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INTUITIONISTIC FUZZY COMMUTATIVE IDEALS OF BCK-ALGEBRAS

YOUNG BAE JUN [a], DONG SOO LEE [b,*] AND CHUL HWAN PARK [c]

ABSTRACT. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Aranassov is one among them. In this paper, we apply the concept of an intuitionistic fuzzy set to commutative ideals in BCK-algebras. The notion of an intuitionistic fuzzy commutative ideal of a BCK-algebra is introduced, and some related properties are investigated. Characterizations of an intuitionistic fuzzy commutative ideal are given. Conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy commutative ideal are given. Using a collection of commutative ideals, intuitionistic fuzzy commutative ideals are established.

1. INTRODUCTION

Logic appears in a 'sacred' form (resp. a 'profane' form) which is dominant in proof theory (resp. model theory). The role of logic in mathematics and computer science is two-fold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc, takes the advantage of the classical logic to handle information with various facets of uncertainty (see [12] for a generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Many-valued logic which is a great extension and development of classical logic [4] has emerged as a useful direction in non-classic logic. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. Fuzzy sets, which were introduced by Zadeh [11], deal with possibilistic uncertainty,

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connected with imprecision of states, perceptions and preferences. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] is one among them. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. For more details on intuitionistic fuzzy sets, we refer the reader to [1, 2]. Since then, a great number of theoretical and practical results appeared in the area of IFSs. There are numerical applications of IFSs in various areas of computer science, for example, in artificial intelligence, as well as in medicine, chemistry, economics, astronomy, etc. As applications of intuitionistic fuzzy sets, Davvaz et al. [5] applied the concept of an intuitionistic fuzzy set to H_v-modules. They introduced the notion of an intuitionistic fuzzy H_v-submodule of an H_v-module, and investigated related properties. They provided characterizations of intuitionistic fuzzy H_v-submodules. Jun et al. [7] discussed the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related results. In this paper, we apply the concept of an intuitionistic fuzzy set to commutative ideals in BCK-algebras. We introduce the notion of an intuitionistic fuzzy commutative ideal of a BCK-algebra, and investigate some related properties. We give characterizations of an intuitionistic fuzzy commutative ideal. We also give conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy commutative ideal. Using a collection of commutative ideals, we establish intuitionistic fuzzy commutative ideals.

2. Basic Results on BCK-algebras

By a *BCK-algebra* we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)$ $(x * x = 0, 0 * x = 0)$,

(a4) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *commutative* if it satisfies the equality:

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x)). \tag{2.1}$$

In any BCK-algebra X, the following hold:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b4) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

A nonempty subset I of a BCK-algebra X is called an *ideal* of X if it satisfies

(c1) $0 \in I$,
(c2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

A nonempty subset I of a BCK-algebra X is called a *commutative ideal* of X (see [9]) if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)\ ((x * y) * z \in I, z \in I \Rightarrow x * (y * (y * x)) \in I)$.

Note that every commutative ideal is an ideal, but the converse is not valid in general. Note also that an ideal I is commutative if and only if it satisfies the following implication:

$$(\forall x, y \in X)\,(x * y \in I \Rightarrow x * (y * (y * x)) \in I). \tag{2.2}$$

We refer the reader to the book [10] for details of BCK-algebras.

3. Basic Results on (Intuitionistic) Fuzzy Sets

A mapping $\mu : X \to [0, 1]$, where X is an arbitrary nonempty set, is called a *fuzzy set* in X. For any fuzzy set μ in X and any $t \in [0, 1]$ we define two sets

$$U(\mu; t) = \{x \in X \mid \mu(x) \geq t\} \quad \text{and} \quad L(\mu; t) = \{x \in X \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

A fuzzy set μ in a BCK-algebra X is called a *fuzzy commutative ideal* of X if it satisfies

(1) $(\forall x \in X)\ (\mu(0) \geq \mu(x))$.
(2) $(\forall x, y, z \in X)\ (\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\})$.

