J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 15, Number 3 (August 2008), Pages 259–296

ISSN 1226-0657

JORDAN DERIVATIONS OF SEMIPRIME RINGS AND NONCOMMUTATIVE BANACH ALGEBRAS, II

BYUNG-DO KIM

ABSTRACT. Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)^2[D(x), x] \in \mathrm{rad}(A)$ or $[D(x), x]D(x)^2 \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

1. INTRODUCTION

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let X be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of a, denoted by $r(a)$, is defined by $r(a) = \inf\{||a^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if a be an element of a normed algebra, then $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$ (see F.F. Bonsall and J. Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B.E. Johnson and A.M. Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I.M. Singer and J. Wermer [9] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.

Received by the editors October 31, 2007. Accepted July 25, 2008.
2000 *Mathematics Subject Classification.* 16N60, 16W25, 17B40.
Key words and phrases. semiprime ring, Banach algebra, Jordan derivation.

M.P. Thomas [10] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

J. Vukman [11] has proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

The author [6] has showed that the following results hold: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, the author [7] has showed that the following results hold:let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \operatorname{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \operatorname{rad}(A)$.

In this paper, our first aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

let R be a 7!-torsion free semiprime ring.

(i) Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)^2[D(x), x] = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^7 = 0$ for all $x \in R$.

(ii) Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$[D(x), x]D(x)^2 = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^7 = 0$ for all $x \in R$.

Using the above results, we generalize J. Vukman's result [11] as follows: let A be a noncommutative Banach algebra and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation.

(iii) Suppose that $D(x)^2[D(x), x] \in \operatorname{rad}(A)$ holds for all $x \in A$. In this case, $D(A) \subseteq \operatorname{rad}(A)$.

(iv) Suppose that $[D(x), x]D(x)^2 \in \operatorname{rad}(A)$ holds for all $x \in A$. In this case, $D(A) \subseteq \operatorname{rad}(A)$.

2. Preliminaries and Results

The following lemma is due to L.O. Chung and J. Luh [4].

Lemma 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to M. Brešar [3].

Theorem 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

We write $Q(A)$ for the set of all quasinilpotent elements in A. M. Brešar [2] has proved the following theorem.

Theorem 2.3. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for every $x \in A$. Then D maps A into rad(A).*

After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer.

We need Theorem 2.4 and 2.5 to obtain the main theorems for Banach algebra theory.

Theorem 2.4. *Let R be a $7!$-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)^2[D(x), x] = 0$$

for all $x \in R$. In this case we have $[D(x), x]^7 = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x,y) = B(y,x),\ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(xy,z) = B(y,z)x + zB(y,x) + D(z)[x,y] + [z,y]D(x),$$
$$B(x,x^2) = 2(f(x)x + xf(x)),\ x,y,z \in R.$$

After this, we use the above relations without specific reference. By assumption,

$$D(x)^2 f(x) = 0,\ x \in R \tag{1}$$

Replacing $x+ty$ for x in (1), we have

$$\begin{aligned} & D(x+ty)^2[D(x+ty), x+ty] \\ \equiv\ & D(x)^2 f(x) + t\{D(y)D(x)f(x) + D(x)D(y)B(x,y) \\ & + D(x)^2 B(x,y)\} + t^2 H_1(x,y) + t^3 H_2(x,y) \\ & + t^4 D(y)^2 f(y) = 0,\ x, y \in R, t \in S_3 \end{aligned} \tag{2}$$

where H_i, $1 \le i \le 2$, denotes the term satisfying the identity (2).
From (1) and (2), we obtain

$$\begin{aligned} & t\{D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2 B(x,y)\} \\ & + t^2 H_1(x,y) + t^3 H_2(x,y) = 0,\ x, y \in R,\ t \in S_3. \end{aligned} \tag{3}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (3) yields

$$D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2 B(x,y) = 0, x, y \in R. \tag{4}$$

Let $y = x^2$ in (4). Then using (1), we have

$$\begin{aligned} & (D(x)x + xD(x))D(x)f(x) + D(x)(D(x)x + xD(x))f(x) \\ & + 2D(x)^2(f(x)x + xf(x)) \\ =\ & f(x)D(x)f(x) + (f(x)D(x) + D(x)f(x))f(x) + f(x)D(x)f(x) \\ & + 2(f(x)D(x) + D(x)f(x))f(x) \\ =\ & 5f(x)D(x)f(x) + 3D(x)f(x)^2 = 0, x \in R. \end{aligned} \tag{5}$$

From (1), we arrive at

$$\begin{aligned} 0 &= [D(x)^2 f(x), x] \\ &= f(x)D(x)f(x) + D(x)f(x)^2 + D(x)^2 g(x),\ x \in R. \end{aligned} \tag{6}$$

From (5) and (6), we get

$$2f(x)D(x)f(x) - 3D(x)^2 g(x) = 0,\ x \in R. \tag{7}$$

Combining (5) with (7),

$$3(2D(x)f(x)^2 + 5D(x)^2 g(x)) = 0,\ x \in R.$$

Since R is 3!-torsion-free, the above relation gives

$$2D(x)f(x)^2 + 5D(x)^2 g(x) = 0,\ x \in R. \tag{8}$$

Left multiplication of (8) by $D(x)$ leads to

$$2D(x)^2 f(x)^2 + 5D(x)^3 g(x) = 0,\ x \in R. \tag{9}$$

Comparing (1) and (9), we arrive at

$$5D(x)^3 g(x) = 0, \ x \in R. \tag{10}$$

Since R is 5!-torsion-free, (10) gives

$$D(x)^3 g(x) = 0, \ x \in R. \tag{11}$$

Writing xy for y in (4), we get

$$\begin{aligned} &xD(y)D(x)f(x) + D(x)yD(x)f(x) + D(x)xD(y)f(x) + D(x)^2 yf(x) \\ &+D(x)^2(xB(x,y) + 2f(x)y + D(x)[y,x]) = 0, \ x, y \in R. \end{aligned} \tag{12}$$

Left multiplication of (4) by x leads to

$$xD(y)D(x)f(x) + xD(x)D(y)f(x) + xD(x)^2 B(x,y) = 0, \ x, y \in R. \tag{13}$$

From (12) and (13), we arrive at

$$\begin{aligned} &D(x)yD(x)f(x) + f(x)D(y)f(x) + D(x)^2 yf(x) + (f(x)D(x) \\ &+D(x)f(x))B(x,y) + 2D(x)^2 f(x)y + D(x)^3[y,x] = 0, \ x, y \in R. \end{aligned} \tag{14}$$

By (1) and (14), it is obvious that

$$\begin{aligned} &D(x)yD(x)f(x) + f(x)D(y)f(x) + D(x)^2 yf(x) + (f(x)D(x) \\ &+D(x)f(x))B(x,y) + D(x)^3[y,x] = 0, \ x, y \in R. \end{aligned} \tag{15}$$

Replacing yx for y in (15), it follows from that

$$\begin{aligned} &D(x)yxD(x)f(x) + f(x)D(y)xf(x) + f(x)yD(x)f(x) \\ &+D(x)^2 yxf(x) + (f(x)D(x) + D(x)f(x))(B(x,y)x + 2yf(x) + [y,x]D(x)) \\ &+D(x)^3[y,x]x = 0, \ x, y \in R. \end{aligned} \tag{16}$$

Right multiplication of (15) by x leads to

$$\begin{aligned} &D(x)yD(x)f(x)x + f(x)D(y)f(x)x + D(x)^2 yf(x)x \\ &+(f(x)D(x) + D(x)f(x))B(x,y)x + D(x)^3[y,x]x = 0, \ x, y \in R. \end{aligned} \tag{17}$$

Combining (16) with (17), we see that

$$\begin{aligned} &-D(x)y(f(x)^2 + D(x)g(x)) - f(x)D(y)g(x) + f(x)yD(x)f(x) \\ &-D(x)^2 yg(x) + 2(f(x)D(x) + D(x)f(x))yf(x) \\ &+(f(x)D(x) + D(x)f(x))[y,x]D(x) = 0, \ x, y \in R. \end{aligned} \tag{18}$$

Left multiplication of (18) by $D(x)^2$ leads to

$$\begin{aligned}&-D(x)^3y(f(x)^2+D(x)g(x))-D(x)^2f(x)D(y)g(x)-D(x)^4yg(x)\\&+D(x)^2f(x)yD(x)f(x)+2D(x)^2(f(x)D(x)\\(19)\quad&+D(x)f(x))yf(x)+D(x)^2(f(x)D(x)+D(x)f(x))[y,x]D(x)=0,\ x,y\in R.\end{aligned}$$

Comparing (1) and (19), we get

$$D(x)^3y(f(x)^2+D(x)g(x))+D(x)^4yg(x)=0,\ x,y\in R. \tag{20}$$

Substituting $yD(x)^2$ for y in (20), we have

$$D(x)^3y(D(x)^2f(x)^2+D(x)^3g(x))+D(x)^4yD(x)^2g(x)=0,\ x,y\in R. \tag{21}$$

From (1), (11) and (21), it is clear that

$$D(x)^4yD(x)^2g(x)=0,\ x,y\in R. \tag{22}$$

Using (8), it follows from (22) that

$$2D(x)^4yD(x)^2g(x)=D(x)^4y(-5D(x)^2g(x))=0,\ x,y\in R. \tag{23}$$

Thus since R is 2-torsion-free, (23) gives

$$D(x)^4yD(x)^2g(x)=0,\ x,y\in R. \tag{24}$$

Combining (8) with (24) that

$$2D(x)^4yD(x)f(x)^2=-5D(x)^4yD(x)^2g(x)=0,\ x,y\in R. \tag{25}$$

And so, since R is 2-torsion-free, (25) yields

$$D(x)^4yD(x)f(x)^2=0,\ x,y\in R. \tag{26}$$

From (7) and (24), we have

$$\begin{aligned}&2D(x)^4yf(x)D(x)f(x)=D(x)^4y(3D(x)^2g(x))\\(27)\quad&=3D(x)^4yD(x)^2g(x)=0,\ x,y\in R.\end{aligned}$$

It follows from (27) that

$$D(x)^4yf(x)D(x)f(x)=0,\ x,y\in R \tag{28}$$

since R is 2-torsion-free. Writing $yf(x)D(x)f(x)z$ for y in (20), we have

$$\begin{aligned}&D(x)^3yf(x)D(x)f(x)z(f(x)^2+D(x)g(x))\\(29)\quad&+D(x)^4yf(x)D(x)f(x)zg(x)=0,\ x,y,z\in R.\end{aligned}$$

