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COMMON FIXED POINT THEOREMS FOR FINITE NUMBER OF MAPPINGS WITHOUT CONTINUITY AND COMPATIBILITY IN MENGER SPACES

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ABSTRACT. The purpose of this paper is to prove some common fixed point theorems for finite number of discontinuous, noncompatible mappings on noncomplete Menger spaces. Our results extend, improve and generalize several known results in Menger spaces. We give formulas for total number of commutativity conditions for finite number of mappings.

1. INTRODUCTION

Sessa [19] generalized the notion of commuting maps given by Jungck [7] and introduced weakly commuting mappings. Further, Jungck [8] introduced more generalized commutativity called compatibility. In 1998, Jungck and Rhoades [10] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true.

Menger [11] introduced the notion of probabilistic metric space, which is generalization of metric space and study of these spaces was expanded rapidly with pioneering work of Schewizer and Sklar [17], [18]. The existence of fixed points for compatible mappings on probabilistic metric space is shown by Mishra [12].

Recently, fixed point theorems in Menger spaces have been proved by many authors including Bylka [1], Pathak, Kang and Baek [13], Stojakovic [24], [25], [26], Hadzic [4], [5], Dadic and Sarapa [3], Rashwan and Hedar [16], Mishra [12], Radu [14], [15]. Sehgal and Bharucha-Reid [20], Cho, Murthy and Stojakovic [2], Sharma and Bagvan [21]Sharma and Deshpande [22].

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Most of the fixed point theorems in Menger spaces deal with conditions of continuity and compatibility or compatibility of type (α) or compatible of type (β).

There are maps which are not continuous but have fixed points. Also weakly compatible maps defined by Jungck and Rhoades [10] are weaker than that of compatibility.

These observations motivated us to prove common fixed point theorem for ten noncompatible, discontinuous mappings in noncomplete Menger spaces. We also extend our results for finite number of mappings. Our main theorems extend, improve and generalize many known results in Menger spaces ([3], [6], [9], [12], [16], [21]-[23], [26]). To prove existence of common fixed point for finite number of mappings some commutativity conditions are required. How many commutativity conditions are necessary? We give answer of this question by giving formulas.

2. PRELIMINARIES

Let R denote the set of reals and R^+ the non-negative reals. A mapping $F : R \to R^+$ is called a distribution function if it is non- decreasing and left continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by L the set of all distribution functions.

A probabilistic metric space is a pair (X, F), where X is non empty set and F is a mapping from $X \times X$ to L.

For $(p, q) \in X \times X$, the distribution function $F(p, q)$ is denoted by $F_{p,q}$. The function $F_{p,q}$are assumed to satisfy the following conditions:

(P$_1$)$F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$,

(P$_2$)$F_{p,q}(0) = 0$ for every $p, q \in X$,

(P$_3$)$F_{p,q}(x) = F_{q,p}(x)$ for every $p, q \in X$,

(P$_4$) if $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$ for every $p, q, r \in X$ and $x, y > 0$.

In metric space (X, d) the metric d induces a mapping $F : X \times X \to L$ such that $F(p, q)(x) = F_{p,q}(x) = H(x - d(p, q))$ for every $p, q \in X$ and $x \in R$, where H is a distributive function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

Definition 1. A function $t : [0,1] \times [0,1] \to [0,1]$ is called a T- norm if it satisfies the following conditions:

(t$_1$) $t(a, 1) = a$ for every $a \in [0,1]$ and $t(0, 0) = 0$,

(t_2) $t(a,b) = t(b,a)$ for every $a, b \in [0,1]$,
(t_3) If $c \geq a$ and $d \geq b$ then $t(c,d) \geq t(a,b)$, for every $a, b, c \in [0,1]$,
(t_4) $t(t(a,b),c) = t(a,t(b,c))$ for every $a, b, c \in [0,1]$.

Definition 2. A Menger space is a triple (X, F, t), where (X, F) is a PM-space and t is a T-norm with the following condition:

(P_5) $F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y))$ for every $p, q, r \in X$ and $x, y \in R^+$.

An important T-norm is the T-norm $t(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$ and this is the unique T-norm such that $t(a,a) \geq a$ for every $a \in [0,1]$. Indeed if it satisfies this condition, we have

$$\begin{aligned}\min\{a,b\} &\leq t(\min\{a,b\}, \min\{a,b\}) \leq t(a,b)\\ &\leq t(\min\{a,b\}, 1) = \min\{a,b\}\end{aligned}$$

Therefore $t = \min$.