An *intuitionistic fuzzy set* (IFS for short) A in X (see [1]) is an object having the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\} \tag{3.1}$$

where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$ denote the degree of membership (namely $\alpha_A(x)$) and the degree of nonmembership (namely $\beta_A(x)$) of each element $x \in X$ to the set A, respectively, and

$$0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \tag{3.2}$$

for each $x \in X$. For the sake of simplicity, we shall use the symbol $A = \langle X, \alpha_A, \beta_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\}$. Obviously, every fuzzy set A' corresponds to the following intuitionistic fuzzy set:

$$A' = \{\langle x, \alpha_{A'}(x), 1 - \alpha_{A'}(x)\rangle \mid x \in X\}. \tag{3.3}$$

Obviously, for an IFS $A = \langle X, \alpha_A, \beta_A\rangle$ in X, when

$$\beta_A(x) = 1 - \alpha_A(x), \quad \text{that is,} \quad \alpha_A(x) + \beta_A(x) = 1 \tag{3.4}$$

for every $x \in X$, the IFS A is a fuzzy set. Hence the notion of intuitionistic fuzzy set theory is a generalization of fuzzy set theory. Let $A = \langle X, \alpha_A, \beta_A\rangle$ be an IFS in X and let $m, n \in [0,1]$ be such that $m + n \leq 1$. Then the set

$$X_A^{(m,n)} := \{x \in X \mid \alpha_A(x) \geq m,\ \beta_A(x) \leq n\}$$

is called an (m,n)*-level subset* of $A = \langle X, \alpha_A, \beta_A\rangle$. Note that

$$\begin{aligned} X_A^{(m,n)} &= \{x \in X \mid \alpha_A(x) \geq m, \beta_A(x) \leq n\} \\ &= \{x \in X \mid \alpha_A(x) \geq m\} \cap \{x \in X \mid \beta_A(x) \leq n\} \\ &= U(\alpha_A; m) \cap L(\beta_A; n). \end{aligned}$$

An IFS $A = \langle X, \alpha_A, \beta_A\rangle$ in a BCK-algebra X is called an *intuitionistic fuzzy ideal* (briefly, IF*-ideal*) of X (see [7]) if it satisfies the following assertions:

(d1) $(\forall x \in X)\ (\alpha_A(0) \geq \alpha_A(x),\ \beta_A(0) \leq \beta_A(x))$,
(d2) $(\forall x, y \in X)\ (\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\})$,
(d3) $(\forall x, y \in X)\ (\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\})$.

4. Intuitionistic Fuzzy Commutative Ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 4.1. An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is called an *intuitionistic fuzzy commutative ideal* (briefly, *IFC-ideal*) of X if it satisfies the condition (d1) and the following assertions:

$$\begin{array}{ll} (4.1) & \begin{array}{l} \alpha_A(x*(y*(y*x))) \geq \min\{\alpha_A((x*y)*z), \alpha_A(z)\}, \\ \beta_A(x*(y*(y*x))) \leq \max\{\beta_A((x*y)*z), \beta_A(z)\} \end{array} \end{array}$$

for all $x, y, z \in X$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set in which the operation $*$ is defined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X; *, 0)$ is a BCK-algebra. Let $(t_0, s_0), (t_1, s_1), (t_2, s_2) \in [0,1] \times [0,1]$ satisfy $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$, that is, $t_0 > t_1 > t_2$, $s_0 < s_1 < s_2$, and $t_i + s_i \leq 1$ for $i = 0, 1, 2$. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X given by

$$A = \left\langle X, \left(\tfrac{0}{t_0}, \tfrac{a}{t_1}, \tfrac{b}{t_2}, \tfrac{c}{t_2}\right), \left(\tfrac{0}{s_0}, \tfrac{a}{s_1}, \tfrac{b}{s_2}, \tfrac{c}{s_2}\right) \right\rangle,$$

that is, $\alpha_A(0) = t_0$, $\alpha_A(a) = t_1$, $\alpha_A(b) = \alpha_A(c) = t_2$, $\beta_A(0) = s_0$, $\beta_A(a) = s_1$, and $\beta_A(b) = \beta_A(c) = s_2$. By routine calculations, we know that $A = \langle X, \alpha_A, \beta_A \rangle$ is an *IFC*-ideal of X.

