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CONTROLLABILITY OF SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper, we find the sufficient conditions of controllability of semilinear neutral functional differential evolution equations with nonlocal conditions using by fractional power of operators and Sadovskii's fixed point theorem.

1. INTRODUCTION

In this paper, we consider the following control system :

$$(1)\qquad \begin{cases} \dfrac{d}{dt}[x(t)+F(t,x(t),x(b_1(t)),\ldots,x(b_m(t)))]+Ax(t) \\ = G(t,x(t),x(a_1(t)),\ldots,x(a_n(t)))+Bu(t), \quad 0\leqslant t\leqslant T \\ x(0)+g(x)=x_0, \end{cases}$$

where X is the Banach space with norm $\|\cdot\|$, the linear operator $-A$ generates an analytic semigroup, $F:[0,T]\times X^{n+1}\to X$ and $G:[0,T]\times X^{n+1}\to X$ are continuous functions. $a_i, b_j \in C([0,T]:[0,T])$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$, $g(x)\in C(E,X)$, $E=C([0,T]:X)$. $B:U\to X$ is bounded linear operator. U is the another Banach space. $u(\cdot)\in L^2([0,T]:U)$ is control function.

The nonlinear Cauchy problem was considered by Byszewski [2]. In the past several years, theorems about existence, uniqueness and stability for differential and functional differential abstract evolution Cauchy problem with nonlinear conditions have been studied by Byszewski and Lakshmikantham [7], Byszewski and Akca [6],

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Byszewski [2-5], Balachandran and Chandrasekaran [1], Ntouyas and Tsamatos [12] and Lin and Liu [10].

Recently, Fu and Ezzinbi [8] are studied the existence of mild and strong solutions of equations (1)($Bu = 0$).

In this paper, we found the sufficient conditions of controllability for the equation (1).

2. PRELIMINARIES

Let X will be a Banach space with norm $\|\cdot\|$. $-A : D(A) \to X$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $S(t)$. Let $0 \in \rho(A)$. Then it is possible to define the fractional power A^α for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Now, we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$. Then for each $0 < \alpha \le 1$, X_α is a Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is comapct. For semigroup $\{S(t)_{t\ge 0}\}$, the following properties will be used :

(a) there is a $M \ge 1$ such that $\|S(t)\| \le M$, for all $t \in [0,T]$;

(b) for any $T > 0$, there exists a positive constant C_α such that

$$\|A^\alpha S(t)\| \le \frac{C_\alpha}{t^\alpha}, \quad 0 < t \le T.$$

For more details about the above preliminaries, we refer to [13].

Now we give the basic assumptions on equation (1).

(H1) $F : [0,T] \times X^{m+1} \to X$ is a continuous function, and there exist a $\beta \in (0,1)$ and $L, L_1 > 0$ such that the function $A^\beta F$ satisfies the Lipschitz condition :

$$\|A^\beta F(s_1, x_0, x_1, \cdots, x_m) - A^\beta F(s_2, \overline{x}_0, \overline{x}_1, \cdots, \overline{x}_m)\|$$
$$\le L(|s_1 - s_2| + \max_{i=0,1,\cdots,m} \|x_i - \overline{x}_i\|)$$

for any $0 \le s_1, s_2 \le T$, $x_i, \overline{x}_i \in X$, $i = 0, 1, \cdots, m$, and the inequality

$$\|A^\beta F(t, x_0, x_1, \cdots, x_m)\| \le L_1(\max\{\|x_i\| \mid i = 0, 1, \ldots, m\} + 1)$$

holds for any $(t, x_0, x_1, \cdots, x_m) \in [0,T] \times X^{m+1}$.