In the sequel, we need the following definitions due to Radu [14].

Definition 3. Let (X, F, t) be a Menger space with continuous T- norm t. A sequence $\{x_n\}$ of points in X is said to be convergent to a point $x \in X$ if for every $\varepsilon > 0$, $\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1$.

Definition 4. Let (X, F, t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda) > 0$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ for all $m, n \in N$.

Definition 5. A Menger space (X, F, t) with the continuous T-norm t is said to be complete if every Cauchy sequence in X converges to a point in X.

Theorem A ([17]). Let t be a T- norm defined by $t(a,b) = \min\{a,b\}$. Then the induced Menger space (X, F, t) is complete if a metric space (X, d) is complete.

Definition 6 ([12]). Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASx_n,SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some $u \in X$ as $n \to \infty$.

Definition 7 ([10]). Two maps A and B are said to be weakly compatible if they commute at coincidence point.

Lemma A ([18], [23]). *Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norm and $t(x,x) \geq x$. Suppose for all $x \in [0,1]$ there exists $k \in (0,1)$ such that for all $x > 0$ and $n \in N$, $F_{x_n,x_{n+1}}(kx) \geq F_{x_{n-1},x_n}(x)$.*

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma B ([12]). *Let (X, F, t) be a Menger space. If there exists $k \in (0,1)$ such that for $p, q \in X$, $F_{p,q}(kx) \geq F_{p,q}(x)$. Then $p = q$.*

2. MAIN RESULTS

Theorem 1. *Let $A, B, S, T, I, J, L, U, P$ and Q be self maps on a Menger space (X, F, t) with $t(a,a) \geq a$ for all $a \in [0,1]$, satisfying*

(1.1) $$P(X) \subset ABIL(X), Q(X) \subset STJU(X)$$

(1.2) *there exists $k \in (0,1)$ such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{ABILy,STJUx}(u), F_{Px,STJUx}(u), F_{Qy,ABILy}(u), F_{Qy,STJUx}(\alpha u), F_{Px,ABILy}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $u > 0$,

(1.3) *if one of $P(X)$, $ABIL(X)$, $STJU(X)$, $Q(X)$ is a complete subspace of X*

then

(i) *P and $STJU$ have a coincidence point and*

(ii) *Q and $ABIL$ have a coincidence point.*

Further if

(1.4) $$AB = BA, AI = IA, AL = LA, BI = IB, BL = LB, IL = LI,$$
$$QL = LQ, QI = IQ, QB = BQ, ST = TS, SJ = JS, SU = US,$$
$$TJ = JT, TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP,$$

(1.5) *the pairs $\{P, STJU\}$ and $\{Q, ABIL\}$ are weakly compatible, then*

$A, B, S, T, I, J, L, U, P$ and Q have a unique common fixed point in X.

Proof. By (1.1) since $P(X) \subset ABIL(X)$ for any point $x_0 \in X$ there exists a point x_1 in X such that $Px_0 = ABILx_1$. Since $Q(X) \subset STJU(X)$, for this point x_1 we can choose a point x_2 in X such that $Qx_1 = STJUx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2, ...,$

$$y_{2n} = Px_{2n} = ABILx_{2n+1} \text{ and } y_{2n+1} = Qx_{2n+1} = STJUx_{2n+2}.$$

By (1.2), for all $u > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we have

$$\begin{aligned} &F_{y_{2n+2},\, y_{2n+1}}(ku) \\ &\geq \min\{Fy_{2n},_{y_{2n+1}}(u), F_{y_{2n+2},y_{2n+1}}(u), F_{y_{2n+1},y_{2n}}(u), F_{y_{2n+1},y_{2n+1}}((1-q)u), \\ &\quad F_{y_{2n+2},y_{2n}}((1+q)u)\} \\ &\geq \min\{Fy_{2n},_{y_{2n+1}}(u), F_{y_{2n+2},y_{2n+1}}(u), F_{y_{2n+1},y_{2n}}(u), \\ &\quad F_{y_{2n+1},y_{2n+2}}(qu)\} \\ &\geq \min\{Fy_{2n},_{y_{2n+1}}(u), F_{y_{2n+2},y_{2n+1}}(u), F_{y_{2n+1},y_{2n+2}}(qu)\}. \end{aligned}$$

