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VAGUE BCK/BCI-ALGEBRAS

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Abstract. The notions of vague BCK/BCI-algebras and vague ideals are introduced, and their properties are investigated. Conditions for a vague set to be a vague ideal are provided. Characterizations of a vague ideal are established.

1. Introduction

Several authors have made a number of generalizations of Zadeh's fuzzy set theory [7]. Of these, the notion of vague set theory introduced by Gau and Buehrer [2] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [1] studied vague groups. Jun and Park [4, 6] studied vague ideals and vague deductive systems in subtraction algebras. In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory on BCK/BCI-algebras. We introduce the notion of vague BCK/BCI-algebras and vague ideals in BCK/BCI-algebras, and then we investigate their properties. We give conditions for a vague set to be a vague ideal. We also establish characterizations of a vague ideal.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)\ (x * x = 0)$,

(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

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A BCI-algebra X satisfying the additional condition:

(a5) $(\forall x \in X)\ (0 * x = 0)$

is called a *BCK-algebra.* In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCK/BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.

(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

(b3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

(b4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

In particular, if X is a BCK-algebra then the following property hold:

(b5) $(\forall x, y \in X)\ ((x * y) * x = 0)$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A subset A of a BCK-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,

(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A \Rightarrow y \in A)$.

Note that every ideal A of a BCK/BCI-algebra X satisfies:

$$(\forall x \in A)\ (\forall y \in X)\ (y \leq x \Rightarrow y \in A). \tag{2.1}$$

We refer the reader to the book [3] and [5] for further information regarding BCK/BCI-algebras.

Definition 2.1 ([1]). A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A true membership function

$$t_A : U \to [0, 1],$$

and

(2) A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0, 1]$, we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. For $\alpha, \beta \in [0, 1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 2.2 ([1]). Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The *(α, β)-cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts of the vague set A are also called *vague-cuts* of A.

Definition 2.3 ([1]). The *α-cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$.

Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Vague BCK-algerbas

In this section we first define the notion of vague BCK/BCI-subalgebras. For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0,1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [1]).

In what follows let X denote a BCK/BCI-algebra unless specified otherwise.

Definition 3.1. A vague set A of X is called a *vague BCK/BCI-algebra* of X if the following condition is true:

$$(\forall x, y \in X)\ (V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\}), \tag{3.1}$$

that is,

$$\begin{aligned} &t_A(x * y) \geq \min\{t_A(x), t_A(y)\}, \\ &1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \end{aligned} \tag{3.2}$$

for all $x, y \in X$.

Example 3.2. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.6, 0.03]\rangle, \langle a, [0.6, 0.03]\rangle, \langle b, [0.3, 0.06]\rangle, \langle c, [0.6, 0.03]\rangle\}.$$

It is routine to verify that A is a vague BCK-algebra of X.

Lemma 3.3. *Every vague BCK/BCI-algebra A of X satisfies:*

$$(\forall x \in X)\ (V_A(0) \succeq V_A(x)). \tag{3.3}$$

Proof. Let $x \in X$. Then

$$t_A(0) = t_A(x * x) \geq \min\{t_A(x), t_A(x)\} = t_A(x)$$

and

$$1 - f_A(0) = 1 - f_A(x * x) \geq \min\{1 - f_A(x), 1 - f_A(x)\} = 1 - f_A(x).$$

This shows that $V_A(0) \succeq V_A(x)$. □

For the convenience of notation, we denote

$$x * \prod_{i=1}^{n} y_i = (\cdots((x * y_1) * y_2) * \cdots) * y_n$$

and

$$x * \coprod_{i=1}^{n} y_i = y_n * (y_{n-1} * (\cdots * (y_1 * x)\cdots)).$$

Proposition 3.4. *Let A be a vague BCK-algebra of X and let $\{x_1, x_2, \cdots, x_n\} \subseteq X$. If there exists $k \in \{1, 2, \cdots, n\}$ such that $x_1 = x_k$, then*

$$(\forall x \in X)\left(V_A\left(x_1 * \prod_{i=2}^{n} x_i\right) \succeq V_A(x)\right). \tag{3.4}$$

Proof. Let k be a fixed number in $\{1, 2, \cdots, n\}$ such that $x_k = x_1$. Using (a3), (a5) and (b2), one can deduce

$$V_A\left(x_1 * \prod_{i=2}^{n} x_i\right) = V_A(0).$$

It follows from Lemma 3.3 that

$$V_A\left(x_1 * \prod_{i=2}^{n} x_i\right) \succeq V_A(x)$$

for all $x \in X$. □

Proposition 3.5. *Let A be a vague BCK-algebra of X and let $k \in \mathbb{N}$. Then*

(i) $(\forall x \in X)\left(V_A\left(x * \coprod_{i=1}^{2k-1} x\right) \succeq V_A(x)\right)$.

