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A FUNCTIONAL EQUATION ON HOMOGENEOUS POLYNOMIALS

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ABSTRACT. In this paper, we obtain the general solution and the stability of the cubic functional equation $f(2x+y, 2z+w) + f(2x-y, 2z-w) = 2f(x+y, z+w) + 2f(x-y, z-w) + 12f(x, z)$. The cubic form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is a solution of the above functional equation.

1. INTRODUCTION

In this paper, let X and Y be real vector spaces. For a mapping $f : X \times X \to Y$, consider the 2-variable cubic functional equation:

$$\begin{aligned} &f(2x+y, 2z+w) + f(2x-y, 2z-w) \\ (1) \qquad &= 2f(x+y, z+w) + 2f(x-y, z-w) + 12f(x, z). \end{aligned}$$

When $X = Y = \mathbb{R}$, the cubic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := ax^3 + bx^2y + cxy^2 + dy^3$ is a solution of (1).

For a mapping $g : X \to Y$, consider the cubic functional equation:

$$(2) \qquad g(2x+y) + g(2x-y) = 2g(x+y) + 2g(x-y) + 12g(x).$$

In 2002, Jun and Kim [2] solved the solution of the equation (2).

In this paper, we investigate the relation between (1) and (2). And we find out the general solution and the generalized Hyers-Ulam stability of (1).

2. RESULTS

The 2-variable cubic functional equation (1) induces the cubic functional equation (2) as follows.

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Theorem 1. *Let $f : X \times X \to Y$ be a mapping satisfying (1) and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{3}$$

for all $x \in X$. Then g satisfies (2).

Proof. By (1) and (3),

$$\begin{aligned} & g(2x+y) + g(2x-y) \\ = & f(2x+y, 2x+y) + f(2x-y, 2x-y) \\ = & 2f(x+y, x+y) + 2f(x-y, x-y) + 12f(x,x) \\ = & 2g(x+y) + 2g(x-y) + 12g(x) \end{aligned}$$

for all $x, y \in X$. □

The cubic functional equation (2) induces the 2-variable cubic functional equation (1) with an additional condition.

Theorem 2. *Let $a, b, c, d \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying (2). If $f : X \times X \to Y$ is the mapping given by*

$$\begin{aligned} f(x,y) := ag(x) + \frac{b}{24}\Big[g(2x+y) - g(2x-y) - 2g(y)\Big] \\ + \frac{c}{24}\Big[g(x+2y) + g(x-2y) - 2g(x)\Big] + dg(y) \end{aligned} \tag{4}$$

for all $x, y \in X$, then f satisfies (1). Furthermore, (3) holds if $a + b + c + d = 1$.

Proof. By (2) and (4), we have

$$\begin{aligned} & f(2x+y, 2z+w) + f(2x-y, 2z-w) \\ = & ag(2x+y) \\ & + \frac{b}{24}\Big[g(4x+2y+2z+w) - g(4x+2y-2z-w) - 2g(2z+w)\Big] \\ & + \frac{c}{24}\Big[g(2x+y+4z+2w) + g(2x+y-4z-2w) - 2g(2x+y)\Big] \\ & + dg(2z+w) + ag(2x-y) \\ & + \frac{b}{24}\Big[g(4x-2y+2z-w) - g(4x-2y-2z+w) - 2g(2z-w)\Big] \\ & + \frac{c}{24}\Big[g(2x-y+4z-2w) + g(2x-y-4z+2w) - 2g(2x-y)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ dg(2z-w)\\
&= a[g(2x+y)+g(2x-y)] + \frac{b}{24}\Big[g(4x+2y+2z+w)+g(4x-2y+2z-w)\\
&\quad - g(4x+2y-2z-w) - g(4x-2y-2z+w) - 2g(2z+w) - 2g(2z-w)\Big]\\
&\quad + \frac{c}{24}\Big[g(2x+y+4z+2w)+g(2x-y+4z-2w)\\
&\quad + g(2x+y-4z-2w)+g(2x-y-4z+2w)\\
&\quad - 2g(2x+y) - 2g(2x-y)\Big] + d[g(2z+w)+g(2z-w)]\\
&= \left(a-\frac{c}{12}\right)[g(2x+y)+g(2x-y)] + \left(d-\frac{b}{12}\right)[g(2z+w)+g(2z-w)]\\
&\quad + \frac{b}{24}\Big[g(2(2x+z)+(2y+w)) + g(2(2x+z)-(2y+w))\\
&\quad - g(2(2x-z)+(2y-w)) - g(2(2x-z)-(2y-w))\Big]\\
&\quad + \frac{c}{24}\Big[g(2(x+2z)+(y+2w)) + g(2(x+2z)-(y+2w))\\
&\quad + g(2(x-2z)+(y-2w)) + g(2(x-2z)-(y-2w))\Big]\\
&= \left(2a-\frac{c}{6}\right)[g(x+y)+g(x-y)+6g(x)]\\
&\quad + \left(2d-\frac{b}{6}\right)[g(z+w)+g(z-w)+6g(z)]\\
&\quad + \frac{b}{12}\Big[g(2x+z+2y+w)+g(2x+z-2y-w)+6g(2x+z)\\
&\quad - g(2x-z+2y-w) - g(2x-z-2y+w) - 6g(2x-z)\Big]\\
&\quad + \frac{c}{12}\Big[g(x+2z+y+2w)+g(x+2z-y-2w)+6g(x+2z)\\
&\quad + g(x-2z+y-2w)+g(x-2z-y+2w)+6g(x-2z)\Big]\\
&= 2ag(x+y)\\
&\quad + \frac{b}{12}\Big[g(2x+2y+z+w) - g(2x+2y-z-w) - 2g(z+w)\Big]\\
&\quad + \frac{c}{12}\Big[g(x+y+2z+2w) + g(x+y-2z-2w) - 2g(x+y)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2dg(z+w) + 2ag(x-y) \\
&+ \frac{b}{12}\Big[g(2x-2y+z-w) - g(2x-2y-z+w) - 2g(z-w)\Big] \\
&+ \frac{c}{12}\Big[g(x-y+2z-2w) + g(x-y-2z+2w) - 2g(x-y)\Big] \\
&+ 2dg(z-w) + 12ag(x) + \frac{b}{2}\Big[g(2x+z) - g(2x-z) - 2g(z)\Big] \\
&+ \frac{c}{2}\Big[g(x+2z) + g(x-2z) - 2g(x)\Big] + 12dg(z) \\
= {}& 2f(x+y, z+w) + 2f(x-y, z-w) + 12f(x,z)
\end{aligned}
$$