Since the t-norm is continuous letting q $\rightarrow$1, we have

$$F_{y_{2n+1},y_{2n+2}}(ku) \geq \min\{F_{y_{2n},y_{2n+1}}(u), F_{y_{2n+1},y_{2n+2}}(u)\}.$$

Similarly, we also have

$$F_{y_{2n+2},y_{2n+3}}(ku) \geq \min\{F_{y_{2n+1},y_{2n+2}}(u), F_{y_{2n+2},y_{2n+3}}(u)\}.$$

In general, we have for m = 1, 2, ...

$$F_{y_{m+1},y_{m+2}}(ku) \geq \min\{F_{y_m,y_{m+1}}(u), F_{y_{m+1},y_{m+2}}(u).$$

Consequently, it follows that for m = 1, 2, ..., p = 1, 2, ...

$$F_{y_{m+1},y_{m+2}}(ku) \geq \min\{F_{y_m,y_{m+1}}(k^{-1}u), F_{y_{m+1},y_{m+2}}(k^{-p}u)\}.$$

By noting that $F_{y_{m+1},y_{m+2}}(k^{-p}u) \rightarrow 1$ as $p \rightarrow \infty$, we have for m = 1, 2, ...

$$F_{y_{m+1},y_{m+2}}(ku) \geq F_{y_m,y_{m+1}}(u).$$

Hence by Lemma A, $\{y_n\}$ is a Cauchy sequence in X. Now suppose $STJU(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $STJU(X)$ and has a limit in $STJU(X)$ call it z. Let $w \in STJU^{-1}(z)$. Then $STJUw = z$. We shall use the fact that subsequence $\{y_{2n}\}$ also converges to z. By (1.2), with $\alpha = 1$ we have

$$\begin{aligned} &F_{Pw,y_{2n+1}}(ku) \\ &\geq \min\{F_{y_{2n},STJUw}(u), F_{Pw,STJUw}(u),\ F_{y_{2n+1},y_{2n}}(u), F_{y_{2n+1},STJUw}(u), F_{Pw,y_{2n}}(u)\} \end{aligned}$$

which implies that as $n \rightarrow \infty$, $F_{Pw,z}(ku) \geq F_{Pw,z}(u)$.

Therefore by Lemma B, we have $Pw = z$. Since $STJUw = z$ thus we have $Pw = z = STJUw$ that is w is coincidence point of P and $STJU$. This proves (i).

Since $P(X) \subset ABIL(X), Pw = z$ implies that $z \in ABIL(X)$.

Let $v \in ABIL^{-1}z$. Then $ABILv = z$. By (1.2), with $\alpha = 1$ we have

$$F_{Px_{2n+2},Qv}(ku)$$
$$\geq \min\{F_{ABILv,y_{2n+1}}(u), F_{y_{2n+2},y_{2n+1}}(u)F_{Qv,ABILv}(u), F_{Qv,y_{2n+1}}(u), F_{y_{2n+2},ABILv}(u),$$

which implies that as $n \to \infty$, $F_{Qv,z}(ku) \geq F_{Qv,z}(u)$.

Therefore by Lemma B, we have $Qv = z$. Since $ABILv = z$, we have $Qv = z = ABILv$ that is v is coincidence point of Q and $ABIL$. This proves (ii).

The remaining two cases pertain essentially to the previous cases. Indeed if $P(X)$ or $Q(X)$ is complete then by (1.1), $z \in P(X) \subset ABIL(X)$ or $z \in Q(X) \subset STJU(X)$. Thus (i) and (ii) are completely established.

Since the pair $\{P, STJU\}$ is weakly compatible therefore P and $STJU$ commute at their coincidence point that is $P(STJUw) = (STJU)Pw$ or $Pz = STJUz$.

Since the pair $\{Q, ABIL\}$ is weakly compatible therefore Q and $ABIL$ commute at their coincidence point that is $Q(ABILv) = (ABIL)QvorQz = ABILz$.

