J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 1 (February 2008), Pages 85–92

ON THE WEAK FORMS OF CHOICE IN TOPOI

IG SUNG KIM

ABSTRACT. In topoi, there are various forms of the axiom of choice such as (ES), (AC) and (WO). And also there are various weak forms of the axiom of choice such as (DES), (IAC) and (ASC). First we investigate the relation between (IAC) and (ASC), and then we study the relation between (AC) and (WO). We get equivalent forms of the axiom of choice in a well-pointed topos.

1. INTRODUCTION

There are various forms of the axiom of choice in topoi:

(ES) Every epimorphism is a section.

(AC) For any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(WO) For any P and $q : U \to \Omega^P$, if there exist $\alpha : V \to U$ and $p : V \to P$ such that $(q\alpha, p)$ factors through $\in_P \to \Omega^P \times P$, then there exists $\alpha_0 : V_0 \to U$ and $p_0 : V_0 \to P$ such that $(q\alpha_0, p_0)$ factors through $\in_P$, and such that for all $\beta : W \to V_0$ and all $p_1 : W \to P$, if $(q\alpha_0\beta, p_1)$ factors through $\in_P$, then $(p_0\beta, p_1)$ factors through a monomorphism $P_1 \to P \times P$.

Also there are various weak forms of the axiom of choice in a topos:

(DES) Every decidable epimorphism is a section.

(ASC) Every separated epimorphism of $\mathcal{E}$ is a section.

(IAC) Every object of $\mathcal{E}$ is internally projective.

Mawanda [7] showed that (ES), (DES) and (ASC) are equivalent in a well-pointed topos. Goldblatt [4] showed that (ES) and (AC) are equivalent in a well-pointed topos. In this paper we show that (IAC) is not equivalent to (ASC) in some topoi, but (IAC) is equivalent to (ASC) in a well-pointed topos. Also we show that (AC)

Received by the editors April 27, 2007 and, in revised form October 8, 2007.
2000 *Mathematics Subject Classification.* 18B25.
Key words and phrases. axiom of choice, implicit axiom of choice, axiom of separated choice.

is not equivalent to (WO) in some topoi, but (AC) is equivalent to (WO) in a well-pointed topos. Therefore we know that (ES), (AC), (WO), (DES), (ASC) and (IAC) are equivalent in a well-pointed topos.

2. PRELIMINARIES

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following:
(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times i_A) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g\times i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ B^A \times A & \xrightarrow{ev_A} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : \mathbf{1} \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow{\chi_h} & \Omega \end{array}$$

Example 2.2. If M_2 is a monoid with two elements, then the category $M_2 - Set$ is a topos.

Consider $(M_2, \circ, e)$ where $M_2 = \{e, a\}$ and $\circ$ is defined by $e \circ e = e$, $e \circ a = a \circ e = a \circ a = a$. Then M_2 is a monoid with identity e, in which a has no inverse. The set L_2 of left ideals of M_2 has three elements, that is, M_2, $\emptyset$, and $\{a\}$. Thus in $M_2 - Set$, $\Omega = (L_2, \omega)$, where the action $\omega : M_2 \times L_2 \to L_2$ is defined by

$\omega(m, B) = \{n | n \circ m \in B\}$.

In fact, $M - Set$ is a topos for any monoid M (see [4]).

Definition 2.3. We say that *supports split* (SS) for every $\mathcal{E}$-object A in the topos $\mathcal{E}$ if the epic part of the epi-monic factorization of $! : A \to \mathbf{1}$ has a right inverse.

