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SOLUTION AND STABILITY OF AN EXPONENTIAL TYPE FUNCTIONAL EQUATION

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ABSTRACT. In this paper we generalize the superstability of the exponential functional equation proved by J. Baker et al. [2], that is, we solve an exponential type functional equation

$$f(x+y) = a^{xy} f(x) f(y)$$

and obtain the superstability of this equation. Also we generalize the stability of the exponential type equation in the spirt of R. Ger[4] of the following setting :

$$\left| \frac{f(x+y)}{a^{xy} f(x) f(y)} - 1 \right| \leq \delta.$$

1. INTRODUCTION

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems ([22]). One of those was the question concerning the stability of homomorphisms :

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the next year, D. H. Hyers [5] answered the Ulam's question for the case of the additive mapping on the Banach spaces G_1, G_2. Thereafter, the result of Hyers has been generalized by Th. M. Rassias [15]. Since then, the stability problems of various functional equations have been investigated by many authors (see [1, 3, 4, 6-13, 16-21]).

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In particular, J. Baker, J. Lawrence and F. Zorzitto in [4] introduced the stability of the exponential functional equation in the following form : if f satisfies the inequality $|f(x+y)-f(x)f(y)| \le \varepsilon$, then either f is bounded or $f(x+y) = f(x)f(y)$. This is frequently referred to as *Superstability.*

In this paper, we will investigate the solution and the superstability of the exponential type functional equation

$$f(x+y) = a^{xy} f(x)f(y), \tag{1}$$

which is a generalization of the superstability of the exponential functional equation given by J. Baker et al.[2]. And also we obtain stability in the sense of R. Ger [4].

2. SOLUTION OF THE EQUATION (1)

Proposition 1. *$f(x) = a^{\frac{x^2}{2}}$ is a solution of the equation (1). In particular, this f is the unique continuous solution with $f(1) = \sqrt{a} = f(-1)$.*

Proof. It is trivial to show that $f(x) = a^{\frac{x^2}{2}}$ is a solution of the equation (1). Let g be any continuous solution of the equation (1) and $g(1) = \sqrt{a} = g(-1)$. Then $g(2) = g(1+1) = ag(1)g(1) = a^{\frac{2^2}{2}}$. If $g(n) = a^{\frac{n^2}{2}}$, then

$$g(n+1) = a^n g(n)g(1) = a^n \cdot a^{\frac{n^2}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{(n+1)^2}{2}}.$$

An induction argument implies that $g(n) = a^{\frac{n^2}{2}}$ for every nonnegative integer n. And similarly $g(-n) = a^{\frac{n^2}{2}}$ for every positive integer n. Also we have

$$g\left(\frac{1}{2}\right)^2 = g(1) \cdot a^{-\frac{1}{4}} = a^{\frac{1}{4}}.$$

Thus $g(\frac{1}{2}) = a^{\frac{(\frac{1}{2})^2}{2}}$. If $g(\frac{1}{2^n}) = a^{\frac{(\frac{1}{2^n})^2}{2}}$ for some nonnegative integer n, then

$$g\left(\frac{1}{2^{n+1}}\right)^2 = g\left(\frac{1}{2^n}\right) \cdot a^{-\frac{1}{2^{(n+1)2}}} = a^{\frac{1}{2^{2n+2}}}$$

and so $g(\frac{1}{2^{n+1}}) = (a^{\frac{1}{2^{2n+2}}})^{\frac{1}{2}} = a^{\frac{(\frac{1}{2^{n+1}})^2}{2}}$. Also an induction argument implies that $g(\frac{1}{2^n}) = a^{\frac{(\frac{1}{2^n})^2}{2}}$ for all nonnegative integer n. Note that for every positive integer m, $m = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$, where $a_i = 0$ or 1 for each $i = 0, 1, \cdots, k$. We may assume that $a_i = 1$ for all $i = 1, 2, \cdots, k$. Then for any positive integer n,

$$\begin{aligned} g\Big(\frac{m}{2^n}\Big) &= g\Big(\frac{1 + 2^1 + \cdots + 2^k}{2^n}\Big) \\ &= g\Big(\frac{1}{2^n}\Big) g\Big(\frac{2^1 + \cdots + 2^k}{2^n}\Big) \cdot a^{\frac{1}{2^n}\cdot\frac{2^1+\cdots+2^k}{2^n}} \\ &= g\Big(\frac{1}{2^n}\Big) g\Big(\frac{2^1}{2^n}\Big) g\Big(\frac{2^2}{2^n}\Big) \cdots g\Big(\frac{2^k}{2^n}\Big) \cdot \prod_{i=0}^{k-1} a^{\frac{2^i}{2^n}\cdot\sum_{j=i+1}^{k}\frac{2^j}{2^n}} \\ &= a^{\frac{(\frac{m}{2^n})^2}{2}} \end{aligned}$$

