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COVERING COVER PEBBLING NUMBER OF A HYPERCUBE & DIAMETER d GRAPHS

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ABSTRACT. A pebbling step on a graph consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. The covering cover pebbling number of a graph is the smallest number of pebbles, such that, however the pebbles are initially placed on the vertices of the graph, after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of G. In this paper we find the covering cover pebbling number of n-cube and diameter two graphs. Finally we give an upperbound for the covering cover pebbling number of graphs of diameter d.

1. INTRODUCTION

Pebbling, one of the latest evolutions in graph theory proposed by Lagarias and Saks has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hurlbert published a survey of pebbling results in [4]. Given a connected graph $G = (V, X)$, where V is the set of all vertices and X is the set of all edges, we distribute certain number of pebbles on the vertices in some configuration. Precisely, a configuration on a graph G is a function from $V(G)$ to $N \cup \{0\}$ representing a placement of pebbles on G. The size of the configuration is the total number of pebbles placed on the vertices. A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In pebbling, a target vertex is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that, regardless of their initial placement and regardless of the target vertex, we can pebble target vertex is called the pebbling number of G. In cover pebbling, the aim is to cover all the vertices with pebbles, *i. e.*, to move a pebble to every vertex of the graph simultaneously. The

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minimum number of pebbles required such that, regardless of their initial placement on G, there is a sequence of pebbling moves, at the end of which, every vertex has at least one pebble on it, is called the cover pebbling number of G. A set of vertices K in G is a covering, if every edge of G has at least one end in K. A covering K is called a minimum covering of G, if G has no covering K' with $|K'| < |K|$. A subset S of V of a graph G is an independent set if no two vertices of S are adjacent in G. The covering cover pebbling number, denoted by $\sigma(G)$, of a graph G is the smallest number of pebbles, such that, however the pebbles are initially placed on the vertices of the graph, after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of G [6]. The pebbles may be placed on any of the vertices of G.

Different coverings may be produced for different initial configurations of pebbles, which is one of the causes that makes this problem, a little difficult. For example, take a cycle C_4 on four vertices, namely v_1, v_2, v_3 and v_4. Place four pebbles only

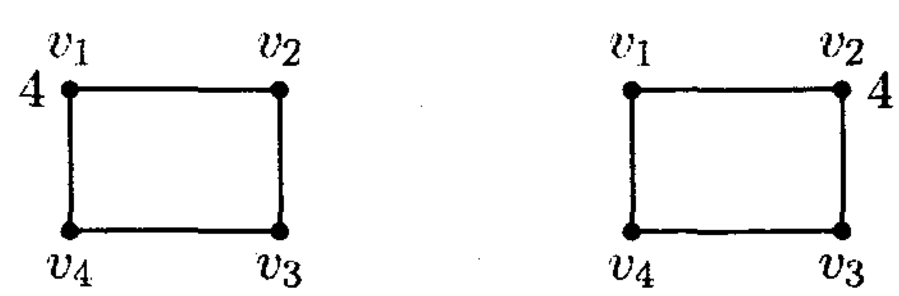

Figure 1.1. An example where two different initial configurations produce different coverings

on v_1. For C_4 on the left, after a sequence of pebbling moves, the vertices v_2 and v_4 are pebbled so that the set $\{v_2, v_4\}$ is a covering of C_4 whereas for C_4 on the right, if four pebbles are placed on v_2, after a sequence of pebbling moves, the vertices v_1 and v_3 are pebbled so that the set $\{v_1, v_3\}$ is a covering of C_4. This is one justification that covering cover pebbling is nontrivial.

In cover pebbling, after all pebbling moves, we reach a stage, at which every vertex of a graph has at least one pebble on it. In covering cover pebbling, after all pebbling moves, we reach a stage, at which the set of vertices with pebbles form a covering of the graph G.

2. The Covering Cover Pebbling Number of Hypercubes

The n-cube Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$, where K_2 is the complete graph on two vertices. Thus Q_n has 2^n vertices each of which may be labeled $a_1 a_2 \cdots a_n$, where each a_i is either 0 or 1. Two vertices of Q_n are adjacent if their binary representations differ at exactly one place.

First, we find the covering cover pebbling number of Q_2. Fig. 2.1 shows the 2-cube appropriately labeled.

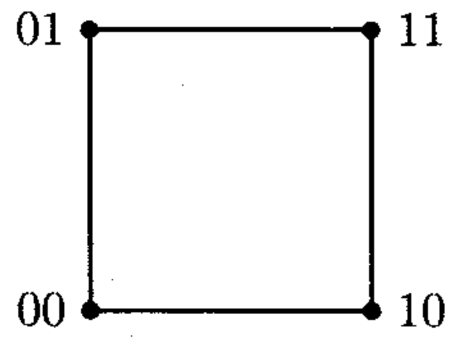

Figure 2.1. Q_2

Theorem 2.1. *The covering cover pebbling number of* Q_2 *is* $\sigma(Q_2) = 4$.

Proof. A single vertex covers two edges which are incident on it. Since there are four edges, the minimum covering C of Q_2 is a set of two vertices and clearly the two vertices are opposite corners (*i. e.*, {01, 10} or {00, 11}).