(H2) The function $G : [0,T] \times X^{n+1} \to X$ is continuous function and satisfies the following conditions :

(i) For $(s, x_0, x_1, \cdots, x_n), (\overline{s}, \overline{x}_0, \cdots, \overline{x}_n) \in [0,T] \times X^{n+1}$, there exists positive number L_2 such that

$$\|G(s, x_0, x_1, \cdots, x_n) - G(\overline{s}, \overline{x}_0, \ldots, \overline{x}_n)\| \\ \leq L_2(|s - \overline{s}| + \max_{i=0,\cdots,n} \|x_i - \overline{x}_i\|).$$

(ii) For $(t, x_0, x_1, \cdots, x_n) \in [0,T] \times X^{n+1}$, there exists positive L_3 such that

$$\|G(t, x_0, x_1, \cdots, x_n)\| \leq L_3(\max\{\|x_i\| \mid i = 0, 1, \cdots, n\} + 1).$$

(H3) $a_i, b_j \in C([0,T] : [0,T]), i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. $g \in C(E : X)$, here and hereafter $E = C([0,T] : X)$ and g satisfies that

(i) There exist positive constants L_4 and L_4' such that

$$\|g(x)\| \leq L_4\|x\| + L_4'$$

for all $x \in E$.

(ii) g is completely continuous map.

3. Existence and Uniqueness of Mild Solution

In this section, we prove the existence and uniqueness of solution, norm estimate and continuation of solutions for the equations (1).

Definition 3.1 ([8]). A continuous function $x(\cdot) : [0,T] \to X$ is said to be a mild solution of the equations (1), if the function $AS(t-s)F(s, x(s), x(b_1(s)), \cdots, x(b_m(s)))$, $s \in [0,T)$ is integrable on $[0,T)$ and the following integral equation verified :

$$(2) \quad \begin{cases} x(t) = S(t)[x_0 + F(0, x(0), x(b_1(0)), \ldots, x(b_m(0))) - g(x)] \\ \qquad - F(t, x(t), x(b_1(t)), \cdots, x(b_m(t))) \\ \qquad + \int_0^t AS(t-s)F(s, x(s), x(b_1(s)), \cdots, x(b_m(s)))ds \\ \qquad + \int_0^t S(t-s)[G(s, x(s), x(a_1(s)), \cdots, x(a_n(s))) + Bu(s)]ds, \\ 0 \leq t \leq T \end{cases}$$

Theorem 3.2. *If hypotheses $(H1) - (H3)$ are satisfied, $x_0 \in X$ and $u \in L^2([0,T] : U)$, then the equations (1) has a mild solution $x(\cdot) \in C([0,T] : X)$ provided that*

$$(3) \qquad L_0 := (M+1)M_0L + \frac{1}{\beta}T^{\beta}C_{1-\beta}L + ML_2 < 1, \quad M_0 = \|A^{-\beta}\|.$$

Proof. For $T, \delta > 0$ such that $0 \le t \le T$, let

$$\|(S(t) - I)(x_0 - g(x))\| \le \frac{\delta}{2},$$

$$L_1(k+1)\Big\{MM_0 + M_0 + \frac{1}{\beta}T^\beta C_{1-\beta}\Big\} + ML_3(k+1)T + M_B\|u\|_{L^2(0,T:U)}\sqrt{T} \le \frac{\delta}{2},$$

where $M_B = M\|B\|$. For simple proof, putting

$$(t, x(t), x(b_1(t)), \dots, x(b_m(t))) = (t, p(t))$$
$$(t, x(t), x(a_1(t)), \dots, x(a_n(t))) = (t, q(t)).$$

Defined S_δ by

$$S_\delta = \big\{\eta \in C([0,T] : X) : \eta(0) = x(0) + g(x),\ \|\eta - x_0 + g(x)\| \le \delta,\ 0 \leqslant t \leqslant T\big\},$$

then S_δ is nonempty closed bounded subset on $C([0,T] : X)$. Also, ϕ is a function on S_δ defined by

$$\begin{aligned}\phi x(t) = & S(t)[x_0 + F(0, p(0)) - g(x)] - F(t, p(t))\\ &+ \int_0^t AS(t-s)F(s, p(s))ds + \int_0^t S(t-s)G(s, q(s))ds\\ &+ \int_0^t S(t-s)Bu(s)ds, \quad 0 \le t \le T.\end{aligned}$$