Similarly, $g(\frac{-m}{2^n}) = a^{\frac{(\frac{m}{2^n})^2}{2}}$ for all positive integer m and n. Now let r be any real number. For each $\epsilon > 0$ we can choose an integer n with $\frac{1}{2\epsilon} < 2^n$ and also choose an integer m such that

$$m < 2^n(r + \epsilon) \le m + 1.$$

Then we have

$$r - \epsilon = r + \epsilon - 2\epsilon < \frac{m+1}{2^n} - \frac{1}{2^n} = \frac{m}{2^n} < r + \epsilon.$$

Thus for any real number $r > 0$, there exist an integer m and a positive integer n such that

$$\Big|\frac{m}{2^n} - r\Big| < \epsilon.$$

By the continuity of g, we have $g(r) = a^{\frac{r^2}{2}}$ for every real number r. Thus $f(x) = g(x)$ for every real number x. □

3. Superstability of the Functional Equation (1)

J. Baker, J. Lawrence and F. Zorzitto [2] proved the superstability of Cauchy's exponential equation

$$f(x + y) = f(x)f(y).$$

That is, if the Cauchy difference $f(x + y) - f(x)(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y, then f is either bounded or exponential. Their result was generalized by J. Baker [1] : let S be a semi-group and let f be a complex-valued function defined on S such that

$$|f(xy) - f(x)f(y)| < \delta$$

for all x, $y \in S$, then f is either bounded or multiplicative.

Theorem 2. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f : (0,\infty) \to (0,\infty)$ be an unbounded function (in particular, $f(m) \geq \max(2, 2\sqrt{\delta})$ for some positive integer m) such that*

$$|f(x+y) - a^{xy} f(x)f(y)| < \delta \tag{2}$$

for all $x, y \in (0,\infty)$. Then

$$f(x+y) = a^{xy} f(x)f(y)$$

for all $x, y \in (0,\infty)$.

Proof. If we replace x and y by m in (2), simultaneously, we get

$$|f(2m) - a^{m^2} f(m)^2| < \delta.$$

An induction argument implies that for all $m \geq 2$

$$\begin{aligned} &|f(nm) - a^{m^2} a^{2m^2} \cdots a^{(n-1)m^2} f(m)^n| \\ &\leq \delta + a^{(n-1)m^2} f(m)\delta + a^{(n-1)m^2} a^{(n-2)m^2} f(m)^2 \delta \\ &\quad + \cdots + a^{(n-1)m^2} a^{(n-2)m^2} \cdots a^{2m^2} f(m)^{n-2} \delta. \end{aligned} \tag{3}$$

Indeed, if the inequality (3) holds, we have

$$\begin{aligned} &|f((n+1)m) - a^{m^2} a^{2m^2} \cdots a^{nm^2} f(m)^{n+1}| \\ &\leq |f((n+1)m) - a^{nm^2} f(nm) f(m)| \\ &\quad + |f(nm) - a^{m^2} a^{2m^2} \cdots a^{(n-1)m^2} f(m)^n| a^{nm^2} f(m) \\ &\leq \delta + a^{nm^2} f(m)\delta + a^{nm^2} a^{(n-1)m^2} f(m)^2 \delta \\ &\quad + \cdots + a^{nm^2} a^{(n-1)m^2} \cdots a^{2m^2} f(m)^{n-1} \delta \end{aligned}$$

for all $n \geq 2$. By (3) with $a \geq 1$, we get

$$\begin{aligned} &\left| \frac{f(nm)}{a^{(n-1)m^2} a^{(n-2)m^2} \cdots a^{2m^2} a^{m^2} f(m)^n} - 1 \right| \\ &\leq \left(\frac{1}{f(m)^n} + \frac{1}{f(m)^{n-1}} + \cdots + \frac{1}{f(m)^2} \right) \delta \\ &< \frac{1}{f(m)^2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) \delta = \frac{2\delta}{f(m)^2} \leq \frac{1}{2} \end{aligned}$$

for all positive integer n. Since $f(m)^n \to \infty$ and

$$a^{(n-1)m^2} a^{(n-2)m^2} \cdots a^{2m^2} a^{m^2} \to \infty \ \text{ as } \ n \to \infty, \quad f(nm) \to \infty \ \text{ as } \ n \to \infty.$$