(ii) $(\forall x \in X)\left(V_A\left(x * \coprod_{i=1}^{2k} x\right) = V_A(x)\right)$.

(iii) $(\forall x \in X)\left(V_A\left(x * \coprod_{i=1}^{k-1} x\right) \succeq V_A(x)\right)$.

Proof. Straightforward. □

Theorem 3.6. *Let A be a vague BCK/BCI-algebra of X. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ of A is a crisp subalgebra of X.*

Proof. Let $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x) \succeq [\alpha, \beta]$, that is, $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$; and $V_A(y) \succeq [\alpha, \beta]$, that is, $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$. It follows from (3.2) that

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta,$$

which means that $V_A(x * y) \succeq [\alpha, \beta]$. Hence $x * y \in A_{(\alpha,\beta)}$. This completes the proof. □

4. VAGUE IDEALS ON BCK/BCI-ALGEBRAS

Definition 4.1. A vague set A of X is called a *vague ideal* of X if the following conditions are true:

(d1) $(\forall x \in X)\ (V_A(0) \succeq V_A(x))$,

(d2) $(\forall x, y \in X)\ (V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$,

that is,

$$t_A(0) \geq t_A(x),\ 1 - f_A(0) \geq 1 - f_A(x), \tag{4.1}$$

and

$$\begin{aligned} &t_A(x) \geq \min\{t_A(x * y), t_A(y)\}, \\ &1 - f_A(x) \geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \end{aligned} \tag{4.2}$$

for all $x, y \in X$.

Example 4.2. Consider a BCK/BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.6, 0.2]\rangle, \langle a, [0.3, 0.6]\rangle, \langle b, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is a vague ideal of X.

Example 4.3. Consider a BCK/BCI-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.7, 0.2]\rangle\}.$$

It is routine to verify that A is a vague ideal of X.

Proposition 4.4. *Every vague ideal A of X satisfies:*

$$(\forall x, y \in X)\ (x \leq y \Rightarrow V_A(x) \succeq V_A(y)). \tag{4.3}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$t_A(x) \geq \min\{t_A(x * y), t_A(y)\} = \min\{t_A(0), t_A(y)\} = t_A(y)$$

and

$$\begin{aligned} 1 - f_A(x) &\geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \\ &= \min\{1 - f_A(0), 1 - f_A(y)\} \\ &= 1 - f_A(y). \end{aligned}$$

This shows that $V_A(x) \succeq V_A(y)$. □

Proposition 4.5. *Every vague ideal A of X satisfies:*

$$(\forall x, y, z \in X)\ (V_A(x * z) \succeq \operatorname{imin}\{V_A((x * y) * z), V_A(y)\}). \tag{4.4}$$

Proof. Using (d2) and (b2), we have

$$\begin{aligned} V_A(x * z) &\succeq \operatorname{imin}\{V_A((x * z) * y), V_A(y)\} \\ &= \operatorname{imin}\{V_A((x * y) * z), V_A(y)\} \end{aligned}$$

for all $x, y, z \in X$. □

We now give conditions for a vague set of X to be a vague ideal of X.

Theorem 4.6. *If A is a vague set of X satisfying* (d1) *and* (4.4), *then A is a vague ideal of X.*

Proof. Taking $z = 0$ in (4.4) and using (b1), we have

$$\begin{aligned} V_A(x) &= V_A(x * 0) \\ &\succeq \operatorname{imin}\{V_A((x * y) * 0), V_A(y)\} \\ &= \operatorname{imin}\{V_A(x * y), V_A(y)\} \end{aligned}$$

for all $x, y \in X$. Hence A is a vague ideal of X. □

Corollary 4.7. *Let A be a vague set of X. Then A is a vague ideal of X if and only if it satisfies conditions* (d1) *and* (4.4).

The following are characterizations of a vague ideal of X.