Since

$$\begin{aligned}&\|\phi x(t) - x_0 + g(x)\|\\ \le&\frac{\delta}{2} + L_1(k+1)\Big\{MM_0 + M_0 + \frac{1}{\beta}T^\beta C_{1-\beta}\Big\} + ML_3(k+1)T + M_B\|u\|_{L^2(0,T:U)}\sqrt{T}\\ \le&\frac{\delta}{2} + \frac{\delta}{2} = \delta\end{aligned}$$

for all $0 \le t \le T$. Therefore

$$\|\phi x(t) - x_0 + g(x)\|_C = \sup_{0 \le t \le T} \|\phi x(t) - x_0 + g(x)\| \le \delta.$$

Hence $\phi x(t) \in S_\delta$. That is, $\phi : S_\delta \to S_\delta$.

Furthermore, If $x(t), y(t) \in S_\delta$, and let

$$(t, y(t), y(b_1(t)), \dots, y(b_m(t))) = (t, v(t))$$
$$(t, y(t), y(a_1(t)), \dots, y(a_n(t))) = (t, r(t)),$$

then

$$\begin{aligned}&\|\phi x(t) - \phi y(t)\|\\ \le& MM_0L \sup_{0 \leqslant s \le T} \|x(s) - y(s)\| + M_0L \sup_{0 \le s \leqslant T} \|x(s) - y(s)\|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\beta}T^{\beta}C_{1-\beta}L\sup_{0\le s\le T}\|x(s)-y(s)\|+ML_2\sup_{0\le s\le T}\|x(s)-y(s)\|\\
=&\Big\{(M+1)M_0L+\frac{1}{\beta}T^{\beta}C_{1-\beta}L+ML_2\Big\}\cdot\sup_{0\le s\le T}\|x(s)-y(s)\|\\
=&L_0\cdot\sup_{0\le t\le T}\|x(s)-y(s)\|.
\end{aligned}
$$

Therefore

$$\|\phi x-\phi y\|_C=\sup_{0\le t\le T}\|\phi x(t)-\phi y(t)\|\leqslant L_0\|x-y\|_C.$$

Since $L_0<1$, ϕ is a contraction mapping, and ϕ has a unique fixed point $x(t)\in S_\delta$. Hence equation (1) has a mild solution (2). □

From Theorem 1, we define the solution mapping $W:L^2([0,T]:U)\to C([0,T]:X)$ by

$$(Wu)(t)=x(x_0:u)\in C([0,T]:X).$$

Theorem 3.3. *Let $x_0\in X$ and $u(\cdot)\in L^2([0,T]:U)$. Solution mapping $(Wu)(t)=x(x_0:u)$ is satisfied the following condition from assumptions (H1)-(H3),*

$$\|x(x_0:u)\|_C\le C(M\|x_0\|_{C(0,T:X)}+M_B\|u\|_{L^2(0,T:U)}\sqrt{T}),\quad 0\le t\le T,$$

where C depends on M, β, L_1, L_3, L_4, L_4', k and T.

Proof. By assumptions,

$$
\begin{aligned}
&\|x(x_0:u)(t)\|\\
\le&M\|x_0\|+M\|A^{-\beta}\|L_1(k+1)+M(L_4\|x\|+L_4')+\|A^{-\beta}\|L_1(k+1)\\
&+\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(k+1)ds+M\int_0^tL_3(k+1)ds+M\|B\|\|u\|_{L^2(0,T:U)}\sqrt{t}\\
\le&M\|x_0\|+M_B\|u\|_{L^2(0,T:U)}\sqrt{T}+\Big\{MM_0L_1(k+1)+M(L_4k+L_4')\\
&+M_0L_1(k+1)+\frac{1}{\beta}T^{\beta}C_{1-\beta}L_1(k+1)+ML_3(k+1)T\Big\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|x(x_0:u)\|_{C(0,T:X)}&=\sup_{0\le t\le T}\|x(x_0:u)(t)\|\\
&\le C(M\|x_0\|_C+M_B\|u\|_{L^2(0,T:U)}\sqrt{T}).
\end{aligned}
$$

□

Theorem 3.4. *Let $x_0\in X$ and $u(\cdot)\in L^2([0,T]:U)$. Then solution (2) can extend on interval $[0,2T]$ from assumptions $(H1)-(H3)$ and satisfy the following condition.*

$$\|x(x_0:u)\|_{C(0,2T:X)}\le C_1(M\|x(x_0:u)\|_{C(0,2T:X)}+M_B\|u\|_{L^2(0,2T:U)}\sqrt{2T}),$$

where C_1 depended on M, β, L_1, L_3, L_4, L_4', k and T.