From (28) and (29), we obtain

$$D(x)^3yf(x)D(x)f(x)z(f(x)^2+D(x)g(x))=0,\ x,y,z\in R. \tag{30}$$

Letting $zD(x)$ in z in (30), we get

$$D(x)^3yf(x)D(x)f(x)z(D(x)f(x)^2+D(x)^2g(x))=0,\ x,y,z\in R. \tag{31}$$

Using (6), the relation (31) gives

$$D(x)^3yf(x)D(x)f(x)zf(x)D(x)f(x)=0,\ x,y,z\in R. \tag{32}$$

Putting $zD(x)^3y$ instead of z in (32),

$$D(x)^3yf(x)D(x)f(x)z(D(x)^3yf(x)D(x)f(x))=0,\ x,y,z\in R. \tag{33}$$

Since R is semiprime, we have

$$D(x)^3yf(x)D(x)f(x)=0,\ x,y\in R. \tag{34}$$

Replacing $x+tz$ for x in (34), we have

$$\begin{aligned}
&D(x+tz)^3yf(x+tz)D(x+tz)f(x+tz)\\
&\equiv D(x)^3yf(x)D(x)f(x)\\
&+t\{(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))yf(x)D(x)f(x)\\
&+D(x)^3y(B(x,z)D(x)f(x)+f(x)D(z)f(x)+f(x)D(x)B(x,z))\}\\
&+t^2I_1(x,y,z)+t^3I_2(x,y,z)+t^4I_3(x,y,z)\\
&+t^5I_4(x,y,z)+t^6I_5(x,y,z)+t^7I_6(x,y,z)\\
&+t^8D(z)^3yf(z)D(z)f(z)=0,\ x,y,z\in R,\ t\in S_7
\end{aligned} \tag{35}$$

where I_i, $1\le i\le 6$, denotes the term satisfying the identity (35).
From (34) and (35), we obtain

$$\begin{aligned}
&t\{(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))yf(x)D(x)f(x)\\
&+D(x)^3y(B(x,z)D(x)f(x)+f(x)D(z)f(x)+f(x)D(x)B(x,z))\}\\
&+t^2I_1(x,y,z)+t^3I_2(x,y,z)+t^4I_3(x,y,z)\\
&+t^5I_4(x,y,z)+t^6I_5(x,y,z)+t^7I_6(x,y,z)=0,\ x,y,z\in R,\ t\in S_7.
\end{aligned} \tag{36}$$

Since R is 7!-torsion-free by assumption, by Lemma 2.1 the relation (36) yields

$$\begin{aligned}
&(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))yf(x)D(x)f(x)\\
&+D(x)^3y(B(x,z)D(x)f(x)+f(x)D(z)f(x)+f(x)D(x)B(x,z))\\
&=0,\ x,y,z\in R.
\end{aligned} \tag{37}$$

Substituting $yf(x)D(x)f(x)w$ for y in (37), we obtain

$$\begin{aligned}&(D(z)D(x)^2 + D(x)D(z)D(x)\\&+D(x)^2D(z))yf(x)D(x)f(x)w(f(x)D(x)f(x)\\&+D(x)^3yf(x)D(x)f(x)w(B(x,z)D(x)f(x)\\&+f(x)D(z)f(x) + f(x)D(x)B(x,z)) = 0,\ x,y,z,w \in R.\end{aligned} \tag{38}$$

Combining (34) with (38), we get

$$\begin{aligned}&(D(z)D(x)^2 + D(x)D(z)D(x)\\&+D(x)^2D(z))yf(x)D(x)f(x))wf(x)D(x)f(x) = 0,\ w,x,y,z \in R.\end{aligned} \tag{39}$$

Writing $w(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))y$ for w in (39), we obtain

$$\begin{aligned}&(D(z)D(x)^2 + D(x)D(z)D(x)\\&+D(x)^2D(z))yf(x)D(x)f(x))w(D(z)D(x)^2 + D(x)D(z)D(x)\\&+D(x)^2D(z))yf(x)D(x)f(x) = 0,\ w,x,y,z \in R.\end{aligned} \tag{40}$$

Since R is semiprime, it follows from (40) that

$$\begin{aligned}&(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))yf(x)D(x)f(x)\\&= 0,\ x,y,z \in R,\ t \in S_7.\end{aligned} \tag{41}$$

Replacing $x + tu$ for x in (41), we have

$$\begin{aligned}&(D(z)D(x+tu)^2 + D(x+tu)D(z)D(x+tu)\\&+D(x+tu)^2D(z))yf(x+tu)D(x+tu)f(x+tu)\\\equiv\ &(D(z)D(x)^2 + D(x)D(z)D(x)\\&+D(x)^2D(z))yf(x)D(x)f(x)\\&+t\{(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x)\\&+D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x)\\&+(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))y(B(x,u)D(x)f(x)\\&+f(x)D(u)f(x) + f(x)D(x)B(x,u))\}\\&+t^2J_1(x,y,z,u) + t^3J_2(x,y,z,u) + t^4J_3(x,y,z,u)\\&+t^5J_4(x,y,z,u) + t^6J_5(x,y,z,u)\\&+t^7(D(z)D(u)^2 + D(u)D(z)D(u)\\&+D(u)^2D(z))yf(u)D(u)f(u) = 0,\ x,y,z,u \in R,\ t \in S_6\end{aligned} \tag{42}$$

where J_i, $1 \leq i \leq 5$, denotes the term satisfying the identity (42).
From (41) and (42), we obtain

$$\begin{aligned} & t\{(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) \\ & +D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x) \\ & +(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))y(B(x,u)D(x)f(x) \\ & +f(x)D(u)f(x) + f(x)D(x)B(x,u))\} \\ & +t^2J_1(x,y,z,u) + t^3J_2(x,y,z,u) + t^4J_3(x,y,z,u) \\ (43) \qquad & +t^5J_4(x,y,z,u) + t^6J_5(x,y,z,u) = 0, \ x,y,z,u \in R, \ t \in S_6. \end{aligned}$$

Since R is 6!-torsion-free by assumption, by Lemma 2.1 the relation (43) yields

$$\begin{aligned} & (D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) \\ & +D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x) \\ & +(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))y(B(x,u)D(x)f(x) \\ (44) \qquad & +f(x)D(u)f(x) + f(x)D(x)B(x,u)) = 0, \ x,y,z,u \in R. \end{aligned}$$

Substituting $yf(x)D(x)f(x)w$ for y in (44), we obtain

$$\begin{aligned} & (D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u) \\ & +(D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x)wf(x)D(x)f(x) \\ & +(D(z)D(x)^2 + D(x)D(z)D(x) \\ & +D(x)^2D(z))yf(x)D(x)f(x)w(B(x,u)D(x)f(x) \\ (45) \qquad & +f(x)D(u)f(x) + f(x)D(x)B(x,u)) = 0, \ x,y,z,u,w \in R. \end{aligned}$$

Combining (41) with (45), we get

$$\begin{aligned} & (D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u) \\ & +(D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x)wf(x)D(x)f(x) \\ (46) \qquad & = 0 \quad x,y,z,u,w \in R, \ t \in S_6. \end{aligned}$$

Replacing $w(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))y$ for w in (46), we obtain

$$\begin{aligned} & (D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) \\ & +D(x)D(z)D(u) + (D(u)D(x) \\ & +D(x)D(u))D(z))yf(x)D(x)f(x)w(D(z)(D(u)D(x) \end{aligned}$$

$$+D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u)$$
$$+(D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x) = 0,\ u, w, x, y, z \in R. \tag{47}$$

Since R is semiprime, it follows from (47) that

$$(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u)$$
$$+(D(u)D(x) + D(x)D(u))D(z))yf(x)D(x)f(x) = 0,\ x, y, z, u \in R. \tag{48}$$

Starting from (48), we have the following relation

$$(D(z)(D(u)D(v) + D(v)D(u)) + D(u)D(z)D(v)$$
$$+D(v)D(z)D(u) + (D(u)D(v)$$
$$+D(v)D(u))D(z))yf(x)D(x)f(x) = 0,\ x, y, z, u, v \in R \tag{49}$$

in the same fashion that makes it possible to obtain (48) from (41). And so, setting $v = u = z$ in (49), it is obvious that

$$6D(z)^3yf(x)D(x)f(x) = 0,\ x, y, z \in R.$$

Since R is 3!-torsion-free, replacing x, z, w for z, x, y in the above relation respectively, we obtain

$$D(x)^3wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{50}$$

Replacing $x + ty$ for x in (50), we have

$$D(x + ty)^3wf(z)D(z)f(z)$$
$$\equiv D(x)^3wf(z)D(z)f(z) + t\{(D(y)D(x)^2$$
$$+D(x)D(y)D(x) + D(x)^2D(y))wf(z)D(z)f(z)\}$$
$$+t^2K(x, y)wf(z)D(z)f(z) + t^3D(y)^3wf(z)D(z)f(z)$$
$$= 0,\ w, x, y, z \in R,\ t \in S_3 \tag{51}$$

where $K(x, y)$ denotes the term satisfying the identity (51).
From (50) and (51), we obtain

$$t\{(D(y)D(x)^2 + D(x)D(y)D(x) + D(x)^2D(y))wf(z)D(z)f(z)\}$$
$$+t^2K(x, y)wf(z)D(z)f(z) = 0,\ w, x, y, z \in R,\ t \in S_3. \tag{52}$$

Since R is 2!-torsion-free by assumption, by Lemma 2.1 the relation (52) yields

$$(D(y)D(x)^2 + D(x)D(y)D(x) + D(x)^2D(y))wf(z)D(z)f(z)$$
$$= 0,\ w, x, y, z \in R. \tag{53}$$

Substituting x^2 for y in (53), we obtain

$$(54)\quad \begin{aligned}&(D(x)xD(x)^2 + xD(x)^3 + D(x)(D(x)x + xD(x))D(x)\\&+D(x)^2(D(x)x + xD(x))wf(z)D(z)f(z) = 0,\ w,x,z \in R.\end{aligned}$$

Combining (50) with (54), we get

$$(55)\quad 2(f(x)D(x)^2 - D(x)^2f(x))wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Since R is 2!-torsion-free, the relation (55) yields

$$(56)\quad (f(x)D(x)^2 - D(x)^2f(x))wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