Now we prove that $Pz = z$. By (1.2), with $\alpha = 1$ we have

$$F_{Pz,Qx_{2n+1}}(ku)$$
$$\geq \min\{F_{y_{2n},STJUz}(u), F_{Pz,STJUz}(u), F_{y_{2n+1},y_{2n}}(u), \ F_{y_{2n+1},STJUz}(u), F_{Pz,y_{2n}}(u)\}.$$

Proceeding limit as $n \to \infty$, we have

$$F_{Pz,z}(ku) \geq F_{Pz,z}(u).$$

Therefore by Lemma B, we have $Pz = z$. So $Pz = STJUz = z$. By (1.2), with $\alpha = 1$ we have

$$F_{Px_{2n+2},Qz}(ku)$$
$$\geq \min\{F_{ABILz,y_{2n+1}}(u), F_{y_{2n+2},y_{2n+1}}(u), F_{Qz,ABILz}(u), F_{Qz,y_{2n+1}}(u),$$
$$F_{y_{2n+2},ABILz}(u).$$

Proceeding limit as $n \to \infty$, we have

$$F_{z,Qz}(ku) \geq F_{Qz,z}(u).$$

Therefore by Lemma B, we have $Qz = z$, so $Qz = ABILz = z$. By (1.2), with $\alpha = 1$ and using (1.4), we have

$$F_{Pz,Q(Lz)}(ku)$$
$$\geq \min\{F_{ABIL(Lz),STJUz}(u), F_{Pz,STJUz}(u), F_{Q(Lz),ABIL(Lz)}(u), F_{Q(Lz),STJUz}(u),$$

$$F_{Pz,ABIL(Lz)}(u)\}.$$

Thus we have

$$F_{z,Lz}(ku) \geq \min\{F_{Lz,z}(u), F_{z,z}(u), F_{Lz,Lz}(u), F_{Lz,z}(u), F_{Lz,z}(u)\}.$$

Therefore by Lemma B, we have $Lz = z$. Since $ABILz = z$ therefore $ABIz = z$. By (1.2), with $\alpha = 1$ and using (1.4), we have

$$\begin{aligned} &F_{Pz,Q(Iz)}(ku) \\ &\geq \min\{F_{ABIL(Iz),STJUz}(u), F_{Pz,STJUz}(u), F_{Q(Iz),ABIL(Iz)}(u), F_{Q(Iz),STJUz}(u), \\ &\qquad F_{Pz,ABIL(Iz)}(u)\}. \end{aligned}$$

Thus we have

$$\begin{aligned} &F_{Iz,z}(ku) \\ &\geq \min\{F_{Iz,z}(u), F_{z,z}(u), F_{Iz,Iz}(u), F_{Iz,z}(u), F_{Iz,z}(u)\}. \end{aligned}$$

Therefore by Lemma B, we have $Iz = z$. Since $ABIz = z$ therefore $ABz = z$. Now to prove $Bz = z$ we put $x = z$, $y = Bz$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$\begin{aligned} &F_{Pz,Q(Bz)}(ku) \\ &\geq \min\{F_{ABIL(Bz),STJUz}(u), F_{Pz,STJUz}(u), F_{Q(Bz),ABIL(Bz)}(u), F_{Q(Bz),STJUz}(u), \\ &\qquad F_{Pz,ABIL(Bz)}(u)\}. \end{aligned}$$

Thus we have

$$F_{z,Bz}(ku) \geq \min\{F_{Bz,z}(u), F_{z,z}(u), F_{Bz,Bz}(u), F_{Bz,z}(u), F_{Bz,z}(u)\}.$$

Therefore by Lemma B, we have $Bz = z$. Since $ABz = z$ therefore $Az = z$. By (1.2), with $\alpha = 1$ and using (1.4), we have

$$\begin{aligned} &F_{P(Uz),Qz}(ku) \\ &\geq \min\{F_{ABILz,STJU(Uz)}(u), F_{P(Uz),STJU(Uz)}(u), F_{Qz,ABILz}(u), F_{Qz,STJU(Uz)}(u), \\ &\qquad F_{P(Uz),ABILz}(u)\}. \end{aligned}$$

Thus we have

$$F_{Uz,z}(ku) \geq \min\{F_{Uz,z}(u), F_{Uz,Uz}(u), F_{z,z}(u), F_{Uz,z}(u), F_{Uz,z}(u)\}.$$

Therefore by Lemma B, we have $Uz = z$. Since $STJUz = z$ therefore $STJz = z$. To prove $Jz = z$ put $x = Jz$, $y = z$ in (1.2) with $\alpha = 1$ and using (1.4), we have

$$F_{P(Jz),Qz}(ku)$$

$$\geq \min\{F_{ABILz,STJU(Jz)}(u), F_{P(Jz),STJU(Jz)}(u), F_{Qz,ABILz}(u), F_{Qz,STJU(Jz)}(u), F_{P(Jz),ABILz}(u)\}.$$

