) for $M = \mu = l = M_0 = \mu_0 = l_0 = 0$ give:

(52) $$r_N = \frac{2}{K + 2L},$$

(53) $$r_M = \frac{2}{K_0},$$

and

$$r_{MN} = \frac{2}{K_1 + 2L_0}. \tag{54}$$

Let us provide a numerical example for Newton's method:

Example 2.4. Let $X = Y = \mathbf{R}$, $D = U(0, 1)$ and define function F on D by

$$F(x) = e^x - 1. \tag{55}$$

Using (42) and (43) we obtain $K = e$, $L = e - 1, K_0 = e - 1$,

$$r_N = \frac{2}{3e-2} = .324947231,$$

and

$$r_M = \frac{2}{e-1} = 1.163953414.$$

We must set $\bar{r}_m = 1$, so that $U(0, \bar{r}_n) \subseteq D$. Choose $z_0 - \frac{1}{2}$. Then in view of (31) and (33) we obtain

$$K_1 = e \quad \text{and} \quad L_0 = e - 1.$$

That is, by (44) we get

$$r_{MN} = r_N = .324947231.$$

Note that the radius of convergence due to Rheinbold [7] is given by

$$r_R = \frac{2}{3K}.$$

In this case we get

$$r_R = \frac{2}{3e} = .24525296.$$

That is, the radius of convergence given by (43) is the largest.

Note that in general method (2) is the fastest. In practice we can choose an initial gives in the largest ball $U(x^*, r_M)$. After a finite number of steps N (due to the convergence of $\{x_n\}$ to x^*) $x_N \in \bar{U}(x^*, r_N)$. Therefore we can set $y_0 = x_N$ and continue approximating x^* with faster method 2). More precisely let $\varepsilon \in (0, \alpha]$ and $x_0 \in \bar{U}\ (x, r_M)$. We must have

$$\varepsilon^N r_M \leq r_N \tag{56}$$

or

$$N \geq \frac{\ln\left(\frac{r_N}{r_M}\right)}{\ln \varepsilon}. \tag{57}$$

Therefore we can set

$$N = \left[\frac{\ln \frac{r_N}{r_M}}{\ln \varepsilon}\right] + 1, \tag{58}$$

where, $[s]$ denotes the integer part of real number s.

Remark 2.5. As noted in [1], [2], [4], [9] the local results obtained here can be used for projection methods such as Arnoldi's, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR), for combined Newton/finite projection methods and in connection with the mesh independence principle to develop the cheapest and most efficient mesh independence strategies.

Remark 2.6. The local results can also be used to solve equations of the form $F(x) = 0$ (when say $A(x) = F'(x)$) where F' satisfies the autonomous differential equation [2], [6]

$$F'(x) = T(F(x^*)), \tag{59}$$

where $T : Y \to X$ is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply our results without actually knowing the solution x^* of equation $F(x) = 0$. Note that in the case of 2.4 we can set $T(x) = x + 1$, so that condition (59) is satisfied.

Finally we note that although Yamamoto-type [8] conditions were used in this study the ideas/results can be extended under more general conditions introduced by us in [1], [2]. However we leave the details to the motivated reader.

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