First, we will show that four pebbles, placed on one vertex, can produce a covering of Q_2, after a sequence of pebbling moves. Next, we will show that, three pebbles, placed on one vertex, cannot produce a covering. Then we will show that three pebbles can produce a covering if they are placed on more than one vertex.

Suppose four pebbles are placed on a vertex, say v. Then we move a pebble to each of the two vertices, which are adjacent to v and thus covering is produced. Next, let three pebbles be placed on a vertex. Thus a pebble can be moved to one of its adjacent vertices. So, there is an edge between the two unpebbled vertices, which remains uncovered. Suppose three pebbles are placed on more than one vertex. Clearly, two opposite corners can be pebbled.

So, placing all pebbles on one vertex is the worst initial configuration and in this configuration, at least four pebbles are needed in order to cover all the edges of Q_2. Therefore $\sigma(Q_2) = 4$. □

Next we consider Q_3. Fig. 2.2 shows the 3-cube.

Theorem 2.2. *The covering cover pebbling number of* Q_3 *is* $\sigma(Q_3) = 13$.

Proof. Every vertex of Q_3 covers three edges, which are incident on it. Thus, to cover all the twelve edges of Q_3, we need at least four vertices, such that, each of which is non-adjacent to every other, in a covering of Q_3. Clearly, the two minimum coverings of Q_3 are $C_1 = \{010, 111, 001, 100\}$ and $C_2 = \{011, 110, 000, 101\}$. Note

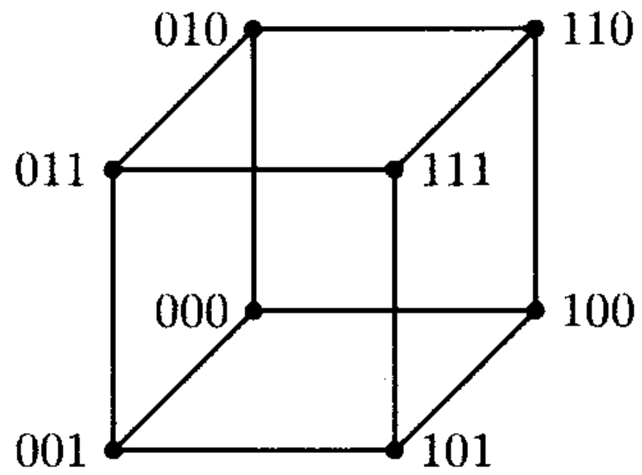

Figure 2.2. Q_3

that every vertex of C_1 is adjacent to three vertices of C_2 and every vertex of C_2 is adjacent to three vertices of C_1.

First, we will show that thirteen pebbles placed on one vertex can cover all the edges of Q_3 after a sequence of pebbling moves. Next, we will show that twelve pebbles placed on one vertex is not enough to produce a covering of Q_3. Then we will show that any configuration of twelve pebbles with at least two occupied vertices can cover all the edges of Q_3.

STEP 1: Suppose thirteen pebbles are placed on a vertex, say v. Without loss of generality, we assume $v \in C_1$. As every other vertex of C_1 is at a distance of two from v, we use twelve pebbles to pebble the other three vertices of C_1. This leaves a pebble on v and thus the covering C_1 is produced.

STEP 2: Let twelve pebbles be placed on a vertex, say v. Suppose $v \in C_1$. The covering C_1 cannot be produced as we need thirteen pebbles from v for C_1 to be produced. Three vertices of C_2 are adjacent to v but the fourth one is at a distance of three from v and so fourteen pebbles would be needed from v to produce the covering C_2. Thus, twelve pebbles placed on a vertex is not sufficient to produce a covering.

STEP 3: Now, let twelve pebbles be placed on two vertices v_1 and v_2. Suppose $v_1, v_2 \in C_1$. Let v_3 and v_4 be the other two vertices of C_1. Clearly either v_1 or v_2 has at least five pebbles. Since every vertex of C_2 is adjacent to three vertices of C_1, there is a vertex, say w of C_2, which is adjacent to v_1, v_2 and v_3. So, we move two pebbles to w and hence we pebble v_3. After these moves either v_1 or v_2 has at least five pebbles or both have at least three pebbles. Now, there is a vertex w_1 of C_2 adjacent to v_1, v_2 and v_4 and thus we move two pebbles to w_1 without making v_1 or v_2 empty and so we move a pebble to v_4. Thus the covering C_1 is produced.

Suppose $v_1 \in C_1$ and $v_2 \in C_2$ and m_1, m_2 are number of pebbles placed on v_1, v_2 respectively. Let $m_1 \geq m_2$. Clearly $m_1 \geq 6$. If v_1 and v_2 are adjacent then there is a vertex, say, v of C_1 which is non-adjacent to v_2. As v is at a distance of two from v_1, we use four pebbles from v_1 to pebble v. After this move, we keep one pebble on v_1 and move the rest to v_2. As the other two unpebbled vertices of C_1 are adjacent to v_2, we need only four pebbles on v_2 to pebble them. Also $\frac{m_1-5}{2} + m_2 \geq \frac{7+m_2}{2} \geq 4$ as $m_2 \geq 1$. Thus the covering C_1 is produced. If v_1 and v_2 are non-adjacent then v_2 is at a distance of three from v_1 and the other three vertices of C_2 are adjacent to v_1. So, we pebble the three unpebbled vertices of C_2 using six pebbles on v_1 and thus the covering C_2 is produced in this case.