From (1) and (56), we have

$$(57)\quad f(x)D(x)^2wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

On the other hand, substituting xw for w in (50), we obtain

$$(58)\quad D(x)^3xwf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Left multiplication of (50) by x gives

$$(59)\quad xD(x)^3wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Combining (58) with (59), we get

$$(60)\quad \begin{aligned}&[D(x)^3, x]wf(z)D(z)f(z)\\&= (f(x)D(x)^2 + D(x)f(x)D(x) + D(x)^2f(x))wf(z)D(z)f(z) = 0,\ x \in R.\end{aligned}$$

From (1),(57) and (60), it is obvious that

$$(61)\quad D(x)f(x)D(x)wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Writing xw for w in (57), we obtain

$$(62)\quad f(x)D(x)^2xwf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Left multiplication of (57) by x gives

$$(63)\quad xf(x)D(x)^2wf(z)D(z)f(z) = 0,\ w,x,z \in R.$$

Comparing (62) and (63), we get

$$(64)\quad \begin{aligned}&[f(x)D(x)^2, x]wf(z)D(z)f(z)\\&= (g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x))wf(z)D(z)f(z) = 0,\ x \in R.\end{aligned}$$

Left multiplication of (53) by $D(x)^2$ gives

$$(D(x)^2D(y)D(x)^2 + D(x)^3D(y)D(x) + D(x)^4D(y))wf(z)D(z)f(z) = 0,\ w, x, y, z \in R. \tag{65}$$

From (50) and (65), we obtain

$$D(x)^2D(y)D(x)^2wf(z)D(z)f(z) = 0,\ w, x, y, z \in R. \tag{66}$$

Putting $yD(x)^2u$ instead of y in (66), we have

$$D(x)^2(D(y)D(x)^2u + yD(D(x)^2)u + yD(x)^2D(u))D(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, y, z \in R. \tag{67}$$

Combining (66) with (67), we get

$$D(x)^2yD(D(x)^2)uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, y, z \in R. \tag{68}$$

Substituting $wf(z)D(z)f(z)y$ for y in (53), we obtain

$$D(x)^2wf(z)D(z)f(z)yD(D(x)^2)uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, y, z \in R. \tag{69}$$

Left multiplication of (69) by $D(D(x)^2)u$ gives

$$D(D(x)^2)uD(x)^2wf(z)D(z)f(z)y(D(D(x)^2)uD(x)^2wf(z)D(z)f(z)) = 0,\ u, w, x, y, z \in R. \tag{70}$$

Since R is semiprime, the relation (70) yields

$$D(D(x)^2)uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, z \in R. \tag{71}$$

The relation (71) yields

$$(D^2(x)D(x) + D(x)D^2(x))uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, z \in R. \tag{72}$$

Left multiplication of (72) by $D(x)$ gives

$$(D(x)D^2(x)D(x) + D(x)^2D^2(x))uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, z \in R. \tag{73}$$

Writing $D(x)$ for y in (53), we have

$$(D^2(x)D(x)^2 + D(x)D^2(x)D(x) + D(x)^2D^2(x))wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{74}$$

Substituting $uD(x)^2w$ for w in (74), we obtain

$$(D^2(x)D(x)^2 + D(x)D^2(x)D(x) + D(x)^2D^2(x))uD(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, z \in R. \tag{75}$$

From (73) and (75), we have

$$D^2(x)D(x)^2uD(x)^2wf(z)D(z)f(z) = 0,\ x, z, u, w \in R. \tag{76}$$

And, putting $wf(z)D(z)f(z)uD^2(x)$ instead of u in (76), it follows that obtain

$$D^2(x)D(x)^2wf(z)D(z)f(z)uD^2(x)D(x)^2wf(z)D(z)f(z) = 0,\ u, w, x, z \in R. \tag{77}$$

Since R is semiprime, we get

$$D^2(x)D(x)^2wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{78}$$

Replacing $x + ty$ for x in (78), we have

$$\begin{aligned} &D^2(x+ty)D(x+ty)^2wf(z)D(z)f(z) \\ \equiv\ &D^2(x)D(x)^2wf(z)D(z)f(z) \\ &+t\{(D^2(y)D(x)^2 + D^2(x)D(y)D(x) \\ &+D^2(x)D(x)D(y))wf(z)D(z)f(z)\} \\ &+t^2L(x,y)wf(z)D(z)f(z) \\ &+t^3D^2(y)D(y)^2wf(z)D(z)f(z) = 0,\ x, y, z, w \in R,\ t \in S_3 \end{aligned} \tag{79}$$

where $L(x, y)$ denotes the term satisfying the identity (79).
From (78) and (79), we obtain

$$\begin{aligned} &t\{(D^2(y)D(x)^2 + D^2(x)D(y)D(x) \\ &+D^2(x)D(x)D(y))wf(z)D(z)f(z)\} \\ &+t^2L(x,y) = 0,\ x, y, z, w \in R,\ t \in S_3. \end{aligned} \tag{80}$$

Since R is 2!-torsion-free by assumption, by Lemma 2.1 the relation (80) yields

$$\begin{aligned} &(D^2(y)D(x)^2 + D^2(x)D(y)D(x) \\ &+D^2(x)D(x)D(y))wf(z)D(z)f(z) = 0,\ w, x, y, z \in R \end{aligned} \tag{81}$$

Substituting yx for y in (81), it follows that

$$\begin{aligned} &(D^2(y)xD(x)^2 + 2D(y)D(x)^3 + yD^2(x)D(x)^2 \\ &+D^2(x)D(y)xD(x) + D^2(x)yD(x)^2 \end{aligned}$$

$$+D^2(x)D(x)D(y)x+D^2(x)D(x)yD(x))wf(z)D(z)f(z)$$
$$=0,\quad w,x,y,z\in R. \tag{82}$$

Writing xw for w in (81), we get

$$(D^2(y)D(x)^2x+D^2(x)D(y)D(x)x$$
$$+D^2(x)D(x)D(y)x)wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{83}$$

Combining (50), (82) with (83), we have

$$(-D^2(y)(f(x)D(x)+D(x)f(x))-D^2(x)D(y)f(x)$$
$$+D^2(x)yD(x)^2+D^2(x)D(x)yD(x))wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{84}$$

Replacing $D(x)w$ for w in (84), we obtain

$$(-D^2(y)(f(x)D(x)^2+D(x)f(x)D(x))-D^2(x)D(y)f(x)D(x)$$
$$+D^2(x)yD(x)^3+D^2(x)D(x)yD(x)^2)wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{85}$$

Combining (50), (57), (61) with (85), we have

$$(-D^2(x)D(y)f(x)D(x)+D^2(x)D(x)yD(x)^2)wf(z)D(z)f(z)$$
$$=0,\quad w,x,y,z\in R. \tag{86}$$

Substituting xy for y in (53), we obtain

$$(xD(y)D(x)^2+D(x)yD(x)^2+D(x)xD(y)D(x)$$
$$+D(x)^2yD(x)+D(x)^2xD(y)+D(x)^3y)wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{87}$$

Left multiplication of (53) by x gives

$$(xD(y)D(x)^2+xD(x)D(y)D(x)$$
$$+xD(x)^2D(y))wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{88}$$

Combining (50), (87) with (88), we obtain

$$(D(x)yD(x)^2+f(x)D(y)D(x)$$
$$+D(x)^2yD(x)+(f(x)D(x)+D(x)f(x))D(y))wf(z)D(z)f(z)$$
$$=0,\quad w,x,y,z\in R. \tag{89}$$

Putting yx instead of y in (89), we have

$$(D(x)yxD(x)^2+f(x)D(y)xD(x)+f(x)yD(x)^2$$
$$+D(x)^2yxD(x)+(f(x)D(x)+D(x)f(x))D(y)x$$
$$+(f(x)D(x)+D(x)f(x))yD(x))wf(z)D(z)f(z)=0,\ w,x,y,z\in R. \tag{90}$$

Substituting xw for w in (89), we have

$$(D(x)yD(x)^2x + f(x)D(y)D(x)x$$
$$+D(x)^2yD(x)^2x + (f(x)D(x) + D(x)f(x))D(y)x)wf(z)D(z)f(z)$$
$$= 0,\ w, x, y, z \in R. \tag{91}$$

From (90) and (91), we get

$$(-D(x)y(f(x)D(x) + D(x)f(x)) - f(x)D(y)f(x) + f(x)yD(x)^2$$
$$-D(x)^2yf(x) + (f(x)D(x) + D(x)f(x))yD(x))wf(z)D(z)f(z)$$
$$= 0,\ x, y, z, w \in R. \tag{92}$$

Let $y = x$ in (92). Then we have

$$(-D(x)x(f(x)D(x) + D(x)f(x)) - f(x)D(x)f(x) + f(x)xD(x)^2$$
$$-D(x)^2xf(x) + (f(x)D(x) + D(x)f(x))xD(x))wf(z)D(z)f(z)$$
$$= 0,\ w, x, z \in R. \tag{93}$$

Using (1), (57) and (61), the relation (93)can be written as

$$(-f(x)^2D(x) - 3f(x)D(x)f(x) + g(x)D(x)^2 + D(x)^2g(x)$$
$$+D(x)f(x)^2)wf(z)D(z)f(z)$$
$$= -2f(x)^2D(x) - 5f(x)D(x)f(x) - 2D(x)f(x)^2)wf(z)D(z)f(z)$$
$$= 0,\ w, x, z \in R.$$

Hence the above relation gives

$$(2f(x)^2D(x) + 5f(x)D(x)f(x) + 2D(x)f(x)^2)wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{94}$$

From (5) and (94), we get

$$(2f(x)^2D(x) - D(x)f(x)^2)wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{95}$$

Comparing (5) and (95), we have

$$(5f(x)D(x)f(x) + 6f(x)^2D(x))wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{96}$$

Writing $f(x)w$ for w in (96), it follows that

$$(5f(x)D(x)f(x)^2 + 6f(x)^2D(x)f(x))wf(z)D(z)f(z) = 0,\ w, x, z \in R. \tag{97}$$

Replacing $f(x)w$ for w in (64), we obtain

$$(g(x)D(x)^2f(x) + f(x)^2D(x)f(x) + f(x)D(x)f(x)^2)wf(z)D(z)f(z)$$
$$= 0,\ w, x, z \in R. \tag{98}$$