Thus we have

$$F_{Jz,z}(ku) \geq \min\{F_{Jz,z}(u), F_{Jz,Jz}(u), F_{z,z}(u), F_{Jz,z}(u), F_{Jz,z}(u)\}.$$

Therefore by Lemma B, we have $Jz = z$. Since $STJz = z$ therefore $STz = z$. To prove $Tz = z$ put $x = Tz$, $y = z$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$F_{P(Tz),Qz}(ku)$$

$$\geq \min\{F_{ABILz,STJU(Tz)}(u), F_{P(Tz),STJU(Tz)}(u), F_{Qz,ABILz}(u), F_{Qz,STJU(Tz)}(u), F_{P(Tz),ABILz}(u)\}.$$

Thus we have

$$F_{Tz,z}(ku) \geq \min\{F_{Tz,z}(u), F_{Tz,Tz}(u), F_{z,z}(u), F_{z,Tz}(u), F_{Tz,z}(u)\}.$$

Therefore by Lemma B, we have $Tz = z$. Since $STz = z$ therefore $Sz = z$. By combining the above results we have

$Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z$. that is z is a common fixed point of $A, B, S, T, I, J, L, U, P$ and Q. The uniqueness of the common fixed point of $A, B, S, T, I, J, L, U, P$ and Q follows easily from (1.2). This completes the proof. □

Remark 1. We note that Theorem 1 is still true if we replace the condition (1.2) by the following condition: there exists k ∈ (0, 1) such that

$$F_{Px,Qy}(ku)$$

$$\geq t(F_{ABILy,STJUx}(u), t(F_{Px,STJUx}(u), t(F_{Qy,ABILy}(u), t(_{Qy,STJUx}(\alpha u), F_{Px,ABILy}((2-\alpha)))))$$

for all $x, y \in X, \alpha \in (0,2)$ and $u > 0$.

If we put P = Q in Theorem 1, we have the following result:

Corollary 2. *Let A, B, S, T, I, J, L, U and P be self maps on a Menger space (X, F, t) with $t(a, a) \geq a$ for all $a \in [0, 1]$, satisfying*

(2.1) $P(X) \subset ABIL(X),\ P(X) \subset STJU(X),$

(2.2) *there exists* $k \in (0,1)$ *such that*

$$F_{Px,Py}(ku) \geq \min\{F_{ABILy,STJUx}(u), F_{Px,STJUx}(u), F_{Py,ABILy}(u), F_{Py,STJUx}(\alpha u), F_{Px,ABILy}((2-\alpha)u)\}$$

for all $x, y \in X, \alpha \in (0,2)$ *and* $u > 0$,

(2.3) *if one of* $P(X), ABIL(X), STJU(X)$ *is a complete subspace of* X *then*

(*i*) P *and* $STJU$ *have a coincidence point and*

(*ii*) P *and* $ABIL$ *have a coincidence point.*

Further if

(2.4) $AB = BA,\ AI = IA,\ AL = LA,\ BI = IB,\ BL = LB,\ IL = LI,$
$PL = LP, PI = IP,\ PB = BP,\ ST = TS,\ SJ = JS,\ SU = US,$
$TJ = JT, TU = UT,\ JU = UJ,\ PU = UP,\ PJ = JP,\ PT = TP,$

(2.5) *the pairs* $\{P,\ STJU\}$ *and* $\{P,\ ABIL\}$ *are weakly compatible, then*

(*iii*) A, B, S, T, I, J, L, U *and* P *have a unique common fixed point in* X.

If we put $L = U = I_X$ (the identity map on X) in Theorem 1, we have the following:

Corollary 3. *Let* A, B, S, T, I, J, P *and* Q *be self maps on a Menger space* (X, F, t) *with* $t(a,a) \geq a$ *for all* $a \in [0,1]$, *satisfying*

(3.1) $P(X) \subset ABI(X), Q(X) \subset STJ(X),$

(3.2) *there exists* $k \in (0,1)$ *such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{ABIy,STJx}(u), F_{Px,STJx}(u), F_{Qy,ABIy}(u), F_{Qy,STJx}(\alpha u),\ F_{Px,ABIy}((2-\alpha)u)\ \}$$

for all $x,\ y \in X,\ \alpha \in (0,2)$ *and* $u > 0$,

(3.3) *if one of* $P(X), ABI(X), STJ(X), Q(X)$ *is a complete subspace of* X *then*

(*i*) P *and* STJ *have a coincidence point and*

(*ii*) Q *and* ABI *have a coincidence point.*

Further if

(3.4) $AB = BA, AI = IA,\ BI = IB, QI = IQ, QB = BQ,$
$ST = TS,\ SJ = JS,\ TJ = JT,\ PJ = JP,\ PT = TP,$

(3.5) *the pairs* $\{P,\ STJ\}$ *and* $\{Q,\ ABI\}$ *are weakly compatible, then*

(*iii*) A, B, S, T, I, J, P *and* Q *have a unique common fixed point in* X.