Theorem 4.3. *Every IFC-ideal of X is an IF-ideal of X.*

Proof. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an *IFC*-ideal of X. Using (a3), (b1), and (4.1), we have

$$\begin{aligned} \min\{\alpha_A(x*z), \alpha_A(z)\} &= \min\{\alpha_A((x*0)*z), \alpha_A(z)\} \\ &\leq \alpha_A(x*(0*(0*x))) = \alpha_A(x), \\ \max\{\beta_A(x*z), \beta_A(z)\} &= \max\{\beta_A((x*0)*z), \beta_A(z)\} \\ &\geq \beta_A(x*(0*(0*x))) = \beta_A(x), \end{aligned}$$

for all $x, z \in X$. Hence $A = \langle X, \alpha_A, \beta_A \rangle$ is an *IF*-ideal of X. □

The following example shows that the converse of Theorem 4.3 is not valid in general.

Example 4.4. Let $X = \{0, a, b, c, d\}$ be a set in which the operation $*$ is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Then $(X; *, 0)$ is a BCK-algebra. Let $(t_0, s_0), (t_1, s_1), (t_2, s_2) \in [0,1] \times [0,1]$ satisfy $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$, that is, $t_0 > t_1 > t_2$, $s_0 < s_1 < s_2$, and $t_i + s_i \leq 1$ for $i = 0, 1, 2$. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X given by

$$A = \left\langle X, \left(\tfrac{0}{t_0}, \tfrac{a}{t_1}, \tfrac{b}{t_2}, \tfrac{c}{t_2}, \tfrac{d}{t_2}\right), \left(\tfrac{0}{s_0}, \tfrac{a}{s_1}, \tfrac{b}{s_2}, \tfrac{c}{s_2}, \tfrac{d}{s_2}\right) \right\rangle,$$

that is, $\alpha_A(0) = t_0$, $\alpha_A(a) = t_1$, $\alpha_A(b) = \alpha_A(c) = \alpha_A(d) = t_2$, $\beta_A(0) = s_0$, $\beta_A(a) = s_1$, and $\beta_A(b) = \beta_A(c) = \beta_A(d) = s_2$. By routine calculations, we know that $A = \langle X, \alpha_A, \beta_A \rangle$ is an IF-ideal of X. But it is not an IFC-ideal of X because

$$\alpha_A(b * (c * (c * b))) = t_2 < t_0 = \min\{\alpha_A((b * c) * 0), \alpha_A(0)\}$$

and/or

$$\beta_A(b * (c * (c * b))) = s_2 > s_0 = \max\{\beta_A((b * c) * 0), \beta_A(0)\}.$$

We give conditions for an IF-ideal to be an IFC-ideal.

Theorem 4.5. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IF-ideal of X. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X if and only if it satisfies the conditions:*

$$\alpha_A(x * y) \leq \alpha_A(x * (y * (y * x))) \text{ and } \beta_A(x * y) \geq \beta_A(x * (y * (y * x))) \tag{4.2}$$

for all $x, y \in X$.

Proof. Assume that $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X. Taking $z = 0$ in (4.1) and using (d1) and (b1), we get (4.2). Conversely suppose that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (4.2). As $A = \langle X, \alpha_A, \beta_A \rangle$ is an IF-ideal, hence

$$\begin{aligned} &\alpha_A(x * y) \geq \min\{\alpha_A((x * y) * z), \alpha_A(z)\}, \\ &\beta_A(x * y) \leq \max\{\beta_A((x * y) * z), \beta_A(z)\} \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$. Combining (4.2) and (4.3), we have (4.1). This completes the proof. □

Lemma 4.6 ([7]). *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IF-ideal of X. If the inequality $x*y \leq z$ holds in X, then*

$$\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\},\ \beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\} \tag{4.4}$$

for all $x, y, z \in X$.

Theorem 4.7. *If* X *is commutative, then every* IF*-ideal of* X *is an* IFC*-ideal of* X.