From (1) and (98), we have

$$(f(x)^2D(x)f(x)+f(x)D(x)f(x)^2)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{99}$$

Subtracting $5\times$ (99) from (97), we get

$$f(x)^2D(x)f(x)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{100}$$

From (99) and (100), we have

$$f(x)D(x)f(x)^2wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{101}$$

Writing $f(x)w$ for w in (95), it follows that

$$(2f(x)^2D(x)f(x)-D(x)f(x)^3)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{102}$$

From (100) and (102), we obtain

$$D(x)f(x)^3wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{103}$$

Right multiplication of (103) by $5f(z)$ gives

$$D(x)f(x)^3)w(5f(z)D(z)f(z)^2)=0,\ w,x,z\in R. \tag{104}$$

From (5) and (104), we obtain

$$D(x)f(x)^3w(-3D(z)f(z)^3)=0,\ w,x,z\in R. \tag{105}$$

Thus since R is 3!-torsion-free, setting $z=x$ in (105) we have

$$D(x)f(x)^3wD(x)f(x)^3=0,\ w,x\in R. \tag{106}$$

Since R is semiprime, it follows from (106) that

$$D(x)f(x)^3=0,\ x\in R. \tag{107}$$

Left multiplication of (95) by $f(x)$ gives

$$(2f(x)^3D(x)-f(x)D(x)f(x)^2)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{108}$$

Combining (101) with (108), we obtain

$$2f(x)^3D(x)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{109}$$

Since R is 2!-torsion-free, we get from (109)

$$f(x)^3D(x)wf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{110}$$

Writing xw for w in (110), we obtain

$$f(x)^3D(x)xwf(z)D(z)f(z)=0,\ w,x,z\in R. \tag{111}$$

Left multiplication of (110) by x gives

$$xf(x)^3D(x)wf(z)D(z)f(z) = 0,\ w,x,z \in R. \tag{112}$$

Combining (111) with (112), we obtain

$$\begin{aligned}&(g(x)f(x)^2D(x) + f(x)g(x)f(x)D(x) + f(x)^2g(x)D(x)\\&+f(x)^4)wf(z)D(z)f(z) = 0,\ w,x,z \in R.\end{aligned} \tag{113}$$

Writing $f(x)^3w$ for w in (113), we obtain

$$\begin{aligned}&(g(x)f(x)^2D(x)f(x)^3 + f(x)g(x)f(x)D(x)f(x)^3\\&+f(x)^2g(x)D(x)f(x)^3 + f(x)^7)wf(z)D(z)f(z)\\&= 0,\ w,x,z \in R.\end{aligned} \tag{114}$$

Combining (107) with (114), we obtain

$$f(x)^7wf(z)D(z)f(z) = 0,\ w,x,z \in R. \tag{115}$$

Right multiplication of (115) by z gives

$$f(x)^7wf(z)D(z)f(z)z = 0,\ w,x,z \in R. \tag{116}$$

Writing wz for w in (115), it follows that

$$f(x)^7wzf(z)D(z)f(z) = 0,\ w,x,z \in R. \tag{117}$$

Combining (116) with (117), we get

$$\begin{aligned}&f(x)^7w[f(z)D(z)f(z), z]\\&= f(x)^7w(g(z)D(z)f(z) + f(z)^3 + f(z)D(z)g(z)) = 0,\ w,x,z \in R.\end{aligned} \tag{118}$$

On the other hand, replacing $y + tz$ for z in (103), we have

$$\begin{aligned}&f(x)^7wf(y+tz)D(y+tz)f(y+tz)\\&\equiv f(x)^7wf(y)D(y)f(y)\\&+t\{f(x)^7w(B(y,z)D(y)f(y) + f(y)D(z)f(y) + f(y)D(y)B(y.z))\}\\&+t^2f(x)^7wM_1(y,z) + t^3f(x)^7wM_2(y,z) + t^4f(x)^7wM_4(y,z)\\&+t^5f(x)^7wf(z)D(z)f(z) = 0,\ w,x,y,z \in R,\ t \in S_3\end{aligned} \tag{119}$$

where $M_i(y,z)$, $1 \le i \le 4$, denotes the term satisfying the identity (119).
From (103) and (119), we obtain

$$\begin{aligned} &t\{f(x)^7w(B(y,z)D(y)f(y)+f(y)D(z)f(y)+f(y)D(z)B(y,z))\} \\ &+t^2f(x)^7wM_1(y,z)+t^3f(x)^7wM_2(y,z)+t^4f(x)^7wM_4(y,z) \\ (120) \qquad &=0, \ w,x,y,z\in R, \ t\in S_4. \end{aligned}$$

Since R is 4!-torsion-free by assumption, by Lemma 2.1 the relation (120) yields

$$\begin{aligned} &f(x)^7w(B(y,z)D(y)f(y)+f(y)D(z)f(y)+f(y)D(y)B(y,z)) \\ (121) \qquad &=0, \ w,x,y,z\in R. \end{aligned}$$

Substituting y^2 for z in (121), we obtain

$$\begin{aligned} &f(x)^7w(3f(y)yD(y)f(y)+2yf(y)D(y)f(y)+2f(y)D(y)f(y)y \\ (122) \qquad &+3f(y)D(y)yf(y))=0, \ w,x,y\in R. \end{aligned}$$

Combining (115) with (122), using (113) we have

$$(123) \qquad f(x)^7w(3(g(y)D(y)f(y)-f(y)D(y)g(y)))=0, \ w,x,y\in R.$$

Since R is 3!-torsion-free, we get (123)

$$(124) \qquad f(x)^7w(g(y)D(y)f(y)-f(y)D(y)g(y))=0, \ w,x,y\in R.$$

Combining (118) with (124), we obtain

$$(125) \qquad f(x)^7w(2g(z)D(z)f(z)+f(z)^3)=0, \ w,x,z\in R.$$

Right multiplication of (125) by $f(z)^4$ gives

$$(126) \qquad f(x)^7w(2g(z)D(z)f(z)^5+f(z)^7)=0, \ w,x,z\in R.$$

Combining (107) with (126), and setting $z=x$ in (126) we obtain

$$(127) \qquad f(x)^7wf(x)^7=0, \ w,x\in R.$$

Since R is semiprime, we obtain from (127)

$$f(x)^7=0, \ x\in R.$$

□

Theorem 2.5. *Let R be a 7!-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D: R \longrightarrow R$ such that*

$$[D(x),x]D(x)^2=0$$

for all $x\in R$. In this case we have $[D(x),x]^7=0$ for all $x\in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x,y) \equiv [D(x),y] + [D(y),x]$, $f(x) \equiv [D(x),x]$, $g(x) \equiv [f(x),x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x,y) = B(y,x),\ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(xy,z) = B(y,z)x + zB(y,x) + D(z)[x,y] + [z,y]D(x),$$
$$B(x,x^2) = 2(f(x)x + xf(x)),\ x,y,z \in R.$$

After this, we use the above relations without specific reference. By assumption,

$$f(x)D(x)^2 = 0,\ x \in R \tag{128}$$

Replacing $x + ty$ for x in (128), we have

$$\begin{aligned} &f(x+ty)D(x+ty)^2 \\ &\equiv f(x)D(x)^2 + t\{B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\} \\ &\quad +t^2N_1(x,y) + t^3N_2(x,y) + t^4 f(y)D(y)^2 \\ &= 0,\ x,y \in R,\ t \in S_3 \end{aligned} \tag{129}$$

where N_i, $1 \le i \le 2$, denotes the term satisfying the identity (129).

From (128) and (129), we obtain

$$\begin{aligned} &t\{B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\} \\ &+t^2N_1(x,y) + t^3N_2(x,y) = 0,\ x,y \in R,\ t \in S_3 \end{aligned} \tag{130}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (130) yields

$$B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y) = 0,\ x,y \in R. \tag{131}$$

Let $y = x^2$ in (131). Then using (128), we have

$$\begin{aligned} &2(f(x)x + xf(x))D(x)^2 + f(x)(D(x)x + xD(x))D(x) \\ &+f(x)D(x)(D(x)x + xD(x)) \\ &= -2f(x)(f(x)D(x) + D(x)f(x)) - f(x)D(x)f(x) \\ &\quad -f(x)(f(x)D(x) + D(x)f(x)) - f(x)D(x)f(x) \\ &= -3f(x)^2D(x) - 5f(x)D(x)f(x) = 0,\ x \in R. \end{aligned}$$

Thus the above relation gives

$$3f(x)^2D(x) + 5f(x)D(x)f(x) = 0,\ x \in R. \tag{132}$$

Using (128), we arrive at

$$\begin{aligned} 0 &= [f(x)D(x)^2, x] \\ &= g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x), \ x \in R. \end{aligned} \tag{133}$$

From (132) and (133), we get

$$2f(x)D(x)f(x) - 3g(x)D(x)^2 = 0, \ x \in R. \tag{134}$$

Combining (132) with (134),

$$3(2f(x)^2D(x) + 5g(x)D(x)^2) = 0, \ x \in R.$$

Since R is 3!-torsion-free, the above relation gives

$$2f(x)^2D(x) + 5g(x)D(x)^2 = 0, \ x \in R. \tag{135}$$

Right multiplication of (135) by $D(x)$ leads to

$$2f(x)^2D(x)^2 + 5g(x)D(x)^3 = 0, \ x \in R. \tag{136}$$

Comparing (128) and (136), we arrive at

$$5g(x)D(x)^3 = 0, \ x \in R. \tag{137}$$

Since R is 5!-torsion-free, (137) gives

$$g(x)D(x)^3 = 0, \ x \in R. \tag{138}$$

Writing yx for y in (131), we get

$$\begin{aligned} &(B(x,y)x + 2yf(x) + [y,x]D(x))D(x)^2 + f(x)(D(y)x + yD(x))D(x) \\ &f(x)D(x)(D(y)x + yD(x)) = 0, \ x,y \in R. \end{aligned} \tag{139}$$

Right multiplication of (131) by x leads to

$$B(x,y)D(x)^2x + f(x)D(y)D(x)x + f(x)D(x)D(y)x = 0, \ x,y \in R. \tag{140}$$

From (139) and (140), we arrive at

$$\begin{aligned} &-B(x,y)(f(x)D(x) + D(x)f(x)) + 2yf(x)D(x)^2 + [y,x]D(x)^3 \\ &-f(x)D(y)f(x) + f(x)yD(x)^2 + f(x)D(x)yD(x) \\ &= 0, \ x,y \in R. \end{aligned} \tag{141}$$