Theorem 4.8. *Let A be a vague set of a BCK-algebra X. Then A is a vague ideal of X if and only if it satisfies the following conditions:*

$$(\forall x, y \in X)\ (V_A(x * y) \succeq V_A(x)), \tag{4.5}$$

$$(\forall x, a, b \in X)\ (V_A(x * ((x * a) * b)) \succeq \text{imin}\{V_A(a), V_A(b)\}). \tag{4.6}$$

Proof. Assume that A is a vague ideal of a BCK-algebra X. Since $(x * y) * x = 0$ for all $x, y \in X$, it follows from (d1) and (d2) that

$$V_A(x * y) \succeq \text{imin}\{V_A((x * y) * x), V_A(x)\} = \text{imin}\{V_A(0), V_A(x)\} = V_A(x)$$

for all $x, y \in X$. Since

$$(x * ((x * a) * b)) * a = (x * a) * ((x * a) * b) \leq b,$$

it follows from (4.3) that $V_A((x * ((x * a) * b)) * a) \succeq V_A(b)$. And, from (d2) we have

$$\begin{aligned} V_A(x * ((x * a)) * b)) &\succeq \text{imin}\{V_A((x * ((x * a) * b)) * a), V_A(a)\} \\ &\succeq \text{imin}\{V_A(a), V_A(b)\}. \end{aligned}$$

Conversely let A be a vague set of X satisfying conditions (4.5) and (4.6). If we take $y = x$ in (4.5), then $V_A(0) = V_A(x * x) \succeq V_A(x)$ for all $x \in X$, proving (d1). And, by using (4.6), we obtain

$$\begin{aligned} V_A(x) &= V_A(x * 0) \\ &= V_A(x * ((x * y) * (x * y))) \\ &= V_A(x * ((x * (x * y)) * y)) \\ &\succeq \text{imin}\{V_A(x * y), V_A(y)\} \end{aligned}$$

for all $x, y \in X$. This proves (d2). Hence A is a vague ideal of X. □

Proposition 4.9. *Let A be a vague set of a BCK-algebra X. Then A is a vague ideal of X if and only if it satisfies:*

$$(\forall x, y, z \in X)\ (x * y \leq z \Rightarrow V_A(x) \succeq \text{imin}\{V_A(y), V_A(z)\}). \tag{4.7}$$

Proof. Assume that A is a vague ideal of a BCK-algebra X and let $x, y, z \in X$ be such that $x * y \leq z$. Then $V_A(z) \preceq V_A(x * y)$ by (4.3). It follows from (d2) that $V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\} \succeq \text{imin}\{V_A(y), V_A(z)\}$. Conversely suppose that A satisfies (4.7). Since $0 * y \leq y$ for all $y \in X$, we have

$$V_A(0) \succeq \text{imin}\{V_A(y), V_A(y)\} = V_A(y)$$

by (4.7). Thus (d1) is valid. Combining (a2) and (4.7), we have

$$V_A(x) \succeq \text{imin}\{V_A(x*y), V_A(y)\},$$

proving (d2). Hence A is a vague ideal of X. □

As generalizations of Proposition 4.9, we have the following results.

Theorem 4.10. *If a vague set A of a BCK-algebra X is a vague ideal of X, then*

$$x * \prod_{i=1}^{n} w_i = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(w_i) \mid i = 1, 2, \cdots, n\} \tag{4.8}$$

for all $x, w_1, w_2, \cdots, w_n \in X$.

Proof. The proof is by induction on n. Let A be a vague ideal of X. By (4.3) and (4.7), we know that the condition (4.8) is valid for $n = 1, 2$. Assume that A satisfies the condition (4.8) for $n = k$, that is,

$$x * \prod_{i=1}^{k} w_i = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(w_i) \mid i = 1, 2, \cdots, k\}$$

for all $x, w_1, w_2, \cdots, w_k \in X$. Let $x, w_1, w_2, \cdots, w_k, w_{k+1} \in X$ be such that $x * \prod_{i=1}^{k+1} w_i = 0$. Then

$$V_A(x*w_1) \succeq \text{imin}\{V_A(w_j) \mid j = 2, 3, \cdots, k+1\}.$$

Since A is a vague ideal of X, it follows from (d2) that

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A(x*w_1),\ V_A(w_1)\} \\ &\succeq \text{imin}\big\{V_A(w_1),\ \min\{V_A(w_j) \mid j = 2, 3, \cdots, k+1\}\big\} \\ &= \text{imin}\{V_A(w_i) \mid i = 1, 2, \cdots, k+1\}. \end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem 4.10.

Theorem 4.11. *Let A be a vague set of a BCK-algebra X satisfying the condition (4.8). Then A is a vague ideal of X.*

Proof. Note that $\big(\cdots((0*\underbrace{x)*x)*\cdots\big)*x}_{n \text{ times}} = 0$ for all $x \in X$. It follows from (4.8) that $V_A(0) \succeq V_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$0 = (x*y)*z = \big(\cdots(((x*y)*z)*\underbrace{0)*\cdots\big)*0}_{n-2 \text{ times}},$$

and so $V_A(x) \succeq \text{imin}\{V_A(y),\ V_A(z),\ V_A(0)\} = \text{imin}\{V_A(y),\ V_A(z)\}$. Hence, by Proposition 4.9, we conclude that A is a vague ideal of X. □

Combining Theorem 4.10 and 4.11, we immediately obtain the following result.