If we put $P = Q$ in Corollary 3, we get the following:

Corollary 4. *Let A, B, S, T, I, J and P be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(4.1) $$P(X) \subset ABI(X), \quad P(X) \subset STJ(X),$$

(4.2) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Py}(ku) \geq \min\{F_{ABIy,STJx}(u), F_{Px,STJx}(u), F_{Py,ABIy}(u), F_{Py,STJx}(\alpha u), F_{Px,ABIy}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$,

(4.3) *if one of* $P(X), ABI(X), STJ(X)$ *is a complete subspace of* X *then*

(*i*) *P and STJ have a coincidence point and*

(*ii*) *P and ABI have a coincidence point.*

Further if

(4.4) $AB = BA$, $AI = IA$, $BI = IB$, $PI = IP$, $PB = BP$,
$ST = TS$, $SJ = JS$, $TJ = JT$, $PJ = JP$, $PT = TP$,

(4.5) *the pairs* $\{P, STJ\}$*and*$\{P, ABI\}$ *are weakly compatible, then*

(*iii*) A, B, S, T, I, J *and P have a unique common fixed point in* X.

Remark 2. Theorem 1 and Corollaries 2- 4 improve extend and generalize the results of Mishra [12], Dedeic and Sarapa [3], Rashwan and Hadar [16], Sharma and Bagwan [21] Sharma and Deshpande [22] and many others.

If we put $I = J = I_X$ (the identity map on X) in Corollary 3 we have the following:

Corollary 5. *Let A, B, S, T, P and Q be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(5.1) $$P(X) \subset AB(X), \quad Q(X) \subset ST(X),$$

(5.2) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{ABy,STx}(u), F_{Px,STx}(u), F_{Qy,ABy}(u), F_{Qy,STx}(\alpha u), F_{Px,ABy}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$,

(5.3) *if one of* $P(X), AB(X), ST(X), Q(X)$ *is a complete subspace of* X *then*

(*i*) *P and ST have a coincidence point and*

(*ii*) *P and AB have a coincidence point.*

Further if

(5.4) $$AB = BA, \quad QB = BQ, \; ST = TS, \; PT = TP,$$

(5.5) *the pairs* $\{P, ST\}$ *and* $\{Q, AB\}$ *are weakly compatible, then*

(*iii*) A, B, S, T, P *and* Q *have a unique common fixed point in* X.

Remark 3. (*i*) Corollary 5 improves the result of Sharma and Bagwan [21].

(*ii*) In view of Remark 1, Theorem 1 extends Theorem 3.1 of Sharma and Deshpande [22].

(*iii*) Corollary 5 also extends, improves and generalizes the results of Mishra [12] Dedeic and Sarapa [3], Rashwan and Hader [16] and many others.

If we put $P = Q$ in Corollary 5 we have the following:

Corollary 6. *Let* A, B, S, T *and* P *be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(6.1) $$P(X) \subset AB(X), P(X) \subset ST(X)$$

(6.2) *there exists* $k \in 0, 1)$ *such that*

$$F_{Px,Py}(ku) \geq \min\{F_{ABy,STx}(u), F_{Px,STx}(u), F_{Py,ABy}(u), F_{Py,STx}(\alpha u), F_{Px,ABy}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$,

(6.3) *if one of* $P(X), AB(X), ST(X), P(X)$ *is a complete subspace of* X *then*

(*i*) P *and* ST *have a coincidence point and*

(*ii*) P *and* AB *have a coincidence point.*

Further if

(6.4) $$AB = BA, \quad PB = BP, \; ST = TS, \; PT = TP,$$

(6.5) *the pairs* $\{P, ST\}$ *and* $\{P, AB\}$ *are weakly compatible, then*

(*iii*) A, B, S, T *and* P *have a unique common fixed point in* X.