STEP 4: Let twelve pebbles be placed on three vertices v_1, v_2 and v_3. Suppose $v_1, v_2, v_3 \in C_1$. We produce the covering C_1 by pebbling the only unpebbled vertex, say w, of C_1 as follows: If there is a vertex with at least five pebbles then we move a pebble to w as it is at a distance of two from every other vertex of C_1. Otherwise, there are at least two vertices, say v_1 and v_2, each with at least three pebbles. We have a vertex, say u of C_2 which is adjacent to v_1, v_2 and w and hence we move two pebbles, *i. e.*, one from v_1, another from v_2 to u and so we move a pebble to w.

Now, if $v_1, v_2 \in C_1$ and $v_3 \in C_2$. Suppose both v_1 and v_2 are adjacent to v_3 and each of v_1, v_2 has exactly two pebbles on it. As v_1 is adjacent to three vertices of C_2 we pebble an unpebbled vertex of C_2 using the two pebbles on v_1. Similarly we pebble another unpebbled vertex of C_2 using the two pebbles on v_2. We pebble the other unpebbled vertex of C_2 by using four pebbles on v_3 and thus the covering C_2 is produced in this case. In all the other cases, we produce the covering C_1 as follows: If v_3 has only one pebble on it, then in all possible distribution of eleven pebbles on v_1 and v_2, we pebble the other two unpebbled vertices of C_1 using eight pebbles without making v_1, v_2 empty as in Step 3. If v_3 has more than one pebble on it and if we consider the case that each of v_1 and v_2 has exactly one pebble on it, then as there is at least one unpebbled vertex of C_1 adjacent to v_3 we pebble it using two pebbles from v_3 and the other unpebbled vertex of C_1 is at a distance of at most three from v_3 and so we use at most eight pebbles to pebble it. Thus the covering C_1 is produced. Otherwise, either v_1 or v_2 has at least two pebbles. If we consider the worst case of having one pebble on v_1, two pebbles on v_2 then we use two pebbles from v_3 to pebble an unpebbled vertex of C_1 as v_3 is adjacent to three vertices of C_1 and so we use at most six pebbles from v_3 (as v_2 has two pebbles) to pebble the other unpebbled vertex of C_1 as it is either adjacent to v_3 or at a distance three from v_3.

STEP 5: Now, let twelve pebbles be placed on four vertices v_1, v_2, v_3 and v_4. Suppose v_1, v_2, $v_3 \in C_1$ and $v_4 \in C_2$. If each of v_1,v_2, v_3, has exactly two pebbles on it, we produce the covering C_2 as follows: Clearly, at least two pebbled vertices of C_1 are adjacent to v_4. We choose one, say v_1, among them. Now there is an unpebbled vertex of C_2 which is at a distance of three from v_1. We pebble this vertex by using the two pebbles on v_2 as it is adjacent to both v_2 and v_3. Now, among the two unpebbled vertices which are adjacent to v_1, if one is at a distance of three from v_3, we pebble that vertex using the two pebbles on v_1 and hence the other unpebbled vertex is pebbled by using the two pebbles on v_3 as v_3 is adjacent to that vertex. If not, both the unpebbled vertices will be adjacent to v_3 also and hence one is pebbled using the two pebbles on v_1 and the other by using the pebbles on v_3 and thus C_2 is produced. If at least one of v_1, v_2, v_3 has either one or more than two pebbles, we produce the covering C_1 by pebbling the only one unpebbled vertex, say w of C_1 as follows: We assume w is non-adjacent to v_4. If either v_1 or v_2 or v_3 has at least five pebbles then we are done. If there are at least two of v_1, v_2, v_3, say v_1 and v_2, each with at least three pebbles, then as there is a vertex, say u, of C_2 adjacent to v_1, v_2 and w_1, we move two pebbles to u and hence we move a pebble to w. If not, suppose each of v_1, v_2 and v_3 has exactly one pebble on it, then since v_4 has nine pebbles and w is at a distance of three from v_4 we move a pebble to w. Among all other cases, we consider the worst case that two of v_1, v_2 and v_3, say v_1 and v_2, each has exactly two pebbles and therefore v_3 has either three or four pebbles. Therefore v_4 has at least four pebbles and thus we move two pebbles to v_3 and now v_3 has at least five pebbles. We use only four pebbles from v_3 to pebble w and thus C_1 is produced. Next, we assume that w is adjacent to v_4. If v_4 has at least two pebbles then we are done. Otherwise v_4 has at most one pebble and therefore eleven pebbles are distributed on v_1, v_2 and v_3. In this case, clearly there is a vertex with at least five pebbles or there are two vertices each with at least three pebbles and so we can pebble w.