Then for any $x, y \in (0,\infty)$ we have

$$\begin{aligned}
&f(nm)|f(x+y) - a^{xy}f(x)f(y)| \\
&\leq |a^{nm(x+y)}f(nm)f(x+y) - f(nm+x+y)|\frac{1}{a^{nm(x+y)}} \\
&\quad + |f(nm+x+y) - a^{x(y+nm)}f(x)f(y+nm)|\frac{1}{a^{nm(x+y)}} \\
&\quad + |f(y+nm) - a^{ynm}f(y)f(nm)|f(x)\cdot\frac{a^{xy}}{a^{ynm}}
\end{aligned}$$

and so

$$\begin{aligned}
&|f(x+y) - a^{xy}f(x)f(y)| \\
&\leq \frac{2\delta}{f(nm)a^{nm(x+y)}} + \frac{a^{xy}\delta f(x)}{f(nm)a^{ynm}} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$. Thus it follows that

$$f(x+y) = a^{xy}f(x)f(y).$$

for any $x, y \in 0,\infty)$. □

Suppose that $H_p : (0,\infty)\times(0,\infty) \longrightarrow (0,\infty)$ be monotonically increasing (in both variables) homogeneous mapping, for which $H_p(tx,ty) = t^pH_p(x,y)$ holds for some $p > 1$, and for all $t, x, y \in (0,\infty)$. For examples $H_p(x,y) = ax^p + by^p$ for $a, b, x, y \in (0,\infty)$.

Theorem 3. *Let $a \geq 1$ be given. If $f : (0,\infty) \longrightarrow (0,\infty)$ satisfies the functional inequality*

$$|f(x+y) - a^{xy}f(x)f(y)| \leq H_p(x,y),$$

then either $f(x) = o(x^p)$ as $x \longrightarrow \infty$ or $f(x+y) = a^{xy}f(x)f(y)$ for every $x, y \in (0,\infty)$.

Proof. By the same method as the proof of Theorem 2, we have

$$\begin{aligned}
&|f(nx) - a^{x^2}a^{2x^2}\cdots a^{(n-1)x^2}f(x)^n| \\
&\leq H_p((n-1)x,x) + H_p((n-2)x,x)f(x)a^{(n-1)x^2} \\
&\quad + H_p((n-3)x,x)f(x)^2a^{(n-1)x^2}a^{(n-2)x^2} \\
&\quad + \cdots + H_p(x,x)f(x)^{n-2}a^{2x^2}\cdots a^{(n-2)x^2}
\end{aligned}$$

for all positive integer $n \geq 2$ and $x > 0$. Since $a \geq 1$, we get

$$\left|\frac{f(nx)}{f(x)^n a^{x^2} a^{2x^2}\cdots a^{(n-1)x^2}}-1\right| \le \sum_{i=1}^{n-1}\frac{H_p(ix,x)}{f(x)^{i+1}} \le \sum_{i=1}^{\infty}\frac{i^p H_p(x,x)}{f(x)^{i+1}}$$
$$\le \frac{H_p(x,x)}{f(x)}\sum_{i=1}^{\infty}\frac{i^p}{f(x)^i}$$

for any positive integer $n \ge 2$ and $x > 0$. Assume that $f(x) \ne o(x^p)$ as $x \to \infty$, that is, there exist some $\alpha > 0$ and a sequence $\{x_k\}$ in $(0,\infty)$ such that $x_k \to \infty$ as $k \to \infty$ and $f(x_k) \ge \alpha x_k{}^p > 1$ for sufficiently large k. Then $\frac{i^p}{f(x_k)^i} \le \frac{i^p}{(\alpha x_k^p)^i}$ and $\sum_{i=1}^{\infty}\frac{i^p}{(\alpha x_k^p)^i}$ converges. We can then let the series $\sum_{i=1}^{\infty}\frac{i^p}{f(x_k)^i}$ converge to a value less than $\frac{f(x)}{2H_p(x,x)}$ by taking sufficiently large k. Thus for some sufficiently large k and any $n \ge 2$, we have

$$\left|\frac{f(nx_k)}{f(x_k)^n\prod_{i=1}^{n-1}a^{ix_k^2}}-1\right| < \frac{1}{2}.$$

Since $\frac{n^pH_p(x_k,x_k)}{f(x_k)^n} \to 0$ and $\prod_{i=1}^{n-1}a^{ix_k{}^2} \to \infty$ as $n \to \infty$, we have

$$\left|\frac{\frac{f(nx_k)}{n^pH_p(x_k,x_k)}}{\frac{f(x_k)^n\prod_{i=1}^{n-1}a^{ix_k{}^2}}{n^pH_p(x_k,x_k)}}-1\right| < \frac{1}{2}$$

and so $\frac{n^pH_p(x_k,x_k)}{f(nx_k)} \to 0$ as $n \to \infty$.