for all $x, y, z, w \in X$.

Letting $x = y = 0$ in (2), we get $g(0) = 0$. Putting $y = x$ in (2), we obtain $g(3x) = 27g(x)$ for all $x \in X$. Setting $x = 0$ in (2), we obtain that g is an odd mapping. If $a + b + c + d = 1$, by (4), we have

$$
\begin{aligned}
f(x,x) &= ag(x) + \frac{b}{24}\Big[g(3x) - g(x) - 2g(x)\Big] + \frac{c}{24}\Big[g(3x) + g(-x) - 2g(x)\Big] + dg(x) \\
&= (a+b+c+d)g(x) = g(x)
\end{aligned}
$$

for all $x \in X$. □

In the following theorem, we find out the general solution of the 2-variable cubic functional equation (1).

Theorem 3. *A mapping $f : X \times X \to Y$ satisfies* (1) *if and only if there exist two symmetric multi-additive mappings $S_1, S_2 : X \times X \times X \to Y$ and two multi-additive mappings $M_1, M_2 : X \times X \times X \to Y$ such that*

$$f(x,y) = S_1(x,x,x) + M_1(y,x,x) + M_2(x,y,y) + S_2(y,y,y),$$

$M_1(x,y,z) = M_1(x,z,y)$ and $M_2(x,y,z) = M_2(x,z,y)$ for all $x, y, z \in X$.

Proof. We first assume that f is a solution of (1). Define $f_1, f_2 : X \to Y$ by $f_1(x) := f(x,0)$ and $f_2(x) := f(0,x)$ for all $x \in X$. Since f_1 and f_2 satisfy (2), by [2], there exist two symmetric multi-additive mappings $S_1, S_2 : X \times X \times X \to Y$ such that $f_1(x) = S_1(x,x,x)$ and $f_2(x) = S_2(x,x,x)$ for all $x \in X$. Define $h, h_1, h_2 : X \times X \to Y$ by

$$
\begin{aligned}
h(x,y) &:= f(x,y) - [f(x,0) + f(0,y)], \\
h_1(y,x) &:= \frac{1}{2}[h(x,y) + h(-x,y)],
\end{aligned}
$$

$$h_2(x,y) := \frac{1}{2}[h(x,y) + h(x,-y)]$$

for all $x, y \in X$. One can easily verify that h_1 and h_2 satisfy the systems of functional equations

$$h_1(x+y,z) = h_1(x,z) + h_1(y,z), \quad h_1(x,y+z) + h_1(x,y-z) = 2h_1(x,y) + 2h_1(x,z)$$

and

$$h_2(x+y,z) = h_2(x,z) + h_2(y,z), \quad h_2(x,y+z) + h_2(x,y-z) = 2h_2(x,y) + 2h_2(x,z)$$