By (128) and (141), it follows that

$$\begin{aligned} &-B(x,y)(f(x)D(x) + D(x)f(x)) + [y,x]D(x)^3 \\ &-f(x)D(y)f(x) + f(x)yD(x)^2 + f(x)D(x)yD(x) = 0, \ x,y \in R. \end{aligned} \tag{142}$$

Replacing xy for y in (142), it follows from that

$$\begin{aligned}&-(xB(x,y)+2f(x)y+D(x)[y,x])(f(x)D(x)+D(x)f(x))+x[y,x]D(x)^3\\&-f(x)(xD(y)+D(x)y)f(x)+f(x)xyD(x)^2+f(x)D(x)xyD(x)\\(143)\quad &=0,\ x,y\in R.\end{aligned}$$

Left multiplication of (142) by x leads to

$$\begin{aligned}&-xB(x,y)(f(x)D(x)+D(x)f(x))+x[y,x]D(x)^3\\(144)\quad &-xf(x)D(y)f(x)+xf(x)yD(x)^2+xf(x)D(x)yD(x)=0,\ x,y\in R.\end{aligned}$$

Comparing (143) and (144), we see that

$$\begin{aligned}&-2f(x)y(f(x)D(x)+D(x)f(x))-D(x)[y,x](f(x)D(x)+D(x)f(x))\\&-g(x)D(y)f(x)-f(x)D(x)yf(x)+g(x)yD(x)^2\\(145)\quad &+(g(x)D(x)+f(x)^2)yD(x)=0,\ x,y\in R.\end{aligned}$$

Right multiplication of (145) by $D(x)^2$ leads to

$$\begin{aligned}&-2f(x)y(f(x)D(x)^3+D(x)f(x)D(x)^2)\\&-D(x)[y,x](f(x)D(x)^3+D(x)f(x)D(x)^2)\\&-g(x)D(y)f(x)D(x)^2-f(x)D(x)yf(x)D(x)^2+g(x)yD(x)^4\\(146)\quad &+(g(x)D(x)+f(x)^2)yD(x)^3=0,\ x,y\in R.\end{aligned}$$

Combining (128) with (146), we get

$$(147)\qquad g(x)yD(x)^4+(g(x)D(x)+f(x)^2)yD(x)^3=0,\ x,y\in R.$$

Replacing $D(x)^2y$ for y in (147), we have

$$(148)\qquad g(x)D(x)^2yD(x)^4+(g(x)D(x)^3+f(x)^2D(x)^2)yD(x)^3=0,\ x,y\in R.$$

From (128),(138) and (148), it is obvious that

$$(149)\qquad g(x)D(x)^2yD(x)^4=0,\ x,y\in R.$$

Using (135), it follows from (149) that

$$(150)\qquad 2f(x)^2D(x)yD(x)^4=-5g(x)D(x)^2yD(x)^4=0,\ x,y\in R.$$

Thus since R is 2-torsion-free, (150) gives

$$(151)\qquad f(x)^2D(x)yD(x)^4=0,\ x,y\in R.$$

From (134) and (151), we have

$$(152)\qquad 2f(x)D(x)f(x)yD(x)^4=3g(x)D(x)^2yD(x)^4=0,\ x,y\in R.$$

And so, since R is 2-torsion-free, (152) yields

$$f(x)D(x)f(x)yD(x)^4 = 0, \ x, y \in R. \tag{153}$$

Substituting $zf(x)D(x)f(x)y$ for y in (147), we have

$$\begin{aligned} &g(x)zf(x)D(x)f(x)yD(x)^4 + (g(x)D(x) \\ &+f(x)^2)zf(x)D(x)f(x)yD(x)^3 = 0, \ x, y, z \in R. \end{aligned} \tag{154}$$

From (153) and (154), we obtain

$$(g(x)D(x) + f(x)^2)zf(x)D(x)f(x)yD(x)^3 = 0, \ x, y, z \in R. \tag{155}$$

Putting $D(x)z$ instead of z in (155), we get

$$(g(x)D(x)^2 + f(x)^2D(x))zf(x)D(x)f(x)yD(x)^3 = 0, \ x, y, z \in R. \tag{156}$$

Using (133), the relation (156) gives

$$\begin{aligned} &f(x)D(x)f(x)zf(x)D(x)f(x)yD(x)^3 \\ &= -(g(x)D(x)^2 + f(x)^2D(x))zf(x)D(x)f(x)yD(x)^3 = 0, \ x, y, z \in R. \end{aligned} \tag{157}$$

Writing $yD(x)^3z$ for z in (157),

$$\begin{aligned} &f(x)D(x)f(x)yD(x)^3zf(x)D(x)f(x)yD(x)^3 \\ &= 0, x, y, z \in R. \end{aligned} \tag{158}$$

Since R is semiprime, we have from (158)

$$f(x)D(x)f(x)yD(x)^3 = 0, x, y \in R. \tag{159}$$

Replacing $x + tz$ for x in (159), we have

$$\begin{aligned} &f(x+tz)D(x+tz)f(x+tz)yD(x+tz)^3 \\ &\equiv f(x)D(x)f(x)yD(x)^3 \\ &+t\{(B(x,z)D(x)f(x) + f(x)D(z)f(x) + f(x)D(x)B(x,z))yD(x)^3 \\ &+f(x)D(x)f(x)y(D(z)D(x)^2 + D(x)D(z)D(x) + D(x)^2D(z))\} \\ &+t^2P_1(x,y,z) + t^3P_2(x,y,z) + t^4P_3(x,y,z) \\ &+t^5P_4(x,y,z) + t^6P_5(x,y,z) + t^7P_6(x,y,z) \\ &+t^8f(z)D(z)f(z)yD(z)^3 = 0, \ x, y, z \in R, \ t \in S_7 \end{aligned} \tag{160}$$

where $P_i, 1 \le i \le 6$, denotes the term satisfying the identity (160).

From (159) and (160), we obtain

$$t\{(B(x,z)D(x)f(x)+f(x)D(z)f(x)+f(x)D(x)B(x,z))yD(x)^3$$
$$+f(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))\}$$
$$+t^2P_1(x,y,z)+t^3P_2(x,y,z)+t^4P_3(x,y,z)$$
$$+t^5P_4(x,y,z)+t^6P_5(x,y,z)+t^7P_6(x,y,z)$$
$$=0,\ x,y,z\in R,\ t\in S_7. \tag{161}$$

Since R is 7!-torsion-free by assumption, by Lemma 2.1 the relation (161) yields

$$(B(x,z)D(x)f(x)+f(x)D(z)f(x)+f(x)D(x)B(x,z))yD(x)^3$$
$$+f(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))$$
$$=0,\ x,y,z\in R. \tag{162}$$

Substituting $wf(x)D(x)f(x)y$ for y in (162), we obtain

$$(B(x,z)D(x)f(x)+f(x)D(z)f(x)$$
$$+f(x)D(x)B(x,z))wf(x)D(x)f(x)yD(x)^3$$
$$+f(x)D(x)f(x)wf(x)D(x)f(x)y(D(z)D(x)^2$$
$$+D(x)D(z)D(x)+D(x)^2D(z))=0,\ w,x,y,z\in R. \tag{163}$$

Combining (159) with (163), we get

$$f(x)D(x)f(x)wf(x)D(x)f(x)y(D(z)D(x)^2$$
$$+D(x)D(z)D(x)+D(x)^2D(z))=0,\ w,x,y,z\in R. \tag{164}$$

Writing $y(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))w$ for w in (164), we obtain

$$f(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)$$
$$+D(x)^2D(z))wf(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)$$
$$+D(x)^2D(z))=0,\ w,x,y,z\in R. \tag{165}$$

Since R is semiprime, it follows from (165) that

$$f(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))=0,\ x,y,z\in R. \tag{166}$$

Replacing $x+tu$ for x in (166), we have

$$f(x+tu)D(x+tu)f(x+tu)y(D(z)D(x+tu)^2$$
$$+D(x+tu)D(z)D(x+tu)+D(x+tu)^2D(z))$$
$$\equiv f(x)D(x)f(x)y(D(z)D(x)^2+D(x)D(z)D(x)+D(x)^2D(z))$$

$$\begin{aligned}
&+t\{f(x)D(x)f(x)y(D(z)(D(u)D(x)+D(x)D(u))+D(u)D(z)D(x)\\
&+D(x)D(z)D(u)+(D(u)D(x)+D(x)D(u))D(z))+(B(x,u)D(x)f(x)\\
&+f(x)D(u)f(x)+f(x)D(x)B(x,u))y(D(z)D(x)^2\\
&+D(x)D(z)D(x)+D(x)^2D(z))\}\\
&+t^2Q_1(x,y,z,u)+t^3JQ_2(x,y,z,u)+t^4Q_3(x,y,z,u)\\
&+t^5Q_4(x,y,z,u)+t^6Q_5(x,y,z,u)\\
&+t^7f(u)D(u)f(u)y(D(z)D(u)^2+D(u)D(z)D(u)\\
&+D(u)^2D(z))=0,\ u,x,y,z\in R,\ t\in S_6 \qquad (167)
\end{aligned}$$

where Q_i, $1\le i\le 5$, denotes the term satisfying the identity (167).
From (166) and (167), we obtain

$$\begin{aligned}
&t\{f(x)D(x)f(x)y(D(z)(D(u)D(x)+D(x)D(u))+D(u)D(z)D(x)\\
&+D(x)D(z)D(u)+(D(u)D(x)+D(x)D(u))D(z))+(B(x,u)D(x)f(x)\\
&+f(x)D(u)f(x)+f(x)D(x)B(x,u))y(D(z)D(x)^2+D(x)D(z)D(x)\\
&+D(x)^2D(z))\}+t^2Q_1(x,y,z,u)+t^3JQ_2(x,y,z,u)+t^4Q_3(x,y,z,u)\\
&+t^5Q_4(x,y,z,u)+t^6Q_5(x,y,z,u)=0,\ u,x,y,z\in R,\ t\in S_6. \qquad (168)
\end{aligned}$$