Suppose v_1, $v_2 \in C_1$ and v_3, $v_4 \in C_2$. We produce either C_1 or C_2 in this case. We produce the covering C_1 by pebbling the two unpebbled vertices, say u_1 and u_2 of C_1 as follows: First we assume if each of v_3 and v_4 has at least two pebbles. Suppose u_1 is adjacent to v_3. If u_2 is also adjacent to v_3 first we pebble u_2 using the two pebbles from v_3 in the case if u_2 is non-adjacent to v_4 and then we pebble u_1 using the two pebbles from v_4 as it is adjacent to v_4. Otherwise we pebble u_2 using the two pebbles from v_4 and hence we pebble u_1 using the two pebbles from

v_3. Next, if either v_3 or v_4, say v_3, has at least two pebbles, we pebble any one of u_1, u_2 which is adjacent to v_3 using two pebbles from v_3 and the other using at most four pebbles from v_3 since the other vertex is adjacent to v_4 and v_4 has already one pebble, we add one more pebble to v_4 using only four pebbles from v_3. If v_3 and v_4 have exactly one pebble on each of them, then either v_1 or v_2 or both v_1 and v_2 will contain at least three pebbles. Suppose v_1 contains at least three pebbles. Now, as at least one of the vertices v_3, v_4, say v_3, is adjacent to v_1, we pebble v_3 using two pebbles from v_1. Now v_3 contains two pebbles and v_3 is adjacent to at least one of the vertices u_1, u_2, say u_1, and so we pebble u_1. After this move, clearly, either v_1 or v_2 contains at least five pebbles or both v_1 and v_2 contain at least three pebbles. Now we choose the vertex of C_2 which is adjacent to v_1, v_2 and u_2 and we place two pebbles on it and so we move a pebble to u_2.

STEP 6: Suppose twelve pebbles are distributed on five vertices, say v_i, $i = 1$ to 5. Without loss of generality we assume $v_1, v_2, v_3 \in C_1$ and $v_4, v_5 \in C_2$. We pebble the other unpebbled vertex, say w, of C_1 as follows: Clearly w is adjacent to either v_4 or v_5 or both v_4 and v_5. Suppose w is adjacent to v_4 and is at a distance of three from v_5. If v_4 has at least two pebbles then we are done. Otherwise if v_4 is adjacent to a pebbled vertex with at least three pebbles or if v_5 has at least four pebbles then we are done. If v_5 has three pebbles and if two vertices of C_1, say v_1 and v_2, have exactly two pebbles on each of them, then we choose either v_1 or v_2, say v_1, which is adjacent to v_4 and we put one more pebble on v_1 using two pebbles from v_5 and then we move a pebble to v_4. Now, v_4 has two pebbles and so we pebble w. If all of the above cases fail then v_3 will contain at least five pebbles and so we are done. Suppose w is adjacent to both v_4 and v_5 and both v_4 and v_5 contain exactly one pebble on each of them. Clearly, at least one of the vertices v_1, v_2, v_3, say v_1, contains at least three pebbles. Also, either v_4 or v_5 or both v_4 and v_5 are adjacent to v_1 and so we choose one of the vertices v_4, v_5 which is adjacent to v_1 and we place one more pebble on it and so we pebble w.

STEP 7: Suppose twelve pebbles are distributed on six vertices v_i, $i = 1$ to 6. Without loss of generality we take $v_1, v_2, v_3 \in C_1$ and $v_4, v_5, v_6 \in C_2$. Let $w_1 \in C_1$ and $w_2 \in C_2$ be the unpebbled vertices. First we assume w_1 and w_2 are adjacent. Clearly w_1 is adjacent to two pebbled vertices of C_2 and w_2 is adjacent to two pebbled vertices of C_1. If any one of the adjacent vertices of w_1 has at least two pebbles, we pebble w_1 and thus C_1 is produced. Otherwise, if any one of the adjacent vertices of w_2 has at least two pebbles, we pebble w_2 and so C_2 is produced. Otherwise each of the above four pebbled vertices has exactly one pebble on it. Clearly the other two pebbled vertices, one from C_1 and the other from C_2, are adjacent. If one of them, say the one from C_1, has at least five pebbles then we pebble w_1 using four pebbles and thus C_1 is produced. Otherwise both of them have exactly four pebbles on them. So we pebble either w_1 or w_2.

Suppose w_1 and w_2 are non-adjacent. Clearly w_1 is adjacent to three pebbled vertices of C_2 and w_2 is adjacent to three pebbled vertices of C_1. Clearly there is a vertex with at least two pebbles. Suppose there is a vertex of C_1 with at least two pebbles then we pebble w_2 and thus C_2 is produced.

We have shown that twelve pebbles suffice to produce a covering when initially placed on two or more vertices. Also we have shown that twelve pebbles placed on one vertex is insufficient to produce a covering of Q_3. Furthermore we have shown that thirteen pebbles placed on one vertex is sufficient to cover all the edges of Q_3.

So, placing all pebbles on one vertex is the worst initial configuration and in this configuration at least thirteen pebbles are needed in order to produce a covering of Q_3.