for all $x, y, z \in X$. By [3], there exist two multi-additive mappings $M_1, M_2 : X \times X \times X \to Y$ such that $h_1(x,y) = M_1(x,y,y)$, $h_2(x,y) = M_2(x,y,y)$ and $M_1(x,y,z) = M_1(x,z,y)$, $M_2(x,y,z) = M_2(x,z,y)$ for all $x, y, z \in X$. Letting $x = z = 0$ in (1) and using $f(0,0) = 0$, we get $f(y,w) + f(-y,-w) = 0$ for all $y, w \in X$. Hence, by the above equality, we have

$$\begin{aligned} f(x,y) &= f(x,0) + h(x,y) + f(0,y) \\ &= S_1(x,x,x) + \frac{1}{2}[h(x,y) + h(-x,y)] + \frac{1}{2}[h(x,y) + h(x,-y)] + S_2(y,y,y) \\ &= S_1(x,x,x) + h_1(y,x) + h_2(x,y) + S_2(y,y,y) \\ &= S_1(x,x,x) + M_1(y,x,x) + M_2(x,y,y) + S_2(y,y,y) \end{aligned}$$

for all $x, y \in X$.

Conversely, we assume that there exist two symmetric multi-additive mappings $S_1, S_2 : X \times X \times X \to Y$ and two multi-additive mappings $M_1, M_2 : X \times X \times X \to Y$ such that

$$f(x,y) = S_1(x,x,x) + M_1(y,x,x) + M_2(x,y,y) + S_2(y,y,y),$$

$M_1(x,y,z) = M_1(x,z,y)$ and $M_2(x,y,z) = M_2(x,z,y)$ for all $x, y, z \in X$. Since M_1, M_2 are multi-additive and S_1, S_2 are symmetric multi-additive, we have

$$\begin{aligned} &f(2x+y, 2z+w) + f(2x-y, 2z-w) \\ &= S_1(2x+y, 2x+y, 2x+y) + M_1(2z+w, 2x+y, 2x+y) \\ &\quad + M_2(2x+y, 2z+w, 2z+w) + S_2(2z+w, 2z+w, 2z+w) \\ &\quad + S_1(2x-y, 2x-y, 2x-y) + M_1(2z-w, 2x-y, 2x-y) \\ &\quad + M_2(2x-y, 2z-w, 2z-w) + S_2(2z-w, 2z-w, 2z-w) \\ &= 8S_1(x,x,x) + 12S_1(x,x,y) + 6S_1(x,y,y) + S_1(y,y,y) \\ &\quad + 8M_1(z,x,x) + 8M_1(z,x,y) + 2M_1(z,y,y) + 4M_1(w,x,x) \end{aligned}$$

$$
\begin{aligned}
&+ 4M_1(w,x,y) + M_1(w,y,y) + 8M_2(x,z,z) + 8M_2(x,z,w) \\
&+ 2M_2(x,w,w) + 4M_2(y,z,z) + 4M_2(y,z,w) + M_2(y,w,w) \\
&+ 8S_2(z,z,z) + 12S_2(z,z,w) + 6S_2(z,w,w) + S_2(w,w,w) \\
&+ 8S_1(x,x,x) - 12S_1(x,x,y) + 6S_1(x,y,y) - S_1(y,y,y) \\
&+ 8M_1(z,x,x) - 8M_1(z,x,y) + 2M_1(z,y,y) - 4M_1(w,x,x) \\
&+ 4M_1(w,x,y) - M_1(w,y,y) + 8M_2(x,z,z) - 8M_2(x,z,w) \\
&+ 2M_2(x,w,w) - 4M_2(y,z,z) + 4M_2(y,z,w) - M_2(y,w,w) \\
&+ 8S_2(z,z,z) - 12S_2(z,z,w) + 6S_2(z,w,w) - S_2(w,w,w) \\
= {}& 4[4S_1(x,x,x) + 3S_1(x,y,y) + 4M_1(z,x,x) \\
&+ M_1(z,y,y) + 2M_1(w,x,y) + 4M_2(x,z,z) + M_2(x,w,w) \\
&+ 2M_2(y,z,w) + 4S_2(z,z,z) + 3S_2(z,w,w)] \\
= {}& 2[S_1(x,x,x) + 3S_1(x,x,y) + 3S_1(x,y,y) + S_1(y,y,y) \\
&+ M_1(z,x,x) + 2M_1(z,x,y) + M_1(z,y,y) + M_1(w,x,x) \\
&+ 2M_1(w,x,y) + M_1(w,y,y) + M_2(x,z,z) + 2M_2(x,z,w) \\
&+ M_2(x,w,w) + M_2(y,z,z) + 2M_2(y,z,w) + M_2(y,w,w) \\
&+ S_1(z,z,z) + 3S_1(z,z,w) + 3S_1(z,w,w) + S_1(w,w,w) \\
&+ S_1(x,x,x) - 3S_1(x,x,y) + 3S_1(x,y,y) - S_1(y,y,y) \\
&+ M_1(z,x,x) - 2M_1(z,x,y) + M_1(z,y,y) - M_1(w,x,x) \\
&+ 2M_1(w,x,y) - M_1(w,y,y) + M_2(x,z,z) - 2M_2(x,z,w) \\
&+ M_2(x,w,w) - M_2(y,z,z) + 2M_2(y,z,w) - M_2(y,w,w) \\
&+ S_1(z,z,z) - 3S_1(z,z,w) + 3S_1(z,w,w) - S_1(w,w,w) \\
&+ 12[S_1(x,x,x) + M_1(z,x,x) + M_2(x,z,z) + S_2(z,z,z)] \\
= {}& 2[S_1(x+y,x+y,x+y) + M_1(z+w,x+y,x+y) \\
&+ M_2(x+y,z+w,z+w) + S_2(z+w,z+w,z+w) \\
&+ S_1(x-y,x-y,x-y) + M_1(z-w,x-y,x-y) \\
&+ M_2(x-y,z-w,z-w) + S_2(z-w,z-w,z-w)] \\
&+ 12[S_1(x,x,x) + M_1(z,x,x) + M_2(x,z,z) + S_2(z,z,z)] \\
= {}& 2f(x+y,z+w) + 2f(x-y,z-w) + 12f(x,z)
\end{aligned}
$$