Now let $x, y > 0$ be given. If k is sufficiently large, we have

$$\begin{aligned}
&f(nx_k)|f(x+y)-a^{xy}f(x)f(y)|\\
&\le |a^{nx_k(x+y)}f(nx_k)f(x+y)-f(nx_k+x+y)|\frac{1}{a^{nx_k(x+y)}}\\
&\quad+|f(nx_k+x+y)-a^{x(y+nx_k)}f(x)f(y+nx_k)|\frac{1}{a^{nx_k(x+y)}}\\
&\quad+|f(y+nx_k)-a^{ynx_k}f(y)f(nx_k)|\frac{f(x)a^{xy}}{a^{ynx_k}}\\
&\le C_1H_p(x+y,nx_k)+C_2H_p(x,y+nx_k)+C_3H_p(y,nx_k)\\
&\le (C_1+C_2+C_3)H_p(nx_k,nx_k) = (C_1+C_2+C_3)n^pH_p(x_k,x_k)
\end{aligned}$$

for some $C_1, C_2, C_3 > 0$ and sufficiently large n. Thus

$$|f(x+y)-a^{xy}f(x)f(y)| \le \frac{(C_1+C_2+C_3)n^pH_p(x_k,x_k)}{f(nx_k)} \to 0$$

as $n \to \infty$. □

4. Stability of the Equation (1)

R. Ger [4] introduced a stability for the exponential equation in the following type :

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \delta.$$

Now we prove the stability of the equation (1) in the sense of Ger.

Theorem 4. *Let $0 < \delta < 1$ and $a \geq 1$ be given. If a function $f : (0,\infty) \to (0,\infty)$ satisfies the inequality*

$$\left|\frac{f(x+y)}{a^{xy}f(x)f(y)} - 1\right| \leq \delta \tag{4}$$

for all $x, y \in (0,\infty)$, then there exists a function $F : (0,\infty) \to (0,\infty)$ such that

$$F(x+y) = a^{xy}F(x)F(y)$$

for all $x, y \in (0,\infty)$ and

$$\left|\frac{F(x)}{f(x)} - 1\right| \leq \delta$$

for all $x \in (0,\infty)$.

Proof. If we define a function $G : (0,\infty) \to (0,\infty)$ by

$$G(x) = \ln f(x)$$

for all $x > 0$, then the equality (4) may be transformed into

$$|G(x+y) - \ln a^{xy} - G(x) - G(y)| \leq \ln(1+\delta) := \theta$$

for all $x, y > 0$. Replacing y by x and dividing by 2, we get

$$\left|\frac{G(2x)}{2} - \ln a^{\frac{x^2}{2}} - G(x)\right| \leq \frac{\theta}{2} \tag{5}$$

for all $x > 0$. We use induction on n to prove

$$\left|\frac{G(2^n x)}{2^n} - \ln(a^{\frac{x^2}{2}} \cdot a^{x^2} \cdot a^{2x^2} \cdots a^{2^{n-2}x^2}) - G(x)\right| \leq \theta \sum_{i=1}^{n} \frac{1}{2^i} \tag{6}$$

for all $x > 0$. On account of (5), the inequality holds for $n = 1$. Suppose that inequality (6) holds true for some $n > 1$. Then (5) and (6) imply

$$\begin{aligned}&\left|\frac{G(2^{n+1}x)}{2^{n+1}} - \ln(a^{\frac{x^2}{2}}\cdot a^{x^2}\cdot a^{2x^2}\cdots a^{2^{n-1}x^2}) - G(x)\right| \\ &\le \left|\frac{G(2^{n+1}x)}{2^{n+1}} - \frac{1}{2}\ln(a^{\frac{(2x)^2}{2}}\cdot a^{(2x)^2}\cdots a^{2^{n-2}(2x)^2}) - \frac{G(2x)}{2}\right| \\ &+ \left|\frac{G(2x)}{2} - \ln a^{\frac{x^2}{2}} - G(x)\right| \le \theta\sum_{i=1}^{n+1}\frac{1}{2^i},\end{aligned}$$

which ends the proof of (6). For any $x > 0$ and for every positive integer n we define

$$P_n(x) = \frac{G(2^n x)}{2^n} - \ln(a^{\frac{x^2}{2}}\cdot a^{x^2}\cdot a^{2x^2}\cdots a^{2^{n-2}x^2}).$$