Proof. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IF-ideal of a commutative BCK-algebra X. It is sufficient to show that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies (4.1). Let $x, y, z \in X$. Then

$$\begin{aligned}((x*(y*(y*x)))*((x*y)*z))*z &= ((x*(y*(y*x)))*z)*((x*y)*z) \\ &\leq (x*(y*(y*x)))*(x*y) \\ &= (x*(x*y))*(y*(y*x)) = 0,\end{aligned}$$

that is, $(x*(y*(y*x)))*((x*y)*z) \leq z$. It follows from Lemma 4.6 that

$$\alpha_A(x*(y*(y*x))) \geq \min\{\alpha_A((x*y)*z), \alpha_A(z)\},$$
$$\beta_A(x*(y*(y*x))) \leq \max\{\beta_A((x*y)*z), \beta_A(z)\}$$

for all $x, y, z \in X$. Hence $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X. □

For any $w \in X$ and any IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X, we let

$$A(w) = \{x \in X \mid \alpha_A(x) \geq \alpha_A(w), \beta_A(x) \leq \beta_A(w)\}.$$

Obviously $w \in A(w)$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X, then $0 \in A(w)$. The following is our question: *For an IFS* $A = \langle X, \alpha_A, \beta_A \rangle$ *in* X *satisfying* (d1), *is* $A(w)$ *a commutative ideal of* X? But the following example provides a negative answer, that is, there exists an element $w \in X$ such that $A(w)$ is not a commutative ideal of X.

Example 4.8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X defined by

$$A = \left\langle X, \left(\tfrac{0}{0.9}, \tfrac{a}{0.7}, \tfrac{b}{0.3}, \tfrac{c}{0.2}, \tfrac{d}{0.5}\right), \left(\tfrac{0}{0.02}, \tfrac{a}{0.03}, \tfrac{b}{0.5}, \tfrac{c}{0.6}, \tfrac{d}{0.3}\right)\right\rangle.$$

Then $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies (d1), and it is not an IF-ideal of X because $\alpha_A(b) < \min\{\alpha_A(b*d),\ \alpha_A(d)\}$ and/or $\beta_A(b) > \max\{\beta_A(b*d),\ \beta_A(d)\}$, and so it is not an

IFC-ideal of X. Then $A(d) = \{0, a, d\}$ is not a commutative ideal of X. Note that $A(b) = \{0, a, b, d\}$ is an ideal of X.

We give conditions for the set $A(w)$ to be an ideal. We first deal with the following lemma.

Lemma 4.9 ([8]). *Let $w \in X$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an IF-ideal of X, then $A(w)$ is an ideal of X.*

Theorem 4.10. *Let $w \in X$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X, then $A(w)$ is a commutative ideal of X.*

Proof. Assume that $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X. Using Theorem 4.3 and Lemma 4.9, we know that $A(w)$ is an ideal of X. Let $x, y \in X$ be such that $x * y \in A(w)$. Then $\alpha_A(x*y) \geq \alpha_A(w)$ and $\beta_A(x*y) \leq \beta_A(w)$. It follows from Theorem 4.5 that $\alpha_A(x*(y*(y*x))) \geq \alpha_A(x*y) \geq \alpha_A(w)$, $\beta_A(x*(y*(y*x))) \leq \beta_A(x*y) \leq \beta_A(w)$ so that $x * (y * (y * x)) \in A(w)$. Hence $A(w)$ is a commutative ideal of X. □

Theorem 4.11. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFC-ideal of X and let $m, n \in [0, 1]$ satisfy $m \leq \alpha_A(0)$, $n \geq \beta_A(0)$ and $m + n \leq 1$. Then the (m, n)-level subset $X_A^{(m,n)}$ is a commutative ideal of X.*

Proof. Obviously $0 \in X_A^{(m,n)}$. Let $x, y, z \in X$ be such that $(x * y) * z \in X_A^{(m,n)}$ and $z \in X_A^{(m,n)}$. Then $\alpha_A((x*y)*z) \geq m$, $\beta_A((x*y)*z) \leq n$, $\alpha_A(z) \geq m$, and $\beta_A(z) \leq n$. It follows from (4.1) that

$$\alpha_A(x * (y * (y * x))) \geq \min\{\alpha_A((x * y) * z), \alpha_A(z)\} \geq m,$$

$$\beta_A(x * (y * (y * x))) \leq \max\{\beta_A((x * y) * z), \beta_A(z)\} \leq n$$

so that $x * (y * (y * x)) \in X_A^{(m,n)}$. Hence $X_A^{(m,n)}$ is a commutative ideal of X. □