Lemma 2.4. *The topos $M_2 - Set$ satisfies (SS).*

Proof. Since supports split if and only if every subobject of the terminal object $\mathbf{1} = \{*\}$ is a projective object, we only need to show that the terminal object is a projective object. Let $f : (X, \psi) \to (Y, \phi)$ be an epimorphism in $M_2 - Set$ where $\psi : M_2 \times X \to X$ and $\phi : M_2 \times Y \to Y$ are actions of M_2 on X and Y, respectively. By the property of M_2, we have $\psi(e, x) = x$ for all $x \in X$. Let $h : \{*\} \to Y$ be a morphism in $M_2 - Set$ where $\nu : M_2 \times \{*\} \to \{*\}$ is a trivial action. Since f is an epimorphism, for $h(*) \in Y$ there is a $x' \in X$ such that $f(x') = h(*)$. We define a morphism $k : \{*\} \to X$ by $k(*) = \psi(a, x')$. Then $k(\nu(n, *)) = \psi(n, k(*)) = k(*)$ and $f \circ k(*) = f(\psi(a, x')) = \phi(a, f(x')) = \phi(a, h(*)) = h(*)$. □

Definition 2.5. We say that $\mathcal{E}$ satisfies the axiom (SG) if the subobjects of $\mathbf{1} = \{*\}$ in $\mathcal{E}$ form a class of generators.

Definition 2.6. A topos $\mathcal{E}$ is called *Boolean* if for every object D in $\mathcal{E}$, $(\text{Sub}(D), \in)$ is a Boolean algebra where $\text{Sub}(D)$ is the class of monomorphism with common codomain D, and $g \in f$ if there exists a morphism $h : B \to A$ such that $f \circ h = g$ where $f : A \to D$ and $g : B \to D$ are monomorphisms.

Lemma 2.7 ([4]). *For any topos $\mathcal{E}$, the following statements are equivariant:*

(1) *$\mathcal{E}$ is Boolean.*
(2) *$Sub(\Omega)$ is a Boolean algebra.*
(3) *$\top : \mathbf{1} \to \Omega$ has a complement in $Sub(\Omega)$.*
(4) *$\bot : \mathbf{1} \to \Omega$ is the complement of $\top$ in $Sub(\Omega)$.*
(5) *$\top \cup \bot \simeq 1_\Omega$ in $Sub(\Omega)$.*
(6) *$\mathcal{E}$ is classical.*
(7) *$i_1 : \mathbf{1} \to \mathbf{1} + \mathbf{1}$ is a subobject classifier.*

Example 2.8. If M is non-trivial monoid and not a group, then the category $M - Set$ is a non-Boolean topos.

For the proof see Goldblatt [4], Ebahimi and Mahmoudi [3] and Madanshekaf and Tavakoli [6].

Lemma 2.9. *In any topos $\mathcal{E}$, (ES) implies (SG).*

Proof. Let $f, g : X \to Y$ be two distinct morphisms. Then there is an equalizer (E, e) of f and g. Since a topos satisfying (AC) is Boolean, there is the complement of E, denoted by $\neg E$, such that $m : \neg E \to X$ is a monomorphism. Consider $\sigma_1(\neg E)$ where σ is a support functor and $\mathbf{1} = \{*\}$ is a terminal object. Then $\sigma_1(\neg E)$ is nonzero and $s : \neg E \to \sigma_1(\neg E)$ is an epimorphism. By hypothesis, there is a morphism $t : \sigma_1(\neg E) \to \neg E$ such that $s \circ t = i_{\sigma_1(\neg E)}$. For $m \circ t : \sigma_1(\neg E) \to X$, the equalizer of $f \circ m \circ t$ and $g \circ m \circ t$ is $(m \circ t)^*(E \to X)$ which is isomorphic to $0 \to \sigma_1(\neg E)$ where 0 is an initial object. Therefore $f \circ m \circ t \neq g \circ m \circ t$. □

Definition 2.10. A topos is called *well-pointed* if it satisfies the extentionality principle for morphisms, i.e., If $f, g : A \to B$ are a pair of distinct parallel morphisms, then there is an element $a : \mathbf{1} \to A$ of A such that $f \circ a \neq g \circ a$.

