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MODIFIED HYERS-ULAM STABILITY OF A JENSEN TYPE QUARTIC FUNCTIONAL EQUATION

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ABSTRACT. In the present paper we introduce a Jensen type quartic functional equation and then investigate the generalized Hyers-Ulam stability problem for the equation.

1. INTRODUCTION

In 1940, S.M. Ulam [24] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers [5] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th.M. Rassias [20] provided a generalization of Hyers' Theorem which allows the *Cauchy difference operator* to be unbounded.

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Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$.
Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1991, Z. Gajda [6] following the same approach as in Th. M. Rassias [20], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [6], as well as by Th.M. Rassias and P. Šemrl [22] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [20] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known as *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [4], D.H. Hyers, G. Isac and Th.M. Rassias [8]). P. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem which allows the *Cauchy difference operator* to be controlled by general functions. In 1999 Y. Lee and K. Jun [16] have obtained a generalization of the Hyers–Ulam–Rassias stability of Jensen's equation $f\left(\frac{x+y}{2}\right) = f(x) + f(y)$.

Now, a function $f(x) = cx^2 (x \in \mathbb{R})$, where c is a real constant, clearly satisfies the equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2). \tag{1.3}$$

It is well known that a function $f : X \to Y$ between real vector spaces satisfies the equation (1.3) if and only if there exists a unique symmetric biadditive function $B : X^2 \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$ (see [1, 14]). Hence, the equation (1.3) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation (1.3) is said to be a quadratic function. The Hyers-Ulam stability of the quadratic functional equation was first proved by F. Skof [23]. Later, the generalized Hyers-Ulam stability of the quadratic functional equation was proved by S. Czerwik [3], and then extended by J.M. Rassias [18] and Th.M. Rassias ([12, 21]).

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [9, 10, 13, 17].

In [2], Chung and Sahoo determined the general solution of the quartic equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y)$$

without assuming any regularity conditions on the unknown mapping f. On the other hand, it is easy to see that the solution f of the equation is even. Thus the above equation can be written in the following way

$$f(2x+y)+f(2x-y)+6f(y)=4f(x+y)+4f(x-y)+24f(x), \tag{1.4}$$

of which the general solution is determined by a symmetric biquadratic mapping $B : E_1 \times E_1 \to E_2$ between real vector spaces E_1, E_2 such that $f(x) = V(x) := B(x,x)$ for all $x \in E_1$ [11, 15]. A mapping $B : E_1 \times E_1 \to E_2$ between real vector spaces is called biquadratic if B is quadratic in each variable. It is easy to see that $f(x) = x^4$ is a solution of the equation (1.4) because of the identity

$$(x+2y)^4+(x-2y)^4+6x^4=4(x+y)^4+4(x-y)^4+24y^4.$$

For this obvious reason, the above functional equation (1.4) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping [19].

In this paper, we are going to consider a modified Jensen type quartic functional equation

$$f\left(\frac{3x+y}{2}\right)+f\left(\frac{x+3y}{2}\right)+6f\left(\frac{x-y}{2}\right)=24f\left(\frac{x+y}{2}\right)+4f(x)+4f(y) \tag{1.5}$$

and then investigate the generalized Hyers–Ulam stability of the equation (1.5) using the direct method.

2. Stability of Equation (1.5)

First of all, we establish the general solution of the functional equation (1.5) by elementary change of variables. In fact, let f be a quartic function. Then replacing x, y by $\frac{x+y}{2}, \frac{x-y}{2}$ in (1.4), respectively, we get the equation (1.3).

Conversely, let f satisfy the equation (1.5). Then replacing x, y by $x+y, x-y$ in (1.5), respectively, we get the equation (1.3). Hence we have the following lemma.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation* (1.5) *if and only if f is quartic.*

From now on, let X be a vector space and Y a Banach space. For a given mapping $f : X \to Y$, we define a difference operator of f by

$$\begin{aligned} Df(x,y) := f\left(\frac{3x+y}{2}\right) + f\left(\frac{x+3y}{2}\right) + 6f\left(\frac{x-y}{2}\right) \\ - 24f\left(\frac{x+y}{2}\right) - 4f(x) - 4f(y) \end{aligned}$$

for all $x, y \in X$.