Theorem 4.12. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X. Assume that $X_A^{(m,n)}$ is a commutative ideal of X for every $m, n \in [0, 1]$ with $m+n \leq 1$. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X.*

Proof. For any $x \in X$, let $\alpha_A(x) = m$ and $\beta_A(x) = n$, where $m + n \leq 1$. Since $0 \in X_A^{(m,n)}$, we have $\alpha_A(0) \geq m = \alpha_A(x)$ and $\beta_A(0) \leq n = \beta_A(x)$. For any $x, y, z \in X$, let $A((x * y) * z) = (m_1, n_1)$ and $A(z) = (m_2, n_2)$, i.e., $\alpha_A((x * y) * z) = m_1$, $\beta_A((x * y) * z) = n_1$, $\alpha_A(z) = m_2$, and $\beta_A(z) = n_2$, where $m_i + n_i \leq 1$ for $i = 1, 2$. Then $(x * y) * z, z \in X_A^{(\min(m_1,n_1),\max(m_2,n_2))}$. Using (c3), we have $x * (y * (y * x)) \in X_A^{(\min(m_1,n_1),\max(m_2,n_2))}$, and so

$$\alpha_A(x*(y*(y*x))) \geq \min(m_1, n_1) = \min\{\alpha_A((x*y)*z), \alpha_A(z)\},$$
$$\beta_A(x*(y*(y*x))) \leq \max(m_2, n_2) = \max\{\beta_A((x*y)*z), \beta_A(z)\}.$$

Therefore $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X. □

Corollary 4.13. *An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is an IFC-ideal of X if and only if $U(\alpha_A; m)$ and $L(\beta_A; n)$ are commutative ideals of X for all $m \in [0, \alpha_A(0)]$ and $n \in [\beta_A(0), 1]$ with $m + n \leq 1$.*

Corollary 4.14. *Let I be a commutative ideal of X and let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X defined by*

$$\alpha_A(x) := \begin{cases} m_0 & \text{if } x \in I, \\ m_1 & \text{otherwise,} \end{cases} \qquad \beta_A(x) := \begin{cases} n_0 & \text{if } x \in I, \\ n_1 & \text{otherwise,} \end{cases}$$

for all $x \in X$ where $0 \leq m_1 < m_0$, $0 \leq n_0 < n_1$ and $m_i + n_i \leq 1$ for $i = 0, 1$. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X.

Lemma 4.15. *An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ is an IFC-ideal of X if and only if the fuzzy sets α_A and $\overline{\beta_A}$ are fuzzy commutative ideals of X, where $\overline{\beta_A}$ is a fuzzy set in X defined by $\overline{\beta_A}(x) = 1 - \beta_A(x)$ for all $x \in X$.*

Proof. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFC-ideal of X. Clearly α_A is a fuzzy commutative ideal of X. Let $x, y, z \in X$. Then

$$\overline{\beta_A}(0) = 1 - \beta_A(0) \geq 1 - \beta_A(x) = \overline{\beta_A}(x),$$
$$\begin{aligned} \overline{\beta_A}(x*(y*(y*x))) &= 1 - \beta_A(x*(y*(y*x))) \\ &\geq 1 - \max\{\beta_A((x*y)*z), \beta_A(z)\} \\ &= \min\{1 - \beta_A((x*y)*z), 1 - \beta_A(z)\} \\ &= \min\{\overline{\beta_A}((x*y)*z), \overline{\beta_A}(z)\}. \end{aligned}$$

Hence $\overline{\beta_A}$ is a fuzzy commutative ideal of X. Conversely, assume that α_A and $\overline{\beta_A}$ are fuzzy commutative ideals of X. For every $x \in X$, we have $\alpha_A(0) \geq \alpha_A(x)$ and $1 - \beta_A(0) = \overline{\beta_A}(0) \geq \overline{\beta_A}(x) = 1 - \beta_A(x)$, which shows that $\beta_A(0) \leq \beta_A(x)$. This shows that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (d1). Now let $x, y, z \in X$. Then