Lemma 2.11 ([7]). *In a well-pointed topos, (ES), (DES) and (ASC) are equivalent.*

3. MAIN RESULTS

In this section, we show that (IAC) is not equivalent to (ASC) in some topoi, but (IAC) is equivalent to (ASC) in a well-pointed topos. Also we show that (AC) is not equivalent to (WO) in some topoi, but (AC) is equivalent to (WO) in a well-pointed topos. Therefore we know that (ES), (AC), (WO), (DES), (ASC) and (IAC) are equivalent in a well-pointed topos.

Theorem 3.1. *There is a topos satisfying (IAC) and not satisfying (ASC).*

Proof. Consider the topos $G - \mathcal{S}et$ where $G = \{e, g\}$ is a group with identity e. We claim that, for any object $X \in G - \mathcal{S}et$, $()^X : G - \mathcal{S}et \to G - \mathcal{S}et$ preserves epimorphisms. Let $h : A \to B$ be an epimorphism in $G - \mathcal{S}et$. Since the forgetful functor $U : G - \mathcal{S}et \to \mathcal{S}et$ preserves epimorphisms, $U(h) : U(A) \to U(B)$ is an epimorphism in $\mathcal{S}et$. Since $U(h)^{U(X)} : U(A)^{U(X)} \to U(B)^{U(X)}$ is an epimorphism in $\mathcal{S}et$ and the forgetful functor $U : G - \mathcal{S}et \to \mathcal{S}et$ preserves exponentials, $U(h^X) : U(A^X) \to U(B^X)$ is an epimorphism in $\mathcal{S}et$ ([4]). Since the forgetful functor $U : G - \mathcal{S}et \to \mathcal{S}et$ retracts epimorphisms, $h^X : A^X \to B^X$ is an epimorphism in $G - \mathcal{S}et$.

Thus $G - \mathcal{S}et$ satisfies (IAC). But this topos does not satisfy (ASC) because there is not an absorbing element in G ([7]). □

Proposition 3.2. *In a topos $\mathcal{E}$ which satisfies (SS), (IAC) implies (ASC).*

Proof. Let $f : X \to Y$ be an epimorphism in $\mathcal{E}$. Since $\mathcal{E}$ has pull-backs, there is a pull-back functor $Y^* : \mathcal{E} \to \mathcal{E}/Y$ defined by $Y^*(W) = W \times Y \to Y$ such that makes the following diagram a pull-back.

$$\begin{array}{ccc} W \times Y & \xrightarrow{\pi_1} & W \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{1} \end{array}$$

Since $\mathcal{E}$ is a finitely complete category with exponentiation, a pull-back functor $Y^* : \mathcal{E} \to \mathcal{E}/Y$ has a right adjoint $\Pi_Y : \mathcal{E}/Y \to \mathcal{E}$ for every morphism Y with codomain $\mathbf{1}$. Since Y is internally projective (Johnstone [5]), $f^Y : X^Y \to Y^Y$ is an epimorphism. Hence there is a morphism $u : \Pi_Y(f) \to \mathbf{1}$ which is a pull-back of $f^Y : X^Y \to Y^Y$. Also, by the property of pull-back, $u : \Pi_Y(f) \to \mathbf{1}$ is an epimorphism. By hypothesis, $u : \Pi_Y(f) \to \mathbf{1}$ has a right inverse $v : \mathbf{1} \to \Pi_Y(f)$ such that $u \circ v = i_1$. Since the pull-back functor has a right adjoint, there is a morphism $w : Y^*(\mathbf{1}) \to f$ such that $f \circ w = i_Y$. Thus every epimorphism in $\mathcal{E}$ has a right inverse. Thus $\mathcal{E}$ is Boolean (Diaconescu [2]). Hence every separated epimorphism is a section by Lemma 2.7.(4). Therefore $\mathcal{E}$ satisfies (ASC). □