Therefore $\sigma(Q_3) = 15$. □

Next we find $\sigma(Q_n)$ in general for an n-dimensional cube. For this we use the definition of stacking and the illustration that how any hypercube, Q_n, can be represented as Q_n's arranged as a Q_{n-2} as such in [3].

Definition 2.3 ([3]). Stacking is the idea of "stacking" all of the pebbles on one vertex for the initial configuration. When it is said that stacking holds for a graph, then the worst initial configuration for the graph is when all of the pebbles are stacked on one vertex.

Theorem 2.1 shows stacking holds for Q_2 and Theorem 2.2 shows stacking holds for Q_3.

Next we show stacking holds for a hypercube Q_n. The idea used in [3] is followed here.

For Q_4, it can be represented as four Q_2's arranged as a Q_2 as shown below:

Stacking holds for Q_2 and because Q_4 can be represented in this manner, stacking also holds for Q_4.

This idea can be expanded for all hypercubes. Q_5 can be represented as Q_2's arranged as a Q_3. Since stacking holds for Q_2 and Q_3 then stacking holds for Q_5. Similarly Q_7 can be represented as Q_2's arranged as a Q_5. Stacking holds for Q_2

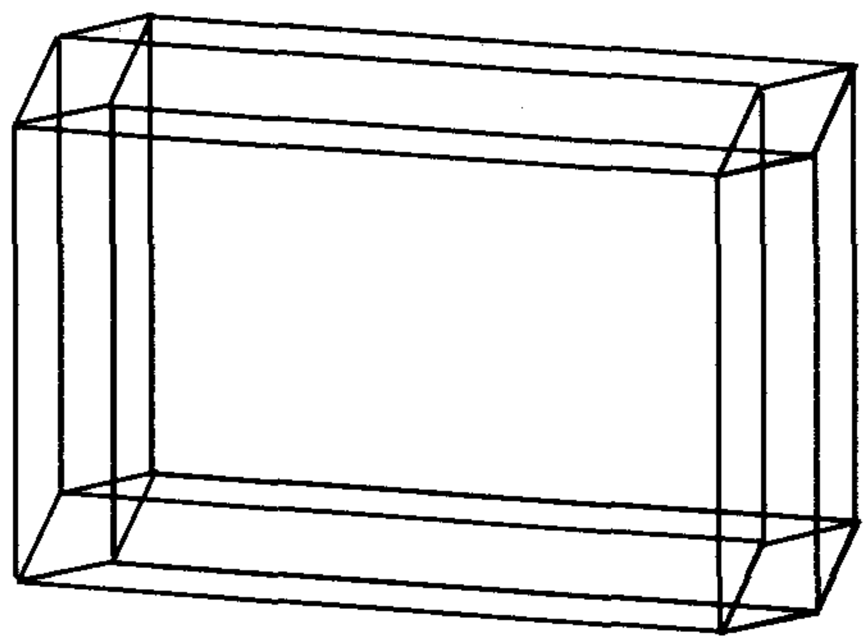

Figure 2.3.

and Q_5, so stacking holds for Q_7. In this way, we let $k = n - 2$, then we can show that Q_k can be represented as Q_2's arranged as a Q_{k-2}, which then leads to Q_n being represented as Q_2's arranged as a Q_k. Using this method it can be shown that stacking holds for any Q_n. Therefore, stacking is true for all hypercubes.

Theorem 2.4. *The covering cover pebbling number of a hypercube Q_n is* $\sigma(Q_n) = \left\lfloor \frac{3^n}{2} \right\rfloor$.

Proof. Clearly Q_n has 2^n vertices and $n2^{n-1}$ edges and each vertex is of degree n. We have shown that stacking is true for Q_n. So, we assume that we place all the pebbles on a single vertex, say v, initially.

CASE (i): n is even. There are n vertices which are adjacent to v, ${}_nC_3$ vertices which are at a distance of three from v, ${}_nC_5$ vertices which are at a distance of five from v, etc., and ${}_nC_{n-1}$ vertices which are at a distance of $n-1$ from v. We assume all these 2^{n-1} vertices form a set C. Clearly C is an independent set and it is a minimum covering of Q_n. Therefore

$$\sigma(Q_n) = {}_nC_1 \cdot 2 + {}_nC_3 \cdot 2^3 + {}_nC_5 \cdot 2^5 + \cdots + {}_nC_{n-1} \cdot 2^{n-1} = \left\lfloor \frac{3^n}{2} \right\rfloor.$$

We note that $Q_n - C$ is also a minimum covering of Q_n but it requires more number of pebbles from v to produce $Q_n - C$ than to produce C.

CASE (ii): n is odd. In this case, we choose in the set C_1 all the ${}_nC_2$ vertices which are at a distance of two from v, all the ${}_nC_4$ vertices which are at a distance of four from v, etc., all the n vertices which are at a distance of $n-1$ from v and finally v also. Clearly C_1 is an independent set and it is a minimum covering of Q_n. Thus

$$\sigma(Q_n) = 1 + {}_nC_2 \cdot 2^2 + {}_nC_4 \cdot 2^4 + \cdots + {}_nC_{n-1} \cdot 2^{n-1} = \left\lfloor \frac{3^n}{2} \right\rfloor.$$

Here also we note that $Q_n - C_1$ is a minimum covering of Q_n but it requires more number of pebbles from v to produce $Q_n - C_1$ than to produce C_1. Thus the proof is complete. □

In the next section, we discuss the covering cover pebbling number of graphs of diameter d.