Theorem 2.2. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.1}$$

and the function $\varphi : X^2 \to [0,\infty)$ satisfy

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{16^i} < \infty \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfying

$$\left\| f(x) - \frac{f(0)}{5} - Q(x) \right\| \le \frac{1}{32}\Phi(x,x) \tag{2.3}$$

for all $x \in X$.

Proof. First, we observe that $\|f(0)\| \le \frac{\varphi(0,0)}{24}$. Put $y := x$ in (2.1) for any fixed $x \in X$. Then we obtain

$$\|2f(2x) + 6f(0) - 32f(x)\| \le \varphi(x,x), \tag{2.4}$$

which yields

$$\left\| \frac{g(2x)}{16} - g(x) \right\| \le \frac{1}{32}\varphi(x,x)$$

for all $x \in X$, where $g(x) := f(x) - \frac{f(0)}{5}$. Thus we have

$$\begin{aligned} \left\| \frac{g(2^m x)}{16^m} - \frac{g(2^n x)}{16^n} \right\| &\le \sum_{i=m}^{n-1} \left\| \frac{g(2^i x)}{16^i} - \frac{g(2^{i+1} x)}{16^{i+1}} \right\| \\ &\le \frac{1}{32} \sum_{i=m}^{n-1} \frac{\varphi(2^i x, 2^i x)}{16^i} \end{aligned} \tag{2.5}$$

for $n > m \geq 0$. Since the right-hand side of the inequality (2.5) tends to 0 as $m \to \infty$ by the convergence of the series (2.2), the sequence $\left\{\frac{g(2^n x)}{16^n}\right\}$ is Cauchy in the Banach space Y. Therefore we may define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{g(2^n x)}{16^n} = \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$$

for all $x \in X$. By letting $n \to \infty$ in (2.5) with $m = 0$, we arrive at the formula (2.3). It follows from the definition of Q and the convergence of the series (2.2) that

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{n\to\infty} \left\|\frac{Df(2^n x, 2^n y)}{16^n}\right\| \\ &\leq \lim_{n\to\infty} \frac{1}{16^n}\varphi(2^n x, 2^n y) = 0\end{aligned}$$

for all $x \in X$. Thus Lemma 2.1 implies that Q is quartic.

Let $Q' : X \to Y$ be another quartic function which satisfies the inequality (2.3). Since Q' is quartic function, we can easily see that

$$Q'(2^n x) = 16^n Q'(x)$$

for any $n \in N$. Thus, it follows from (2.3) that

$$\begin{aligned}\left\|\frac{f(2^n x)}{16^n} - Q'(x)\right\| &= \left\|\frac{f(2^n x)}{16^n} - \frac{Q'(2^n x)}{16^n}\right\| \\ &\leq \frac{1}{16^n}\left\|f(2^n x) - \frac{f(0)}{5} - Q'(2^n x)\right\| + \frac{1}{16^n}\frac{\|f(0)\|}{5} \\ &\leq \frac{1}{16^n}\sum_{i=0}^{\infty}\frac{\varphi(2^{i+n}x, 2^{i+n}x)}{16^i} + \frac{1}{16^n}\left\|\frac{f(0)}{5}\right\|.\end{aligned}$$

By letting $n \to \infty$, then we get that $Q(x) = Q'(x)$ for all $x \in X$, which completes the proof of the theorem. □

Theorem 2.3. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in X$. Suppose that there exists a constant L with $0 < L < 1$ such that the function $\varphi : X^2 \to [0,\infty)$ satisfies

$$\varphi(2x, 2y) \leq 16L\varphi(x,y) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfying

$$\left\|f(x) - \frac{f(0)}{5} - Q(x)\right\| \leq \frac{\varphi(x,x)}{32(1-L)}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\varphi(2^i x, 2^i y) \le (16L)^i \varphi(x, y)$$

for all $x, y \in X$ and all positive integer i. Thus it follows from the inequality (2.5) that for all integers m, n with $n > m \ge 0$

$$\begin{aligned}\left\| \frac{g(2^m x)}{16^m} - \frac{g(2^n x)}{16^n} \right\| &\le \sum_{i=m}^{n-1} \left\| \frac{g(2^i x)}{16^i} - \frac{g(2^{i+1} x)}{16^{i+1}} \right\| \\ &\le \frac{1}{32} \sum_{i=m}^{n-1} \frac{\varphi(2^i x, 2^i x)}{16^i} \\ &\le \frac{1}{32} \sum_{i=m}^{n-1} L^i \varphi(x, x), \quad x \in X.\end{aligned}$$

We observe that the right-hand side of the inequality tends to 0 as $m \to \infty$ since the series $\sum_{i=0}^{\infty} L^i \varphi(x, x)$ converges to $\frac{\varphi(x,x)}{1-L}$ for any $x \in X$.