$$\alpha_A(x*(y*(y*x))) \geq \min\{\alpha_A((x*y)*z), \alpha_A(z)\}$$

and

$$\begin{aligned} 1 - \beta_A(x*(y*(y*x))) &= \overline{\beta_A}(x*(y*(y*x))) \\ &\geq \min\{\overline{\beta_A}((x*y)*z), \overline{\beta_A}(z)\} \end{aligned}$$

$$= \min\{1-\beta_A((x*y)*z), 1-\beta_A(z)\}$$
$$= 1-\max\{\beta_A((x*y)*z), \beta_A(z)\},$$

and so $\beta_A(x*(y*(y*x))) \le \max\{\beta_A((x*y)*z), \beta_A(z)\}$. Hence $A = \langle X, \alpha_A, \beta_A\rangle$ is an IFC-ideal of X. □

Theorem 4.16. *An IFS $A = \langle X, \alpha_A, \beta_A\rangle$ in X is an IFC-ideal of X if and only if $\Box A := \langle X, \alpha_A, \overline{\alpha_A}\rangle$ and $\Diamond A := \langle X, \overline{\beta_A}, \beta_A\rangle$ are IFC-ideals of X.*

Proof. It is straightforward by Lemma 4.15. □

Theorem 4.17. *Let $A = \langle X, \alpha_A, \beta_A\rangle$ be an IFS in X and*

$$\mathrm{Im}(A) = \{(m_0, n_0), (m_1, n_1), \cdots, (m_k, n_k)\}$$

where $(m_i, n_i) < (m_j, n_j)$, that is, $m_i < m_j$ and $n_i > n_j$ whenever $i > j$. Let $\{G_r \mid r = 0, 1, \cdots, k\}$ be a family of commutative ideals of X such that

(1) $G_0 \subset G_1 \subset \cdots \subset G_k = X$,
(2) $A(G_r^*) = (m_r, n_r)$, *i.e.*, $\alpha_A(G_r^*) = m_r$ *and* $\beta_A(G_r^*) = n_r$, *where* $G_r^* = G_r \setminus G_{r-1}$, $G_{-1} = \emptyset$ *for* $r = 0, 1, \cdots, k$.

Then $A = \langle X, \alpha_A, \beta_A\rangle$ is an IFC-ideal of X.

Proof. Since $0 \in G_0$, we have $\alpha_A(0) = m_0 \ge \alpha_A(x)$ and $\beta_A(0) = n_0 \le \beta_A(x)$ for all $x \in X$. Accordig to our suppoition, we have $X = \cup_{r=0}^k G_r^*$. For any $x, y, z \in X$, then there exist r_1 and r_2, such that $(x*y)*z \in G_r^*$ and $z \in G_r^*$, then $\alpha_A((x*y)*z) = m_{r_1}$ and $\alpha_A(z) = m_{r_2}$. There is no harm in assuming $r_1 \le r_2$, then $m_{r_2} \le m_{r_1}$. Since $G_{r_1} \subset G_{r_2}$ and $G_{r_i}^* = G_{r_i} \setminus G_{r_i-1} (i = 1, 2)$, we have $(x*y)*z \in G_{r_2}$ and $z \in G_{r_2}$. Because G_{r_2} is a commutative ideal of X, we obtain $x*(y*(y*x)) \in G_{r_2}$. Hence $\alpha_A(x*(y*(y*x))) \ge m_{r_2} = \min\{m_{r_1}, m_{r_2}\}$, that is, $\alpha_A(x*(y*(y*x))) \ge \min\{\alpha_A((x*y)*z), \alpha_A(z)\}$. Similary, we can prove that $\beta_A(x*(y*(y*x))) \le \max\{\beta_A((x*y)*z), \beta_A(z)\}$. Therefore $A = \langle X, \alpha_A, \beta_A\rangle$ is an IFC-ideal of X. □

Theorem 4.18. *Let $\{G_m \mid m \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a finite collection of commutative ideals of X such that $X = \bigcup_{m\in\Lambda} G_m$, and for every $m, n \in \Lambda$, $m < n$ if and only if $G_n \subset G_m$. Then an IFS $A = \langle X, \alpha_A, \beta_A\rangle$ in X defined by*

$$\alpha_A(x) = \sup\{m \in \Lambda \mid x \in G_m\} \text{ and } \beta_A(x) = \inf\{m \in \Lambda \mid x \in G_m\}$$

for all $x \in X$ is an IFC-ideal of X.