Theorem 3.3. *There is a topos satisfying (ASC) and not satisfying (IAC).*

Proof. Consider the topos $M_2 - \mathcal{S}et$ where $M_2 = \{e, a\}$ is a monoid with identity e, in which a has no inverse. Since a is an absorbent element in M_2, $M_2 - \mathcal{S}et$ satisfies (ASC) (Mawanda [7]). But this topos does not satisfy (IAC). Assume $M_2 - \mathcal{S}et$ satisfies (IAC). By Lemma 2.4, $M_2 - \mathcal{S}et$ satisfies (ES). Also, by Lemma 2.9, $M_2 - \mathcal{S}et$ satisfies (SG). But this is a contradiction. Since we can construct the two different morphism $c, i : \{X, \psi\} \to \{X, \phi\}$ where $X = \{x, y\}$, $c(x) = c(y) = x$ and $i(x) = x, i(y) = y$. Also $\psi : M_2 \times X \to X$ is an action of M_2 on X with $\psi(e, x) = x, \psi(e, y) = y, \psi(a, x) = x, \psi(a, y) = x$ and $\phi : M_2 \times X \to X$ is an action of M_2 on X with $\phi(e, x) = x, \phi(e, y) = y, \phi(a, x) = x, \phi(a, y) = y$. Then $c, i : \{X, \psi\} \to \{X, \phi\}$ are action preserving morphisms. And let $h : \{\{*\}, \nu\} \to \{X, \psi\}$ where $h(*) = x$ and $\nu : M_2 \times \{*\} \to \{*\}$ with trivial action. Since $h(*) = x$ is the unique action preserving morphism, we have $c \circ h = i \circ h$. □

Proposition 3.4. *In a Boolean topos $\mathcal{E}$, (ASC) implies (IAC).*

Proof. By Lemma 2.7 and hypothesis, $\mathcal{E}$ satisfies (ASC) if and only if every epimorphism in $\mathcal{E}$ is a retraction. Let $e : X \to Y$ be an epimorphism in $\mathcal{E}$. We claim that, for any object K in $\mathcal{E}$, $e^K : X^K \to Y^K$ is an epimorphism. For any morphism $s : K \to Y$ in $\mathcal{E}$, since the epimorphism $e : X \to Y$ is a retraction by hypothesis, there is a morphism $r : Y \to X$ in $\mathcal{E}$ such that $e \circ r = i_Y$. Hence we get a morphism $r \circ s : K \to X$ in $\mathcal{E}$ such that $e^K(r \circ s) = e \circ r \circ s = s$. Thus the morphism $e^K : X^K \to Y^K$ is an epimorphism. □

Proposition 3.5 ([7]). *There is a topos satisfying (AC) and not satisfying (WO).*

Proposition 3.6. *In a Boolean topos $\mathcal{E}$, (AC) implies (WO).*

Proof. Let X_0 be a noninitial object in $\mathcal{E}$. By (AC), there is a morphism $\psi : NX_0 \to X_0$ such that $\psi \circ g_i \in g_i'$ where NX_0 is the object of noninitial subobjects of X_0 with the usual ordering, $g_i : \mathbf{1} \to NX_0$ is a morphism and $g_i' : X_0' \to X_0$ is a monomorphism ([8]). By hypothesis, for g_0 such that $g_0(*) = X_0$ we get that $-(\psi \circ g_0) \equiv g_1$ which is the complement of $(\psi \circ g_0)$ in g_0, where the pullback of $\psi \circ g_0$ and $-(\psi \circ g_0)$ is the initial object, $-(\psi \circ g_1) \equiv g_2$ which is the complement of $(\psi \circ g_1)$ in g_1, where the pullback of $\psi \circ g_1$ and $-(\psi \circ g_1)$ is the initial object, etc. Generally, we get that $-(\psi \circ g_{n-1}) \equiv g_n$ which is the complement of $(\psi \circ g_{n-1})$ in g_{n-1}, where the pullback of $\psi \circ g_{n-1}$ and $-(\psi \circ g_{n-1})$ is the initial object. That is,

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle \psi\circ g_0} \\ \mathbf{1} & \xrightarrow{-(\psi\circ g_0)} & X_0 \end{array}$$

is a pullback square.