3. GRAPHS OF DIAMETER d

Definition 3.1 ([7]). Let U and W be subsets of $V(G)$. Let $u \in U$. Then $d(u, W) = \min_{w \in W} d(u, w)$ and $d(U, W) = \min_{u \in U} d(u, W)$.

Theorem 3.2. *Let G be a graph of diameter 2 with n vertices ($n \geq 3$). Then $\sigma(G) \leq 4n - 10$.*

Proof. First, we will exhibit a graph G such that $\sigma(G) > 4n - 11$. Consider the graph G defined as follows: $V(G) = \{v_1, v_2, \ldots v_n\}$ and $E(G) = v_1v_2 \cup E(K_{n-1})$ where K_{n-1} is the complete graph on $n - 1$ vertices $v_2, v_3, \ldots, v_n$. Suppose all the pebbles are placed on vertex v_1. In order to cover all the edges of G, we need to pebble the vertex v_2 and any other $n - 3$ vertices of K_{n-1}. For this we need $4n - 10$ pebbles from the vertex v_1. Hence $\sigma(G) > 4n - 11$. Here is an example for $n = 6$ and $d = 2$.

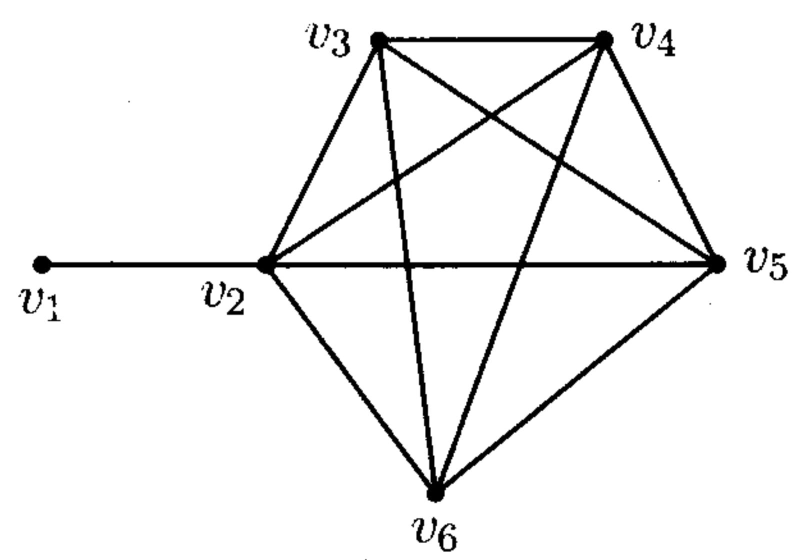

Figure 3.1. A graph where $n = 6$ and $d = 2$ such that $\sigma(G) = 4n - 10$

Suppose we are given a graph G of diameter 2 on n vertices and $4n - 10$ pebbles. To prove the theorem, we will show that after a sequence of pebbling moves every edge of G has at least one of its ends on a vertex that contains pebbles.

For $n = 3$, clearly path P_3 is the only graph of diameter 2. Clearly two pebbles can cover both the edges however the pebbles are distributed on the vertices of P_3.

Now, we assume $n \geq 4$ and we start with a configuration c of $4n - 10$ pebbles. We give a recursively defined algorithm for covering each edge of G through a sequence of pebbling moves. First, to begin the algorithm, we take

$$\begin{aligned} c_0 &= c \\ R_0 &= \{v \in G : c(v) > 0\} \\ S_0 &= \{v \in G : c(v) \geq 3\} \\ T_0 &= V(G) - R_0 \\ Q_0 &= \phi \end{aligned}$$

We will describe our algorithm by defining c_k, R_k, S_k, T_k and Q_k recursively. At each step, we will need to make sure a few conditions hold to ensure that the next step of the algorithm may be performed. For each k, we will insist that:

(1) For every $v \in T_k \cup Q_k$, $c_k(v) = 0$ and for every $v \in R_k$, $c_k(v) > 0$.
(2) $|T_k| = |T_0| - k$.
(3) $S_k = \{v \in G : c_k(v) \geq 3\}$.
(4) R_k, T_k and Q_k are pairwise disjoint and $R_k \cup T_k \cup Q_k = V(G)$.
(5) S_k and T_k are both non-empty. In particular if $d(S_k, T_k) = 2$, then there exists a vertex $v_k \in S_k$ such that $c_k(v_k) > 3$.
(6) For every vertex q of Q_k, every vertex adjacent to q has at least one pebble on it.
(7) c_k is the configuration of pebbles which can be reached from c by a sequence of pebbling moves.