If we put $B = T = I_X$ (the identity mapping on X) in Corollary 5 then (5.4) is satisfied trivially and we have the following:

Corollary 7. *Let* A, S, P *and* Q *be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(7.1) $$P(X) \subset A(X), \quad Q(X) \subset S(X)$$

(7.2) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{Ay,Sx}(u), F_{Px,Sx}(u), F_{Qy,Ay}(u), F_{Qy,Sx}(\alpha u), F_{Px,Ay}((2-\alpha)u)\}$$

for all x, $y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$, (7.3) *if one of* $P(X), A(X), S(X), Q(X)$ *is a complete subspace of* X *then*

(*i*) P *and* S *have a coincidence point and*

(*ii*) P *and* A *have a coincidence point.*

Further if

(7.4) *the pairs* $\{P, S\}$ *and* $\{Q, A\}$ *are weakly compatible, then*

(*iii*) A, S, P *and* Q *have a unique common fixed point in* X.

Remark 4. (*i*) In view of Remark 1, Corollary 7 is generalization of result of Mishra [12], Dedeic and Sarapa [3], Rashwan and Hedar [16].

(*ii*) In view of Remark 1, Corollary 7 is Theorem 3.1 of Sharma and Deshpande [22].

If we put $A = S$ in Corollary 7 we have the following result:

Corollary 8. *Let* A, P *and* Q *be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(8.1) $$P(X) \subset A(X), \quad Q(X) \subset A(X),$$

(8.2) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{Ay,Ax}(u), F_{Px,Ax}(u), F_{Qy,Ay}(u), F_{Qy,Ax}(\alpha u), F_{Px,Ay}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$,

(8.3) *if one of* $P(X), A(X), Q(X)$ *is a complete subspace of* X *then*

(*i*) P *and* A *have a coincidence point.*

Further if

(8.4) *the pairs* $\{P, A\}$ *and* $\{Q, A\}$ *are weakly compatible, then*

(*ii*) A, P *and* Q *have a unique common fixed point in* X.

If we put $A = I_X$(the identity map on X) in Corollary 8 we have the following:

Corollary 9. *Let* P *and* Q *be self maps on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*

(9.1) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{y,x}(u), F_{Px,x}(u), F_{Qy,y}(u), F_{Qy,x}(\alpha u), F_{Px,y}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$, (9.2) *if one of* $P(X), Q(X)$ *is a complete subspace of* X *then* P *and* Q *have a unique common fixed point in* X.

If we put $P = Q$ in Corollary 9, we have the following:

Corollary 10. *Let* P *be a self map on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*
(10.1) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Py}(ku) \geq \min\{F_{y,x}(u), F_{Px,x}(u), F_{Py,y}(u), F_{Py,x}(\alpha u), F_{Px,y}((2-\alpha)u)\}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $u > 0$,
(10.2) *if* $P(X)$ *is a complete subspace of* X *then* P *has a unique common fixed point in* X.

Corollary 11. *Let* P *be a self map on a Menger space* (X, F, t) *with* $t(a, a) \geq a$ *for all* $a \in [0, 1]$, *satisfying*
(11.1) *there exists* $k \in (0, 1)$ *such that*

$$F_{Px,Py}(ku) \geq F_{x,y}(u),$$

for all $x, y \in X$ *and* $u > 0$,
(11.2) *if* $P(X)$ *is a complete subspace of* X, *then*
(*iii*) P *has a unique common fixed point in* X.

The metric version of Theorem 1 is as follows:

Theorem 12. *Let* A, B, S, T, I, J, L, U, P *and* Q *be self maps on a metric space* (X, d) *satisfying* (1.1) *and*
(12.1) *there exists* $k \in (0, 1)$ *such that*

$$d(Px, Qy) \leq k \max\{d(ABILy, STJUx), d(Px, STJUx), d(Qy, ABILy) \frac{1}{2}\{d(Qy, STJUx) + d(Px, ABILy)\}$$

for all $x, y \in X$.

In addition if condition (1.3) is satisfied then we have (i)and (ii). Further if (1.4) and (1.5) are satisfied then $A, B, S, T, I, J, L, U, P$ and Q have a unique common fixed point.

Remark 5. Theorem 12 improves, extends and generalizes the result of Mishra [12], Jungck [9], Hadzic [6], Xieping [27] and many others.