Corollary 4.12. *Let A be a vague set of a BCK-algebra X. Then A is a vague ideal of X if and only if it satisfies* (4.8).

Theorem 4.13. *Let A be a vague ideal of X. Then for any $\alpha, \beta \in [0,1]$, the vague-cut $A_{(\alpha,\beta)}$ of A is a crisp ideal of X.*

Proof. Obviously, $0 \in A_{(\alpha,\beta)}$. Let $x, y \in X$ be such that $y \in A_{(\alpha,\beta)}$ and $x*y \in A_{(\alpha,\beta)}$. Then $V_A(y) \succeq [\alpha,\beta]$, i.e., $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$; and $V_A(x*y) \succeq [\alpha,\beta]$, i.e., $t_A(x*y) \geq \alpha$ and $1 - f_A(x*y) \geq \beta$. It follows from (4.2) that

$$t_A(x) \geq \min\{t_A(x*y), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x) \geq \min\{1 - f_A(x*y), 1 - f_A(y)\} \geq \beta$$

so that $V_A(x) \succeq [\alpha,\beta]$. Hence $x \in A_{(\alpha,\beta)}$, and so $A_{(\alpha,\beta)}$ is an ideal of X. □

The ideals like $A_{(\alpha,\beta)}$ are also called *vague-cut ideals* of X.

Clearly we have the following result.

Proposition 4.14. *Let A be a vague ideal of X. Two vague-cut ideals $A_{(\alpha,\beta)}$ and $A_{(\gamma,\delta)}$ with $[\alpha,\beta] < [\gamma,\delta]$ are equal if and only if there is no $x \in X$ such that*

$$[\alpha,\beta] \preceq V_A(x) \preceq [\gamma,\delta].$$

Theorem 4.15. *Let X be finite and let A be a vague ideal of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) \mid x \in X\}.$$

Then all the vague-cut ideals $A_{(a_1,a_2)}$ of X are the only vague-cut ideals of X satisfying $[a_1,a_2] \in V(A)$.

Proof. Consider $[a_1,a_2] \in I[0,1]$ where $[a_1,a_2] \notin V(A)$. If

$$[\alpha,\beta] < [a_1,a_2] < [\gamma,\delta]$$

where $[\alpha,\beta], [\gamma,\delta] \in V(A)$, then $A_{(\alpha,\beta)} = A_{(a_1,a_2)} = A_{(\gamma,\delta)}$. If

$$[a_1,a_2] < [a_1,a_3]$$

where $[a_1,a_3] = \operatorname{imin}\{V_A(x) \mid x \in X\}$, then $A_{(a_1,a_3)} = X = A_{(a_1,a_2)}$. Hence for any $[a_1,a_2] \in I[0,1]$, the vague-cut ideal $A_{(a_1,a_2)}$ of X is one of the form $A_{(\alpha,\beta)}$ for $[\alpha,\beta] \in V(A)$. This completes the proof. □

Theorem 4.16. *Any ideal D of X is a vague-cut ideal of X with respect to some vague ideal of X.*

Proof. Consider the vague set A of X given by

$$V_A(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in D, \\ [0,0] & \text{if } x \notin D, \end{cases} \tag{4.9}$$

where $\alpha \in (0,1)$. Since $0 \in D$, we have $V_A(0) = [\alpha, \alpha] \succeq V_A(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in D$, then

$$V_A(x) = [\alpha, \alpha] \succeq \text{imin}\{V_A(x * y), V_A(y)\}.$$

Assume that $x \notin D$. Then $y \notin D$ or $x * y \notin D$. It follows that

$$V_A(x) = [0,0] = \text{imin}\{V_A(x * y), V_A(y)\}.$$

Thus A is a vague ideal of X. Clearly $D = A_{(\alpha,\alpha)}$. □

Theorem 4.17. *Let A be a vague ideal of X. Then the set*

$$D := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a crisp ideal of X.

Proof. Obviously $0 \in D$. Let $x, y \in X$ be such that $x * y \in D$ and $y \in D$. Then $V_A(x * y) = V_A(0) = V_A(y)$, and so

$$V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\} = V_A(0)$$

by (d2). Since $V_A(0) \succeq V_A(x)$ for all $x \in X$, it follows that $V_A(x) = V_A(0)$ so that $x \in D$. Therefore D is a crisp ideal of X. □

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