The rest of proof is similar to that of Theorem 2.2. □

For a function $\varphi(t) := t^p, p < 4$, it follows easily that $\varphi(2t) = 16L\varphi(t)$, where $L := 2^{p-4} < 1$. Therefore, we have the following corollary by Theorem 2.2 and Theorem 2.3.

Corollary 2.4. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{2.7}$$

for some $p < 4$ and for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ such that

$$\left\| f(x) - \frac{f(0)}{5} - Q(x) \right\| \le \frac{\varepsilon \|x\|^p}{16 - 2^p}$$

for all $x \in X$.

Corollary 2.5. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \le \varepsilon$$

for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ such that

$$\left\| f(x) - \frac{f(0)}{5} - Q(x) \right\| \le \frac{\varepsilon}{30}$$

for all $x \in X$.

Now, we investigate another stability question controlled by a function $\varphi : X^2 \to [0, \infty)$.

Theorem 2.6. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \le \varphi(x,y)$$

and the function $\varphi : X^2 \to [0,\infty)$ satisfy

$$\Phi_1(x,y) := \sum_{i=1}^{\infty} 16^i \varphi\Big(\frac{x}{2^i}, \frac{y}{2^i}\Big) < \infty$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \le \frac{1}{32}\Phi_1(x,x)$$

for all $x \in X$.

Proof. We note that $\varphi(0,0) = 0$ and so $f(0) = 0$ by the convergence of $\Phi_1(0,0)$. Thus it follows from the inequality (2.4) that

$$\Big\|16^m f\Big(\frac{x}{2^m}\Big) - 16^n f\Big(\frac{x}{2^n}\Big)\Big\| \le \sum_{i=m+1}^{n} \Big\|16^{i-1} f(\frac{x}{2^{i-1}}) - 16^i f\Big(\frac{x}{2^i}\Big)\Big\| \le \frac{1}{32}\sum_{i=m+1}^{n} 16^i \varphi\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big) \tag{2.8}$$

for all integers m, n with $n > m \ge 0$.

The rest of proof is similar to that of Theorem 2.2. □

Theorem 2.7. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in X$. Suppose that there exists a constant L with $0 < L < 1$ such that the function $\varphi : X^2 \to [0,\infty)$ satisfies

$$\varphi(x,y) \le \frac{L}{16}\varphi(2x,2y) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \le \frac{L\varphi(x,x)}{32(1-L)}$$

for all $x \in X$.

Proof. We note that $\varphi(0,0) = 0$ and so $f(0) = 0$ by the inequality $\varphi(0,0) \le \frac{L}{16}\varphi(0,0)$. It follows from (2.9) that

$$\varphi\Big(\frac{x}{2^i}, \frac{y}{2^i}\Big) \le \Big(\frac{L}{16}\Big)^i \varphi(x,y)$$

for all $x, y \in X$. Thus it follows from the inequality (2.8) that for all integers with $n > m \geq 0$

$$\left\|16^m f\left(\frac{x}{2^m}\right) - 16^n f\left(\frac{x}{2^n}\right)\right\| \leq \sum_{i=m+1}^{n} \left\|16^{i-1} f\left(\frac{x}{2^{i-1}}\right) - 16^i f\left(\frac{x}{2^i}\right)\right\|$$

$$\leq \frac{1}{32} \sum_{i=m+1}^{n} 16^i \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right) \leq \frac{1}{32} \sum_{i=m+1}^{n} L^i \varphi(x, x), \ x \in X.$$

The rest of proof is similar to that of Theorem 2.6. □

For a function $\varphi(t) := t^p, p > 4$, it follows easily that $\varphi(\frac{t}{2}) = \frac{L}{16}\varphi(t)$, where $L := 2^{4-p} < 1$.

Corollary 2.8. *If a function $f : X \to Y$ satisfies the inequality* (2.7) *for some $p > 4$ and for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon \|x\|^p}{2^p - 16}$$

for all $x \in X$.

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