Since R is 6!-torsion-free by assumption, by Lemma 2.1 the relation (168) yields

$$\begin{aligned}
&f(x)D(x)f(x)y(D(z)(D(u)D(x)+D(x)D(u))+D(u)D(z)D(x)\\
&+D(x)D(z)D(u)+(D(u)D(x)+D(x)D(u))D(z))+(B(x,u)D(x)f(x)\\
&+f(x)D(u)f(x)+f(x)D(x)B(x,u))y(D(z)D(x)^2\\
&+D(x)D(z)D(x)+D(x)^2D(z))=0,\ u,x,y,z\in R. \qquad (169)
\end{aligned}$$

Substituting $wf(x)D(x)f(x)y$ for y in (169), we obtain

$$\begin{aligned}
&f(x)D(x)f(x)wf(x)D(x)f(x)y(D(z)(D(u)D(x)+D(x)D(u))\\
&+D(u)D(z)D(x)+D(x)D(z)D(u)+(D(u)D(x)+D(x)D(u))D(z))\\
&+(B(x,u)D(x)f(x)+f(x)D(u)f(x)\\
&+f(x)D(x)B(x,u))wf(x)D(x)f(x)y(D(z)D(x)^2\\
&+D(x)D(z)D(x)+D(x)^2D(z))=0,\ x,y,z,u\in R. \qquad (170)
\end{aligned}$$

Combining (166) with (170), we get

$$f(x)D(x)f(x)wf(x)D(x)f(x)y(D(z)(D(u)D(x)+D(x)D(u))$$

$$+D(u)D(z)D(x) + D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))$$

(171) $$= 0, \; u, x, y, z \in R.$$

Putting $y(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x) + D(x)D(z)D(u)$ $+(D(u)D(x) + D(x)D(u))D(z))w$ instead of w in (171), we obtain

$$f(x)D(x)f(x)y(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x)$$
$$+D(x)D(z)D(u) + (D(u)D(x)$$
$$+D(x)D(u))D(z))wf(x)D(x)f(x)y(D(z)(D(u)D(x) + D(x)D(u))$$
$$+D(u)D(z)D(x) + D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))$$

(172) $$= 0, \; u, w, x, y, z \in R.$$

Since R is semiprime, it follows from (172) that

$$f(x)D(x)f(x)y(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x)$$

(173) $$+D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z)) = 0, \; u, w, x, y, z \in R.$$

Starting from (173), we have

$$f(x)D(x)f(x)y(D(z)(D(u)D(x) + D(x)D(u)) + D(u)D(z)D(x)$$
$$+D(x)D(z)D(u) + (D(u)D(x)$$
$$+D(x)D(u))D(z))wf(x)D(x)f(x)y(D(z)(D(u)D(x) + D(x)D(u))$$
$$+D(u)D(z)D(x) + D(x)D(z)D(u) + (D(u)D(x) + D(x)D(u))D(z))$$

(174) $$= 0, \; u, v, w, x, y, z \in R.$$

in the same fashion that makes it possible to obtain (173) from (166).

And so, setting $v = u = z$ in (174), the relation (174) yields

$$6f(x)D(x)f(x)yD(z)^3 = 0, \; x, y, z \in R.$$

Since R is 3!-torsion-free, replacing x, z, w for z, x, y in the above relation, it is obvious that

(175) $$f(z)D(z)f(z)wD(x)^3 = 0, \; x, z, w \in R.$$

Replacing $x + ty$ for x in (175), we have

$$f(z)D(z)f(z)wD(x + ty)^3$$
$$= f(z)D(z)f(z)wD(x)^3 + t\{f(z)D(z)f(z)w(D(y)D(x)^2$$
$$+D(x)D(y)D(x) + D(x)^2D(y))\}$$

$$+t^2 f(z)D(z)f(z)wS(x,y) + t^3 f(z)D(z)f(z)wD(y)^3$$

$$= 0,\ w,x,y,z \in R,\ t \in S_2 \tag{176}$$

where $S(x,y)$ denotes the term satisfying the identity (176).
From (175) and (176), we obtain

$$t\{f(z)D(z)f(z)w(D(y)D(x)^2 + D(x)D(y)D(x) + D(x)^2D(y))\}$$

$$+t^2 f(z)D(z)f(z)wS(x,y) = 0,\ w,x,y,z \in R,\ t \in S_3 \tag{177}$$

Since R is 2!-torsion-free by assumption, by Lemma 2.1 the relation (177) yields

$$f(z)D(z)f(z)w(D(y)D(x)^2 + D(x)D(y)D(x) + D(x)^2D(y))$$

$$= 0,\ w,x,y,z \in R. \tag{178}$$

Substituting x^2 for y in (178), we obtain

$$f(z)D(z)f(z)w(D(x)xD(x)^2 + xD(x)^3 + D(x)(D(x)x + xD(x))D(x)$$

$$+D(x)^2(D(x)x + xD(x))) = 0,\ w,x,z \in R. \tag{179}$$

Combining (175) with (179), we get

$$2f(z)D(z)f(z)w(f(x)D(x)^2 - D(x)^2f(x)) = 0,\ w,x,z \in R. \tag{180}$$

Since R is 2!-torsion-free by assumption, the relation (180) yields

$$f(z)D(z)f(z)w(f(x)D(x)^2 - D(x)^2f(x)) = 0,\ w,x,z \in R. \tag{181}$$

From (128) and (181), we have

$$f(z)D(z)f(z)wD(x)^2f(x) = 0,\ w,x,z \in R. \tag{182}$$

On the other hand, substituting wx for w in (175), we obtain

$$f(z)D(z)f(z)wxD(x)^3 = 0,\ w,x,z \in R. \tag{183}$$

Right multiplication of (175) by x gives

$$f(z)D(z)f(z)wD(x)^3x = 0,\ w,x,z \in R. \tag{184}$$

Combining (183) with (184), we get

$$f(z)D(z)f(z)w[D(x)^3, x]$$

$$= f(z)D(z)f(z)w(f(x)D(x)^2$$

$$+D(x)f(x)D(x) + D(x)^2f(x)) = 0,\ w,x,z \in R. \tag{185}$$

From (128), (182) and (185), it is obvious that

$$f(z)D(z)f(z)wD(x)f(x)D(x) = 0,\ w,x,z \in R. \tag{186}$$

Writing wx for w in (182), we obtain

$$f(z)D(z)f(z)wxD(x)^2f(x) = 0,\ w, x, z \in R. \tag{187}$$

Right multiplication of (182) by x gives

$$f(z)D(z)f(z)wD(x)^2f(x)x = 0,\ w, x, z \in R. \tag{188}$$

Combining (187) with (188), we get

$$\begin{aligned} &f(z)D(z)f(z)w[D(x)^2f(x), x] \\ &= f(z)D(z)f(z)w(f(x)D(x)f(x) \\ &+D(x)f(x)^2 + D(x)^2g(x)) = 0,\ w, x, z \in R. \end{aligned} \tag{189}$$

Right multiplication of (178) by $D(x)^2$ gives

$$\begin{aligned} &f(z)D(z)f(z)w(D(y)D(x)^4 \\ &+D(x)D(y)D(x)^3 + D(x)^2D(y)D(x)^2) = 0,\ w, x, y, z \in R. \end{aligned} \tag{190}$$

From (175) and (190), we obtain

$$f(z)D(z)f(z)wD(x)^2D(y)D(x)^2 = 0,\ w, x, y, z \in R. \tag{191}$$

Putting $uD(x)^2y$ instead of y in (191), we have

$$\begin{aligned} &f(z)D(z)f(z)w(D(x)^2(D(y)D(x)^2uD(x)^2 \\ &+D(x)^2yD(D(x)^2)uD(x)^2 + D(x)^2yD(x)^2D(u)D(x)^2) \\ &= 0,\ u, w, x, y, z, w \in R. \end{aligned} \tag{192}$$

Comparing (191) and (192), we get

$$f(z)D(z)f(z)wD(x)^2yD(D(x)^2)uD(x)^2 = 0,\ u, w, x, y, z \in R. \tag{193}$$

Substituting $vf(z)D(z)f(z)w$ for u in (193), we obtain

$$f(z)D(z)f(z)wD(x)^2yD(D(x)^2)vf(z)D(z)f(z)wD(x)^2 = 0,\ v, w, x, y, z \in R. \tag{194}$$

Right multiplication of (194) by $yD(D(x)^2)$ gives

$$\begin{aligned} &f(z)D(z)f(z)wD(x)^2yD(D(x)^2)vf(z)D(z)f(z)wD(x)^2yD(D(x)^2) \\ &= 0,\ v, w, x, y, z \in R. \end{aligned} \tag{195}$$

Since R is semiprime, the relation (195) yields

$$f(z)D(z)f(z)wD(x)^2yD(D(x)^2) = 0,\ w, x, y, z \in R. \tag{196}$$

The relation (196) yields

$$f(z)D(z)f(z)wD(x)^2y(D^2(x)D(x)+D(x)D^2(x))=0,\ w,x,y,z\in R. \tag{197}$$

Right multiplication of (197) by $D(x)$ gives

$$f(z)D(z)f(z)wD(x)^2y(D^2(x)D(x)^2+D(x)D^2(x)D(x))=0,\ w,x,y,z\in R. \tag{198}$$

Substituting $D(x)$ for y in (178), we obtain

$$\begin{aligned}&f(z)D(z)f(z)w(D^2(x)D(x)^2+D(x)D^2(x)D(x)\\&+D(x)^2D^2(x))=0,\ w,x,y,z\in R.\end{aligned} \tag{199}$$

Substituting $wD(x)^2y$ for w in (199), we obtain

$$\begin{aligned}&f(z)D(z)f(z)wD(x)^2y(D^2(x)D(x)^2+D(x)D^2(x)D(x)\\&+D(x)^2D^2(x))=0,\ w,x,y,z\in R.\end{aligned} \tag{200}$$

From (198) and (200), we have

$$f(z)D(z)f(z)wD(x)^2yD(x)^2D^2(x)=0,\ x,y,z,w\in R. \tag{201}$$

And, substituting $D^2(x)yf(z)D(z)f(z)w$ for y in (201), we obtain

$$\begin{aligned}&f(z)D(z)f(z)wD(x)^2D^2(x)yf(z)D(z)f(z)wD(x)^2D^2(x)\\&=0,\ x,y,z,w\in R.\end{aligned} \tag{202}$$