Theorem 13. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$. Let A_1, A_2, ..., A_n, S_1, S_2, ..., S_n, P and Q be self mappings on a Menger space (X, F, t) with $t(a,a) \geq a$ for all $a \in [0,1]$ satisfying:*

(13.1) $P(X) \subset A_1A_2...A_n(X), \quad Q(X) \subset S_1S_2...S_n(X),$

(13.2) *there exists a constant $k \in (0,1)$ such that*

$$F_{Px,Qy}(ku) \geq \min\{F_{A_1A_2...A_ny,S_1S_2...S_nx}(u), F_{Px,S_1S_2...S_nx}(u), F_{Qy,A_1A_2...A_ny}(u), F_{Qy,S_1S_2...S_nx}(\alpha u), F_{Py,A_1A_2...A_ny}(2-\alpha)u)\}$$

for all $x, y \in X, a \geq 0$, $\alpha \in (0,2)$ and $t > 0$,

(13.3) *if one of $P(X)$, $A_1A_2 \ldots A_n(X)$, $S_1S_2 \ldots S_n(X)$, $Q(X)$ is a complete subspace of X then*

(i) P and $S_1S_2...S_n$ have a coincidence point and

(ii) Q and $A_1A_2...A_n$ have a coincidence point.

Further if

(13.4) *A_1 commutes with $A_2, A_3, ..., A_n$,*
A_2 commutes with A_3, A_4, ..., A_n,
A_3 commutes with A_4, $A_5, ..., A_n$,
... ...,
... ... ,
A_{n-1} commutes with A_n,
similarly S_1 commutes with $S_2, S_3, ..., S_n$,
S_2 commutes with S_3, $S_4..., S_n$,
S_3 commutes with S_4, $S_5..., S_n$,
... ... ,
... ...,
S_{n-1} commutes with S_n,
P commutes with $S_2, S_3, ..., S_n$,
Q commutes with $A_2, A_3, ..., A_n$.

(13.5) *the pairs $\{P, S_1S_2...S_n\}$ and $\{Q, A_1A_2...A_n\}$ are weakly compatible, then*

(iii) $A_1, A_2, ..., A_n, S_1, S_2, ...,_n, P$ and Q have a unique common fixed point in X.

Proof. Since $P(X) \subset A_1A_2...A_n(X)$, for any point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = A_1A_2...A_nx_1$. Since $Q(X) \subset S_1S_2...S_n(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = S_1S_2...S_nx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2, ...$

$$y_{2n} = Px_{2n} = A_1A_2...A_nx_{2n+1},$$
$$y_{2n+1} = Qx_{2n+1} = S_1S_2...S_nx_{2n+2}.$$

By using the method of proof of Theorem 1, we can see that conclusions (i), (ii) and (iii) hold. □

Discussion and Auxiliary Results

In view of above results it is very much clear that we extend, improve and generalize many results in metric spaces and Menger spaces. We prove common fixed point theorems for finite number of mappings in Menger spaces. This is the first effort in the existing literature. To prove common fixed point theorems for contractive type condition with more than four mappings some commutativity conditions for mappings are always essential. How many commutative conditions are necessary? As an answer of this question we are giving the following formulas:

(i) If the number of mappings are even and finite in above theorems and corollaries then there will be $\frac{n^2-2n-8}{4}$ commutativity conditions, where $n = 4, 6, 8, 10, 12, ...$ up to finite values. For example if $n = 10$ then 18 commutativity conditions are required. (See (1.4)).

(ii) If the number of mappings are odd and finite in above theorems and corollaries then there will be $\frac{n^2-9}{4}$ commutativity conditions, where $n = 5, 7, 9, 11......$ up to finite values. For example if $n = 7$ then 10 commutativity conditions are required. (See (4.4)).

(iii) If $n = 1, 2, 3, 4$ then any commutativity condition is not required. See Corollaries 7 to 11.

Our theorems apply to a wider class of mappings than the results on compatible or compatible of type (α) or compatible of type (β) maps since compatible or compatible of type (α) or compatible of type (β) maps constitute a proper subclass of weakly compatible maps.

We point out that common fixed point theorems for finite number of maps can be proved without continuity of any mappings.

In our all results we replace the completeness of the whole space with a set of alternative conditions.

In this way we prove common fixed point theorems for finite number of maps in Menger spaces by relaxing, replacing and omitting some conditions in the analogous results.

Our results contain so many results in the existing literature and will be helpful for the workers in the field.

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