Proof. According to Corollary 4.13, it is sufficient to show that the nonempty sets $U(\alpha_A; m)$ and $L(\beta_A; n)$ are commutative ideals of X for every $m, n \in [0,1]$ with $m+n \leq 1$. In order to show that $U(\alpha_A; m)$ is a commutative ideal, we divide into the following two cases: (i) $m = \sup\{k \in \Lambda \mid k < m\}$ and (ii) $m \neq \sup\{k \in \Lambda \mid k < m\}$. Case (i) implies that $x \in U(\alpha_A; m) \Leftrightarrow x \in G_k$ for all $k < m \Leftrightarrow x \in \bigcap_{k<m} G_k$, so that $U(\alpha_A; m) = \bigcap_{k<m} G_k$, which is a commutative ideal of X. For the case (ii), we claim that $U(\alpha_A; m) = \bigcup_{k \geq m} G_k$. If $x \in \bigcup_{k\geq m} G_k$, then $x \in G_k$ for some $k \geq m$. It follows that $\alpha_A(x) \geq k \geq m$ so that $x \in U(\alpha_A; m)$. This proves that $\bigcup_{k\geq m} G_k \subset U(\alpha_A; m)$. Now assume that $x \notin \bigcup_{k\geq m} G_k$. Then $x \notin G_k$ for all $k \geq m$. Since $m \neq \sup\{k \in \Lambda \mid k < m\}$, there exists $\varepsilon > 0$ such that $(m-\varepsilon, m) \cap \Lambda = \emptyset$. Hence $x \notin G_k$ for all $k > m - \varepsilon$, which means that if $x \in G_k$ then $k \leq m - \varepsilon$. Thus $\alpha_A(x) \leq m - \varepsilon < m$, and so $x \notin U(\alpha_A; m)$. Therefore $U(\alpha_A; m) = \bigcup_{k\geq m} G_k$. Next we show that $L(\beta_A; n)$ is a commutative ideal of X for all $n \in [\beta_A(0), 1]$. We consider the following two cases: (iii)$n = \inf\{k \in \Lambda \mid n < k\}$ and (iv) $n \neq \inf\{k \in \Lambda \mid n < k\}$. For the case (iii) we have $x \in L(\beta_A; n) \Leftrightarrow x \in G_k$ for all $n < k \Leftrightarrow x \in \bigcap_{n<k} G_k$, and hence $L(\beta_A; n) = \bigcap_{n<k} G_k$, which is a commutative ideal of X. For the case (iv), we will show that $L(\beta_A; n) = \bigcup_{n\geq k} G_k$. If $x \in \bigcup_{n\geq k} G_k$, then $x \in G_k$ for some $n \geq k$. It follows that $\beta_A(x) \leq k \leq n$ so that $x \in L(\beta_A; n)$. Hence $\bigcup_{n\geq k} G_k \subset L(\beta_A; n)$. Conversely, if $x \notin \bigcup_{n\geq k} G_k$ then $x \notin G_k$ for all $k \leq n$. Since $n \neq \inf\{k \in \Lambda \mid n < k\}$, there exists $\varepsilon > 0$ such that $(n, n+\varepsilon) \cap \Lambda = \emptyset$, which implies that $x \notin G_k$ for all $k < n + \varepsilon$, that is, if $x \in G_k$ then $k \geq n + \varepsilon$. Thus $\beta_A(x) \geq n + \varepsilon > n$, that is, $x \notin L(\beta_A; n)$. Therefore $L(\beta_A; n) \subset \bigcup_{n\geq k} G_k$ and consequently $L(\beta_A; n) = \bigcup_{n\geq k} G_k$. This completes the proof. □

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[a]DEPARTMENT OF MATHEMATICS EDUCATION (AND RINS), GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA
Email address: `skywine@gmail.com`

[b]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ULSAN, ULSAN 680-749, KOREA
Email address: `dslee@ulsan.ac.kr`

[c]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ULSAN, ULSAN 680-749, KOREA
Email address: `skyroemary@gmail.com` or `chpark@ulsan.ac.kr`