Since R is semiprime, we get from (202)

$$f(z)D(z)f(z)wD(x)^2D^2(x)=0,\ w,x,y,z\in R. \tag{203}$$

Replacing $x+ty$ for x in (203), we have

$$\begin{aligned}&f(z)D(z)f(z)wD^2(x+ty)D(x+ty)^2\\ \equiv\ &f(z)D(z)f(z)wD(x)^2D^2(x)\\&+t\{f(z)D(z)f(z)w(D(y)D(x)D^2(x)+D(x)D(y)D^2(x)\\&+D(x)^2D^2(y))\}\\&+t^2f(z)D(z)f(z)wT(x,y)\\&+t^3f(z)D(z)f(z)wD(y)^2D^2(y)=0,\ w,x,y,z\in R,\ t\in S_3\end{aligned} \tag{204}$$

where $T(x,y)$ denotes the term satisfying the identity (204).
From (203) and (204), we obtain

$$t\{f(z)D(z)f(z)w(D(y)D(x)D^2(x)+D(x)D(y)D^2(x)$$
$$+D(x)^2D^2(y))\}$$
$$+t^2f(z)D(z)f(z)wT(x,y)$$
$$=0,\ w,x,y,z,w\in R,\ t\in S_3. \tag{205}$$

Since R is 2!-torsion-free by assumption, by Lemma 2.1 the relation (205) yields

$$f(z)D(z)f(z)w(D(y)D(x)D^2(x)+D(x)D(y)D^2(x)$$
$$+D(x)^2D^2(y))=0,\ w,x,y,z\in R. \tag{206}$$

Substituting xy for y in (206), we obtain

$$f(z)D(z)f(z)w(xD(y)D(x)D^2(x)+D(x)yD(x)D^2(x)$$
$$+D(x)xD(y)D^2(x)+D(x)^2yD^2(x)$$
$$+D(x)^2xD^2(y)+2D(x)^3D(y)+D(x)^2D^2(x)y$$
$$=0,\ w,x,y,z\in R. \tag{207}$$

Writing wx for w in (206), we get

$$f(z)D(z)f(z)w(xD(y)D(x)D^2(x)+xD(x)D(y)D^2(x)$$
$$+xD(x)^2D^2(y))=0,\ w,x,y,z\in R. \tag{208}$$

Combining (175), (203), (207) with (208), we have

$$f(z)D(z)f(z)w(D(x)yD(x)D^2(x)+f(x)D(y)D^2(x)$$
$$+D(x)^2yD^2(x)+(f(x)D(x)+D(x)f(x))D^2(y)$$
$$=0,\ w,x,y,z\in R. \tag{209}$$

Replacing $wD(x)$ for w in (209), we obtain

$$f(z)D(z)f(z)w(D(x)^2yD(x)D^2(x)+D(x)f(x)D(y)D^2(x)$$
$$+D(x)^3yD^2(x)+(D(x)f(x)D(x)+D(x)^2f(x))D^2(y)$$
$$=0,\ w,x,y,z\in R. \tag{210}$$

Comparing (175),(182), (186), (203) and (210), we have

$$f(z)D(z)f(z)w(D(x)^2yD(x)D^2(x)+D(x)f(x)D(y)D^2(x)$$
$$=0,\ w,x,y,z\in R. \tag{211}$$

Substituting yx for y in (178), we obtain

$$f(z)D(z)f(z)w(D(y)xD(x)^2 + yD(x)^3 + D(x)D(y)xD(x) + D(x)yD(x)^2 + D(x)^2yD(x) + D(x)^2D(y)x) = 0, \ w, x, y, z \in R. \tag{212}$$

Right multiplication of (178) by x gives

$$f(z)D(z)f(z)w(D(y)D(x)^2x + D(x)D(y)D(x)x + D(x)^2D(y)x) = 0, \ w, x, y, z \in R. \tag{213}$$

Combining (175), (212) with (213), we obtain

$$f(z)D(z)f(z)w(-D(y)(f(x)D(x) + D(x)f(x)) - D(x)D(y)f(x) + D(x)yD(x)^2 + D(x)^2yD(x)) = 0, \ w, x, y, z \in R. \tag{214}$$

Putting xy instead of y in (214), we have

$$f(z)D(z)f(z)w(-xD(y)(f(x)D(x) + D(x)f(x)) - D(x)y(f(x)D(x) + D(x)f(x))D(x)^2 - D(x)xD(y)f(x) - D(x)^2yf(x) + D(x)xyD(x)^2 + D(x)^2xyD(x)) = 0, \ w, x, y, z \in R. \tag{215}$$

Substituting wx for w in (214), we have

$$f(z)D(z)f(z)w(-xD(y)(f(x)D(x) + D(x)f(x)) - xD(x)D(y)f(x) + xD(x)yD(x)^2 + xD(x)^2yD(x)) = 0, \ w, x, y, z \in R. \tag{216}$$

From (215) and (216), we get

$$f(z)D(z)f(z)w(-D(x)y(f(x)D(x) + D(x)f(x)) - f(x)D(y)f(x) - D(x)^2yf(x) + f(x)yD(x)^2 + (f(x)D(x) + D(x)f(x))yD(x)) = 0, \ w, x, y, z \in R. \tag{217}$$

Let $y = x$ in (217). Then we have

$$f(z)D(z)f(z)w(-D(x)x(f(x)D(x) + D(x)f(x)) - f(x)D(x)f(x) - D(x)^2xf(x) + f(x)xD(x)^2 + (f(x)D(x) + D(x)f(x))xD(x)) = 0, \ w, x, z \in R. \tag{218}$$

Using (186), (218) gives

$$f(z)D(z)f(z)w(-f(x)(f(x)D(x) + D(x)f(x)) - f(x)D(x)f(x) - (f(x)D(x) + D(x)f(x))f(x) + g(x)D(x)^2$$

$$-(f(x)D(x)+D(x)f(x))f(x))$$
$$= f(z)D(z)f(z)w(-f(x)^2D(x)-4f(x)D(x)f(x)-D(x)f(x)^2$$
$$+g(x)D(x)^2)=0,\ w,x,z\in R. \tag{219}$$

Thus the relation (219) yields

$$f(z)D(z)f(z)w(2f(x)^2D(x)+5f(x)D(x)f(x)+2D(x)f(x)^2)$$
$$=0,\ w,x,z\in R. \tag{220}$$

From (132) and (220), we get

$$f(z)D(z)f(z)w(f(x)^2D(x)-2D(x)f(x)^2)=0,\ w,x,z\in R. \tag{221}$$

According to (132) and (221), we have

$$f(z)D(z)f(z)w(5f(x)D(x)f(x)+6D(x)f(x)^2)=0,\ w,x,z\in R. \tag{222}$$

Writing $wf(x)$ for w in (222), it follows that

$$f(z)D(z)f(z)w(5f(x)^2D(x)f(x)+6f(x)D(x)f(x)^2)=0,\ w,x,z\in R. \tag{223}$$

Replacing $wf(x)$ for w in (189), we obtain

$$f(z)D(z)f(z)w(f(x)^2D(x)f(x)+f(x)D(x)f(x)^2+f(x)D(x)^2g(x))$$
$$=0,\ w,x,z\in R. \tag{224}$$

From (128) and (224), we have

$$f(z)D(z)f(z)w(f(x)^2D(x)f(x)+f(x)D(x)f(x)^2)=0,\ w,x,z\in R. \tag{225}$$

Subtracting $5\times$ (225) from (223), we get

$$f(z)D(z)f(z)wf(x)D(x)f(x)^2=0,\ w,x,z\in R. \tag{226}$$

From (225) and (226), we have

$$f(z)D(z)f(z)wf(x)^2D(x)f(x)=0,\ w,x,z\in R. \tag{227}$$

Writing $wf(x)$ for w in (221), it follows that

$$f(z)D(z)f(z)w(f(x)^3D(x)-2f(x)D(x)f(x)^2)=0,\ w,x,z\in R. \tag{228}$$

From (226) and (228), we obtain

$$f(z)D(z)f(z)wf(x)^3D(x)=0,\ w,x,z\in R. \tag{229}$$

And left multiplication of (229) by $5f(x)$ gives

$$(5f(z)^2D(z)f(z))wf(x)^3D(x)=0,\ w,x,z\in R. \tag{230}$$

Comparing (132) and (230), we have

$$(-3f(z)^3D(z)wf(x)^3D(x) = 0,\ w,x,z \in R. \tag{231}$$

Since R is 3!-torsion-free, we get from (231)

$$f(z)^3D(z)wf(x)^3D(x) = 0,\ w,x,z \in R.$$

Let $z = x$ in the above relation. Then we have

$$f(x)^3D(x)wf(x)^3D(x) = 0,\ w,x \in R. \tag{232}$$

And so, since R is semiprime, we obtain from (232)

$$f(x)^3D(x) = 0, x \in R. \tag{233}$$

Right multiplication of (221) by $f(x)$ gives

$$f(z)D(z)f(z)w(f(x)^2D(x)f(x) - 2D(x)f(x)^3) = 0,\ w,x,z \in R. \tag{234}$$

Combining (227) with (234), we obtain

$$2f(z)D(z)f(z)wD(x)f(x)^3 = 0,\ w,x,z \in R. \tag{235}$$

Since R is 2-tosion-free, we get from (235)

$$f(z)D(z)f(z)wD(x)f(x)^3 = 0,\ w,x,z \in R. \tag{236}$$

Right multiplication of (236) by x gives

$$f(z)D(z)f(z)wD(x)f(x)^3x = 0,\ w,x,z \in R. \tag{237}$$

Writing wx for w in (236),

$$f(z)D(z)f(z)wxD(x)f(x)^3 = 0,\ w,x,z \in R. \tag{238}$$

Comparing (237) and (238), we obtain

$$\begin{aligned} &f(z)D(z)f(z)w[D(x)f(x)^3, x] \\ &= f(z)D(z)f(z)w(f(x)^4 + D(x)g(x)f(x)^2 + D(x)f(x)g(x)f(x) \\ &\quad +D(x)f(x)^2g(x)) = 0,\ w,x,z \in R. \end{aligned} \tag{239}$$

Writing $wf(x)^3$ for w in (239),

$$\begin{aligned} &f(z)D(z)f(z)w(f(x)^7 + f(x)^3D(x)g(x)f(x)^2 \\ &+f(x)^3D(x)f(x)g(x)f(x) + f(x)^3D(x)f(x)^2g(x)) = 0,\ w,x,z \in R. \end{aligned} \tag{240}$$