For $k = 0$, only condition (5) is not immediately clear. If $T_0 = \phi$ we are clearly done for c already covers every edge of G. If $S_0 = \phi$, we claim that $c(v) > 0$ at least for $n - 1$ vertices. Suppose $c(v) > 0$ for at most $n - 2$ vertices. Each of these $n - 2$ vertices can have at most two pebbles. Therefore, size of the configuration c is at most $(n-2)2 = 2n-4 < 4n-10$ as $n \geq 4$, raising a contradiction. Therefore $c(v) > 0$ at least for $n - 1$ vertices and these $n - 1$ vertices can cover all the edges of G. So, if $S_0 = \phi$ we are done. Therefore $T_0 \neq \phi$ and $S_0 \neq \phi$. Suppose $d(S_0, T_0) = 2$. Note that T_0 contains at least two vertices, which are adjacent to each other, otherwise all the edges are covered by c. Suppose $c_0(v) = 3$ for all $v \in S_0$. Then the size of the configuration is at most $(n - 2)3 = 3n - 6 < 4n - 10$ for $n \geq 5$ (as the case is trivial for $n = 4$) raising a contradiction. Therefore, there exists some $v_0 \in S_0$ such that $c_0(v_0) \geq 4$.

Suppose for some $k = m+1$ where $m < n-2$ we have defined c_{m+1}, R_{m+1}, S_{m+1} T_{m+1} and Q_{m+1} and the above conditions hold for $k = m+1$. We shall assume that there is some edge which is not covered by c_{m+1}. Thus $|T_{m+1}| \geq 2$, since for an uncovered edge both the end vertices must be in T_{m+1}. Without loss of generality, we assume $d(S_r, T_r) = 1$ for all $r \leq m$.

CASE (i): $d(S_{m+1}, T_{m+1}) = 1$. Choose $v' \in S_{m+1}$ for which $c_{m+1}(v') = 3$ (if exists), otherwise $c_{m+1}(v') > 3$ and choose $w' \in T_{m+1}$ such that $d(v', w') = 1$. We move a pebble to w' if w' is adjacent to at least one vertex of T_{m+1}. Otherwise we take w' in Q_{m+2}.

In the case if we pebble w', let c_{m+2} be the configuration of pebbles resulting from this move. We again let $S_{m+2} = \{v \in G : c_{m+2}(v) \geq 3\}$. If now $c_{m+2}(v') \geq 3$, then $S_{m+2} = S_{m+1}$. Otherwise $S_{m+2} = S_{m+1} \backslash \{v'\}$. We also let $T_{m+2} = T_{m+1} \backslash \{w'\}$, $R_{m+2} = R_{m+1} \cup \{w'\}$ and $Q_{m+2} = Q_{m+1}$. If $|T_{m+2}|$ is either 0 or 1, we are done and so we may assume that $|T_{m+2}| \geq 2$ and there exist two vertices in T_{m+2} such that they are adjacent to each other. Clearly the conditions 1, 2, 3, 4, 6 and 7 are easily seen to hold for $k = m+2$. We claim that $S_{m+2} \neq \phi$. As $m+1$ steps are carried out so far and in each step at most two pebbles are used, the size of the initial configuration is at most $2(m+1)+2\cdot 0+(n-(m+3))2 = 2n-4 < 4n-10$ raising a contradiction. Therefore $S_{m+2} \neq \phi$. Suppose $d(S_{m+2}, T_{m+2}) = 2$. We claim that there exists a vertex v_{m+2} in S_{m+2} such that $c_{m+2}(v_{m+2}) \geq 4$. Suppose not. Then the size of the initial configuration is at most $2(m+1)+2\cdot 0+(n-(m+3))3 = 3n-m-7 < 4n-10$ as $n \geq 4$ and $k \geq 0$, raising a contradiction. Therefore condition 5 also holds when $k = m+2$.

On the other hand, if we take w' in Q_{m+2}, we let $c_{m+2} = c_{m+1}$, $S_{m+2} = S_{m+1}$, $T_{m+2} = T_{m+1} \backslash \{w'\}$, $R_{m+2} = R_{m+1}$ and $Q_{m+2} = Q_{m+1} \cup \{w'\}$. Clearly all the conditions 1, 2, 3, 4, 5, 6 and 7 are easily seen to hold for $k = m+2$.

CASE (ii): $d(S_{m+1}, T_{m+1}) = 2$. We choose $v \in S_{m+2}$ for which $c_{m+1}(v) > 3$ and $w \in T_{m+1}$ such that $d(v, w) = 2$. We pebble w if w is adjacent to at least one vertex of T_{m+1}. Otherwise we take w in Q_{m+2}.