Let $m, n > 0$ be integers with $n > m$. Then it follows from (6)

$$\begin{aligned}&|P_n(x) - P_m(x)| \\ &= \frac{1}{2^m}\left|\frac{G(2^{n-m}(2^m x))}{2^{n-m}} - \ln\left(\prod_{i=-1}^{n-m-2} a^{2^i(2^m x)}\right) - G(2^m x)\right| \\ &\le \theta\sum_{i=m+1}^{n}\frac{1}{2^i} \to 0\end{aligned}$$

as $m \to \infty$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function $L : (-\infty, \infty) \to (-\infty, \infty)$ by

$$L(x) := \lim_{n\to\infty} P_n(x)$$

and

$$F(x) := e^{L(x)} = \lim_{n\to\infty}\frac{f(2^n x)^{\frac{1}{2^n}}}{a^{\frac{x^2}{2}}a^{x^2}a^{2x^2}\cdots a^{2^{n-2}x^2}}$$

for all $x > 0$. Thus

$$\begin{aligned}&\frac{F(x+y)}{a^{xy}F(x)F(y)} \\ &= \lim_{n\to\infty}\frac{f(2^n x + 2^n y)^{\frac{1}{2^n}}a^{\frac{x^2}{2}}a^{x^2}a^{2x^2}\cdots a^{2^{n-2}x^2}a^{\frac{y^2}{2}}a^{y^2}a^{2y^2}\cdots a^{2^{n-2}y^2}}{a^{xy}f(2^n x)^{\frac{1}{2^n}}f(2^n y)^{\frac{1}{2^n}}a^{\frac{(x+y)^2}{2}}a^{(x+y)^2}a^{2(x+y)^2}\cdots a^{2^{n-2}(x+y)^2}} \\ &= \lim_{n\to\infty}\left[\frac{f(2^n x + 2^n y)}{a^{2^{2n}xy}f(2^n x)f(2^n y)}\right]^{\frac{1}{2^n}}\end{aligned}$$

and

$$(1-\delta)^{\frac{1}{2^n}} \le \left[\frac{f(2^n x + 2^n y)}{a^{2^{2n}xy}f(2^n x)f(2^n y)}\right]^{\frac{1}{2^n}} \le (1+\delta)^{\frac{1}{2^n}}$$

for all $x, y > 0$ and for every positive integer n. Therefore we have

$$F(x+y) = a^{xy}F(x)F(y)$$

for all $x, y > 0$. We can easily see from (4) that

$$(1-\delta)^{\frac{1}{2}} \le \frac{f(2x)^{\frac{1}{2}}}{a^{\frac{x^2}{2}}f(x)} \le (1+\delta)^{\frac{1}{2}}$$

for all $x > 0$. Note that for all $x > 0$ and for every positive integer n

$$\frac{f(2^n x)^{\frac{1}{2^n}}}{a^{\frac{x^2}{2}}a^{x^2}\cdots a^{2^{n-2}x^2}f(x)}$$
$$= \frac{f(2^n x)^{\frac{1}{2^n}}}{(a^{(2^{n-1}x)^2})^{\frac{1}{2^n}}f(2^{n-1}x)^{\frac{1}{2^{n-1}}}} \cdot \frac{f(2^{n-1}x)^{\frac{1}{2^{n-1}}}}{(a^{(2^{n-2}x)^2})^{\frac{1}{2^{n-1}}}f(2^{n-2}x)^{\frac{1}{2^{n-2}}}}$$
$$\cdots \frac{f(2x^{\frac{1}{2}})}{(a^{x^2})^{\frac{1}{2}}f(x)}.$$

Thus we have

$$(1-\delta)^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}} \le \frac{f(2^n x)^{\frac{1}{2^n}}}{a^{\frac{x^2}{2}}a^{x^2}\cdots a^{2^{n-2}x^2}f(x)} \le (1+\delta)^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}$$

and so

$$1-\delta \le \frac{F(x)}{f(x)} \le 1+\delta$$

for all $x > 0$. □

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