Combining (233) with (240), we obtain

$$f(z)D(z)f(z)wf(x)^7 = 0, w,x,z \in R. \tag{241}$$

Left multiplication of (241) by z gives

$$zf(z)D(z)f(z)wf(x)^7 = 0, w, x, z \in R. \tag{242}$$

Writing zw for w in (242), it follows that

$$f(z)D(z)f(z)zwf(x)^7 = 0, w, x, z \in R. \tag{243}$$

Combining (242) with (243), we get

$$\begin{aligned}&(g(z)D(z)f(z) + f(z)^3 + f(z)D(z)g(z))wf(x)^7 \\ &= 0, w, x, z \in R.\end{aligned} \tag{244}$$

On the other hand, replacing $y + tz$ for z in (241), we have

$$\begin{aligned}&f(y+tz)D(y+tz)f(y+tz)wf(x)^7 \\ \equiv\ & f(y)D(y)f(y)wf(x)^7 \\ &+t\{(B(y,z)D(y)f(y) + f(y)D(z)f(y) + f(y)D(y)B(y,z))wf(x)^7\} \\ &+t^2U_1(y,z)wf(x)^7 + t^3U_2(y,z)wf(x)^7 \\ &+t^4U_4(y,z)wf(x)^7 + t^5f(z)D(z)f(z)wf(x)^7 \\ =\ & 0,\ w, x, y, z \in R,\ t \in S_3\end{aligned} \tag{245}$$

where $U_i(y,z), 1 \leq i \leq 4$, denotes the term satisfying the identity (245).
From (241) and (245), we obtain

$$\begin{aligned}&t\{(B(y,z)D(y)f(y) + f(y)D(z)f(y) + f(y)D(y)B(y,z))wf(x)^7\} \\ &+t^2U_1(y,z)wf(x)^7 + t^3U_2(y,z)wf(x)^7 \\ &+t^4U_4(y,z)wf(x)^7 = 0,\ w, x, y, z \in R.\end{aligned} \tag{246}$$

Since R is 4!-torsion-free by assumption, by Lemma 2.1 the relation (246) yields

$$\begin{aligned}&(B(y,z)D(y)f(y) + f(y)D(z)f(y) + f(y)D(y)B(y,z))wf(x)^7 \\ &= 0,\ w, x, y, z \in R.\end{aligned} \tag{247}$$

Substituting y^2 for z in (247), we obtain

$$\begin{aligned}&(3(f(y)yD(y)f(y) + f(y)D(y)yf(y)) + 2(yf(y)D(y)f(y) \\ &+f(y)D(y)f(y)y))wf(x)^7 = 0,\ w, x, y, z \in R.\end{aligned}$$

From (241) and the above relation we obtain

$$3(f(y)yD(y)f(y) + f(y)D(y)yf(y))wf(x)^7 = 0,\ w, x, y, z \in R. \tag{248}$$

Comparing (241) and (248), we have

$$(249) \qquad 3(g(y)D(y)f(y) - f(y)D(y)g(y))wf(x)^7 = 0, \ w, x, y, z \in R.$$

Since R is 3!-torsion-free, using (241) we get form (249)

$$(250) \qquad (g(y)D(y)f(y) - f(y)D(y)g(y))wf(x)^7 = 0, \ w, x, y, z \in R.$$

Let $y = z$ in (244). Then it is obvious that

$$(251) \qquad (g(y)D(y)f(y) + f(y)^3 + f(y)D(y)g(y))wf(x)^7 = 0, \ w, x, z \in R.$$

Combining (250) with (251), we obtain

$$(252) \qquad (f(y)^3 + 2f(y)D(y)g(y))wf(x)^7 = 0, \ w, x, y \in R.$$

Left multiplication of (252) by $f(y)^4$ gives

$$(253) \qquad (f(y)^7 + 2f(y)^5 D(y)g(y))wf(x)^7 = 0, \ w, x, y \in R.$$

Combining (233) with (253), we obtain

$$f(y)^7 wf(x)^7 = 0, \ x, w \in R.$$

Let $y = x$ in the above relation.

$$(254) \qquad f(x)^7 wf(x)^7 = 0, \ w, x \in R.$$

Then since R is semiprime, we obtain from (254)

$$f(x)^7 = 0, \ x \in R.$$

□

The following theorem as our main theorem generalizes the results of J. Vukman's theorem [12].

Theorem 2.6. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)^2[D(x), x] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. By the result of B.E. Johnson and A.M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [8] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. Then D is a derivation on A/P. By the assumption that $D(x)^2f(x) \in \mathrm{rad}(A)$, $x \in A$, we obtain $(D_P(\hat{x}))^2[D_P(\hat{x}),\hat{x}] = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 2.4 are fulfilled. And since the prime and factor algebra A/P is noncommutative, from Theorem 2.4 we have $[D_P(\hat{x}),\hat{x}]^7 = 0$, $\hat{x} \in A/P$. And for each P, by the elementary properties of the spectral radius r_P in a Banach algebra A/P, it follows that $r_P([D_P(\hat{x}),\hat{x}])^7 = r_P([D_P(\hat{x}),\hat{x}]^7) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}),\hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}),\hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the one hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, we obtain $D_P(A/P) \subseteq \mathrm{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$ for all primitive ideals of A. Hence we see that $D(A) \subseteq P$ for all primitive ideals of A. And so, $D(A) \subseteq \mathrm{rad}(A)$. On the other hand. In case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well, since A/P is semisimple and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebra. In other words, $D(x) \in P$ for all primitive ideals of A and all $x \in A$. i.e. we get $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we have $D(A) \subseteq \mathrm{rad}(A)$. □

Theorem 2.7. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x),x]D(x)^2 \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. The proof is similar as in the proof of Theorem 2.6. □

The following theorem generalizes Vukman's result [11].

Theorem 2.8. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)^2[D(x),x] = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. According to the result of B.E. Johnson and A.M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair [8] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x)+P$, $\hat{x} = x+P$. From the given assumptions $D(x)^2f(x) = 0$, $x \in A$, it follows that $(D_P(\hat{x}))^2[D_P(\hat{x}),\hat{x}] = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem

2.4 are fulfilled. And also the prime and factor algebra A/P is noncommutative. Hence by Theorem 2.5 we have $[D_P(\hat{x}),\hat{x}]^7 = 0,\ \hat{x} \in A/P$. Then $r_P([D_P(\hat{x}),\hat{x}])^7 = r([D_P(\hat{x}),\hat{x}]^7) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}),\hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}),\hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the other hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, one obtains $D_P(A/P) \subseteq \mathrm{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \mathrm{rad}(A)$. But since A is semisimple, $\mathrm{rad}(A) = \{0\}$. And so, $D = 0$. On the other hand, in case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well since A/P is semisimple and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebras. In other words $D(x) \in \cap P = \{0\}$ for all primitive ideals and all $x \in A$. And so Thus $D = 0$. Therefore in any case we have $D \equiv 0$. □

We generalize the result of J. Vukman [11].

Theorem 2.9. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x),x]D(x)^2 = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. The proof is similar as in the proof of Theorem 2.8. □

Corollary 2.10. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x),x] = 0$$

for all $x \in A$. Then we have $D = 0$.

Corollary 2.11. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x),x]D(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

Note: By a simple calculation, if R is a 3!-semiprime ring and $D : R \longrightarrow R$ is a Jordan derivation on R with $[D(x),x]D(x) = 0$ or $D(x)[D(x),x] = 0$ for all $x \in R$, we have $[D(x),x]^2 = 0$ for all $x \in R$.

As a special case of Theorem 2.8 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 2.12. *Let A be a semisimple Banach algebra. Suppose*

$$[x,y]^2[[x,y],x]=0$$

for all $x,y\in A$. In this case, A is commutative.

As a special case of Theorem 2.9 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 2.13. *Let A be a semisimple Banach algebra. Suppose*

$$[[x,y],x][x,y]^2=0$$

for all $x,y\in A$. In this case, A is commutative.

As a special cases (Theorem in [11]) of Theorem 2.6 we have the following result.

Corollary 2.14. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D: A\longrightarrow A$ such that*

$$D(x)[D(x),x]\in rad(A)$$

and for all $x\in A$. Then we have $D(A)\subseteq rad(A)$.

As a special case (Theorem in [11]) of Theorem 2.7 we have the following result.

Corollary 2.15. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D: A\longrightarrow A$ such that*

$$[D(x),x]D(x)\in rad(A)$$

and for all $x\in A$. Then we have $D(A)\subseteq rad(A)$.

References

1. F.F. Bonsall & J. Duncan: *Complete Normed Algebras.* Berlin-Heidelberg-New York, 1973.
2. M. Brešar: Derivations of noncommutative Banach algebras II. *Arch. Math.* **63** (1994), 56-59.
3. ——— : Jordan derivations on semiprime rings. *Proc. Amer. Math. Soc.* **104**(4) (1988), 1003-1006.
4. L.O. Chung & J. Luh: Semiprime rings with nilpotent derivatives. *Canad. Math. Bull.* **24**(4) (1981), 415-421.
5. B.E. Johnson & A.M. Sinclair: Continuity of derivations and a problem of Kaplansky. *Amer. J. Math.* **90** (1968), 1067-1073.

6. B.-D. Kim: On the derivations of semiprime rings and noncommutative Banach algebras. *Acta Mathematica Sinica* **16**(1) (2000), 21-28.
7. ______ : Derivations of semiprime rings and noncommutative Banach algebras. *Commun. Korean Math. Soc.* **17**(4) (2002), 607-618.
8. A.M. Sinclair: Jordan homohorphisms and derivations on semisimple Banach algebras. *Proc. Amer. Math. Soc.* **24** (1970), 209-214.
9. I.M. Singer & J. Wermer: Derivations on commutative normed algebras. *Math. Ann.* **129** (1955), 260-264.
10. M.P. Thomas: The image of a derivation is contained in the radical. *Annals of Math.* **128** (1988), 435-460.
11. J. Vukman: A result concerning derivations in noncommutative Banach algebras. *Glasnik Mathematicki* **26**(46) (1991), 83-88.
12. ______ : On derivations in prime rings and Banach algebras. *Proc. Amer. Math. Soc.* **116**(4) (1992), 877-884.

DEPARTMENT OF MATHEMATICS, KANGNUNG NATIONAL UNIVERSITY, KANGNUNG, GANGWON-DO 210-702, KOREA
Email address: bdkim@knusun.kangnung.ac.kr