Thus we construct $\phi : X_0 \to NX_0$ such that $Im(\phi)$ is a subobject of NX_0 consistsing of $g_0, -(\psi \circ g_0), -(\psi \circ g_1), \ldots$ and $-(\psi \circ g_{n-1})$, where $-(\psi \circ g_{n-1})$ is a noninitial object, and $\psi \circ \phi = i_{X_0}$. Then $Im(\phi)$ is a linear ordered with minimal choice. Since ϕ is a monomorphism, X_0 has an ordering with minimal choice. □

Proposition 3.7 ([7]). *There is a topos satisfying (WO) and not satisfying (AC).*

Proposition 3.8. *In a well-pointed topos $\mathcal{E}$, (WO) implies (AC).*

Proof. Since every morphism is epi-monic factorizable in $\mathcal{E}$, for any morphism $f : A \to B$ in $\mathcal{E}$, there exist an epimorpmism $e : A \to X$ and a monomorphism $m :$

$X \to B$ such that $f = m \circ e$. By hypothesis, there exists a morphism $t : B \to X$ such that $t \circ m = i_X$. We only show that there is a morphism $s : X \to A$ such that $f = f \circ (s \circ t) \circ f = f$. Since $e : A \to X$ is an epimorphism, there is a morphism $q : X \to \Omega^A$ which is the interpretation of the term $\{a|e(a) = x\}$. By definition of (WO), we can find an epimorphism $r : V \to X$ and a morphism $n : V \to A$ such that n is a minimal choice of qr. Since every epimorphism is a coequalizer, there are morphisms $u, v : W \to V$ such that the following square

$$\begin{array}{ccc} W & \xrightarrow{u} & V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle r} \\ V & \xrightarrow[r]{} & X \end{array}$$

commutes.

Thus we get $q \circ r \circ v = q \circ r \circ u$. Also nu, nv are both minimal choice of $q \circ r \circ v = q \circ r \circ u$. By definition of (WO), we have $nu = nv$. Since every epimorphism is a coequalizer, there is a morphism $s : X \to A$ such that $s \circ r = n$. Also there is a morphism $c : X \to \in_A$ such that $k \circ s = c$ where $k : A \to \in_A$ and $\in_A$ is the subobject classified by $ev : \Omega^A \times A \to \Omega$. Then we have $(q, s) = l \circ c = l \circ k \circ s = (q \circ e \circ s, s)$ where $l : \in_A \to \Omega^A \times A$. Since q is a monomorphism, we have $f = f \circ (s \circ t) \circ f = f$. □

Corollary 3.9. *In a well-pointed topos $\mathcal{E}$, (ES), (AC), (WO), (DES), (ASC) and (IAC) are equivalent.*

References

1. M. Barr & C. Well: *Topos, Triples and Theories.* Springer-Verlag, N. Y. , 1985.
2. R. Diaconescu: Axiom of Choice and Complementation. *Proc. Amer. Math. Soc.* **51** (1975), no. 1, 176-178.
3. M.M. Ebrahimi & M. Mahmoudi: The category of M-sets. *Ital. J. Pure and Appl. Math.* (2001), no. 9, 123-132.
4. R. Goldblatt: *Topoi.* North-Holland, 1984.
5. P.T. Johnstone: *Topos Theory.* Academic Press, N. Y., 1977.
6. A. Madanshekaf & J. Tavakoli: *On the Category of M-Set.* Far East J. Math. Sci. (2000), no. 2, 251-260.
7. M.M. Mawanda: Well-ordering and Choice. *J. Pure and Appl. Algebra.* **50** (1988), 171-184.
8. A.M. Penk: Two Form of the Axiom of Choice for an Elementary Topos. *J. Symbolic logic.* **40** (1975), no. 2, 192-212.

DEPARTMENT OF DATA INFORMATION, SANGJI UNIVERSITY, WONJU, KANGWON-DO 220-702, KOREA
Email address: iskim@@sangji.ac.kr