for all $x, y, z, w \in X$. □

Let Y be complete and let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying

$$\tilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty \tag{5}$$

for all $x, y, z, w \in X$.

Theorem 4. *Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(2x+y, 2z+w) + f(2x-y, 2z-w) \\ &- 2f(x+y, z+w) - 2f(x-y, z-w) - 12f(x,z)\| \le \varphi(x, y, z, w) \end{aligned} \tag{6}$$

for all $x, y, z, w \in X$. Then there exists a unique cubic mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \le \tilde{\varphi}(x, 0, y, 0) \tag{7}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \lim_{j \to \infty} \frac{1}{8^j} f(2^j x, 2^j y)$$

for all $x, y \in X$.

Proof. Letting $y = 0$ and $w = 0$ in (6), we have

$$\left\| f(x, z) - \frac{1}{8} f(2x, 2z) \right\| \le \frac{1}{16} \varphi(x, 0, z, 0)$$

for all $x, z \in X$. Thus we obtain

$$\left\| \frac{1}{8^j} f(2^j x, 2^j z) - \frac{1}{8^{j+1}} f(2^{j+1} x, 2^{j+1} z) \right\| \le \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x, 0, 2^j z, 0)$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{8^j} f(2^j x, 2^j y) - \frac{1}{8^{j+1}} f(2^{j+1} x, 2^{j+1} y)] \right\| \le \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x, 0, 2^j y, 0)$$

for all $x, y \in X$ and all j. For given integers $l, m (0 \le l < m)$, we get

$$\left\| \frac{1}{8^l} f(2^l x, 2^l y) - \frac{1}{8^m} f(2^m x, 2^m y) \right\| \le \sum_{j=l}^{m-1} \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x, 0, 2^j y, 0) \tag{8}$$

for all $x, y \in X$. By (8), the sequence $\{\frac{1}{8^j} f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{8^j} f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j \to \infty} \frac{1}{8^j} f(2^j x, 2^j y)$$

for all $x, y \in X$. By (6), we have

$$\left\| \frac{1}{8^j} f(2^j(2x+y), 2^j(2z+w)) + \frac{1}{8^j} f(2^j(2x-y), 2^j(2z-w)) \right.$$
$$\left. - \frac{2}{8^j} f(2^j(x+y), 2^j(z+w)) - \frac{2}{8^j} f(2^j(x-y), 2^j(z-w)) - \frac{12}{8^j} f(2^j x, 2^j z) \right\|$$
$$\leq \frac{1}{8^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w)$$

for all $x, y, z, w \in X$ and all j. Letting $j \to \infty$ and using (5), we see that F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (8), one can obtain the inequality (7). If $G : X \times X \to Y$ is another cubic mapping satisfying (7), we obtain that

$$\begin{aligned} &\|F(x,y) - G(x,y)\| \\ &= \frac{1}{8^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\leq \frac{1}{8^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + \frac{1}{8^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\leq \frac{2}{8^n} \tilde{\varphi}(2^n x, 0, 2^n y, 0) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for all $x, y \in X$. Hence the mapping F is the unique cubic mapping, as desired. □

REFERENCES

1. J. Aczél & J. Dhombres: *Functional Equations in Several Variables.* Cambridge Univ. Press, Cambridge, 1989.
2. K.-W. Jun & H.-M. Kim: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. *J. Math. Anal. Appl.* **274** (2002), 867–878.
3. W.-G. Park, J.-H. Bae & B.-H. Chung: On an additive-quadratic functional equation and its stability. *J. Appl. Math. & Computing* **18** (2005), 563–572.

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