In the case if we pebble w, we let c_{m+2} be the configuration of pebbles resulting from this move. We again let $S_{m+2} = \{v \in G : c_{m+2}(v) \geq 3\}$. If now $c_{m+2} \geq 3$, then $S_{m+2} = S_{m+1}$. We also let $T_{m+2} = T_{m+1} \backslash \{w\}$, $R_{m+2} = R_{m+1} \cup \{w\}$ and $Q_{m+2} = Q_{m+1}$. If $|T_{m+2}|$ is either 0 or 1, we are done and so we assume that $|T_{m+2}| \geq 2$ and there exist two vertices in T_{m+2} such that they are adjacent to each other. Clearly the conditions 1, 2, 3, 4, 6 and 7 are easily seen to hold for $k = m + 2$. Clearly $d(S_{m+2}, T_{m+2}) = 2$. So, we claim that $c_{m+2}(v) \geq 4$ for some $v \in S_{m+2}$. Suppose not. The size of the initial configuration is at most $m \cdot 2 + 1 \cdot 4 + 2 \cdot 0 + (n - (m+3))3 = 3n - m - 5 < 4n - 10$ as $m \leq n - 2$ and $n \geq 4$, raising a contradiction. Here, note that if $|S_{m+2}| = 1$, by making $c_{m+3}(v) = 0$ after the next move, we did not cause any edge from v to become non-covered as every vertex adjacent to v has pebbles on it.

The algorithm continues as long as there is some non-covered edge in G. By condition 2, it must terminate after at most $|T_0| - 1$ steps, because when $n - 1 = |T_0|$, we would have $|T_{n-2}| = 1$ and certainly there could be no non-covered edge in G. Thus the algorithm eventually stops, having created some c_k which covers every edge of G. By property 7, c_k is reachable from c by pebbling moves. □

We conclude this section by conjecturing an analogous result for graphs of diameter d, along with a valid upper bound construction for the conjecture.

Conjecture 3.3. *Let G be a graph of diameter d with n vertices.*
Then $\sigma(G) \leq \left\lfloor \frac{2^{d+1}}{3} \right\rfloor + (n - (d+1))2^d.$

To see that the result is reasonable, we will show that $\sigma(G) > \lfloor \frac{2^{d+1}}{3} \rfloor + (n - (d + 1))2^d - 1$ for some class of graphs with diameter d and n vertices. We construct the following class of graphs with vertices $\{v_1, v_2, \ldots, v_n\}$ as follows: Consider the path P_d of length $d - 1$ formed by the vertices $v_1, v_2, \ldots, v_d$. Connect P_d to the complete graph on $(n - (d - 1))$ vertices $v_{d+1}, v_{d+2}, \ldots, v_n$ at v_d. Here is an example for $n = 10$, $d = 5$.

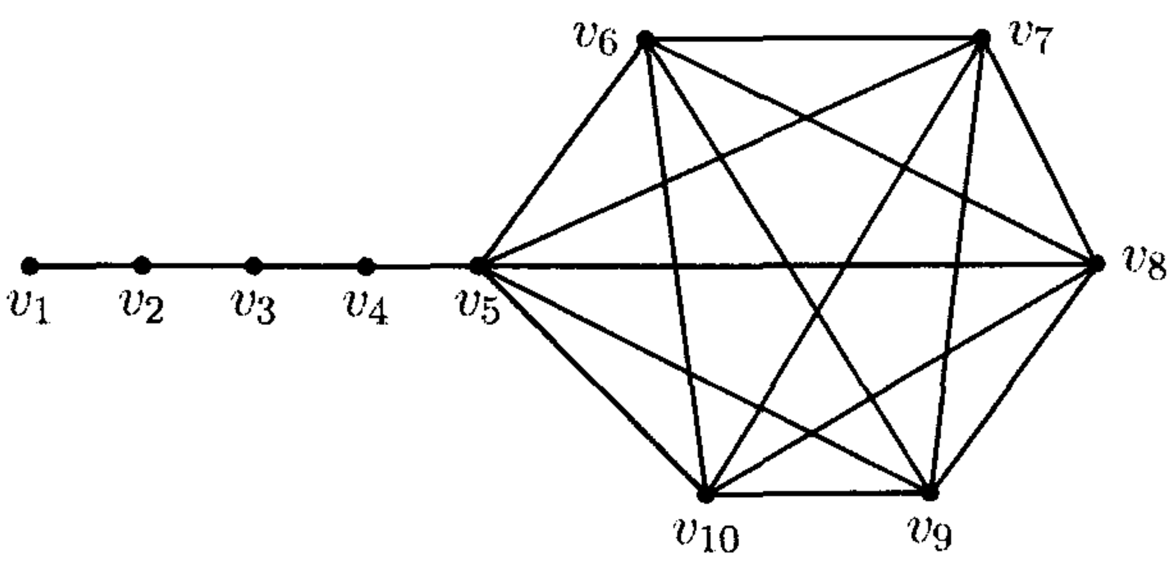

Figure 3.2. A graph where $n = 10$ and $d = 5$ such that $\sigma(G) > \lfloor \frac{2^{d+1}}{3} \rfloor + (n - (d + 1))2^d$

Suppose all the pebbles are placed on vertex v_1. In order to cover all the edges of G, we need to pebble at least the vertices $v_1, v_3, v_5, \ldots, v_d$ of the path and any other $n-d-1$ vertices of the complete graph in the case if d is odd and the vertices $v_2, v_4, \ldots, v_d$ and any other $n-d-1$ vertices of the complete graph in the case if d is even. For this we need $\lfloor \frac{2^{d+1}}{3} \rfloor + \big(n-(d+1)\big)2^d$ pebbles from v_1. □

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