Proof. If $x(x_0 : u)(t)$ satisfy the solution of (2) on $[0, t_1 + t_2]$ such that $t_1, t_2 > 0$. Then, for $t \in [t_1, t_1 + t_2]$,

$$\begin{aligned} x(x_0 : u)(t) &= S(t-t_1)[x(x_0 : u)(t_1) + F(0, p(0)) - g(x)] + F(t, p(t)) \\ &\quad + \int_{t_1}^{t} AS(t-s)F(s, p(s))ds + \int_{t_1}^{t} S(t-s)G(s, q(s))ds \\ &\quad + \int_{t_1}^{t} S(t-s)Bu(s)ds, \end{aligned}$$

and for $t \in [0, t_2]$

$$\begin{aligned} x(x_0 : u)(t_1 + t) &= S(t)[x(x_0 : u)(t_1) + F(0, p(0)) - g(x)] + F(t, p(t)) \\ &\quad + \int_0^t AS(t-s)F(s + t_1, p(s + t_1))ds \\ &\quad + \int_0^t S(t-s)G(s + t_1, q(s + t_1))ds + \int_0^t S(t-s)Bu(s + t_1)ds. \end{aligned}$$

It means that $t \to x(x_0 : u)(t_1 + t)$ is a solution of (2) on $[0, t_2]$ with initial data $x(x_0 : u)(t_1)$. Inversely, let $x^*(x_0 : u)(t)$ be solution of (2) on $[0, t_1]$, and $\widehat{x}^*(x_0 : u)(t)$ be a solution of (2) on $[0, T]$ with initial data $\widehat{x}^*(x_0 : u)(t_1)$. If

$$x(x_0 : u)(t) = \begin{cases} x^*(x_0 : u)(t), & 0 \le t \le t_1 \\ \widehat{x}^*(x_0 : u)(t - t_1), & t_1 \le t \le t_1 + T \end{cases},$$

then $x(x_0 : u)(t)$ is a solution of (2) on $[0, t_1 + T]$. Therefore, let $t_1 = T$. Then $x(x_0 : u)(t)$ is solution of (2) on $[0, 2T]$. From assumptions

$$\begin{aligned} &\|x(x_0 : u)(t + T)\| \\ &\le M\|x(x_0 : u)(T)\| + M_B\|u\|_{L^2(0,2T:U)}\sqrt{2T} + \{MM_0L_1(k+1) + M(L_4k + L_4') \\ &\quad + M_0L_1(k+1) + \frac{1}{\beta}T^{\beta}C_{1-\beta}L_1(k+1) + ML_3(k+1)T\}. \end{aligned}$$

Hence

$$\begin{aligned} \|x(x_0 : u)\|_{C(0,2T:X)} &= \sup_{0 \le t \le T} \|x(x_0 : u)(t + T)\| \\ &\le C_1(M\|x(x_0 : u)(T)\|_{C(0,2T:X)} + M_B\|u\|_{L^2(0,T:U)}\sqrt{2T}). \end{aligned}$$

□

Theorem 3.5. *Let $\lim_{n\to\infty} u_n = u$ on U. Then for each T, $x(x_0 : u_n)(t)$ converges to $x(x_0 : u)(t)$ on $C([0, T] : X)$, as $n \to \infty$.*

Proof. By Theorem 3.2,

$$\|x(x_0 : u)(t) - x(x_0 : u_n)(t)\| \leq M_B \|u - u_n\|_{L^2(0,T:U)} \sqrt{T},$$

and since $u_n \to u$ as $n \to \infty$, $M_B \|u - u_n\|_{L^2(0,T:U)} \sqrt{T} \longrightarrow 0$. Therefore $x(x_0 : u_n)(t)$ converges to $x(x_0 : u)(t)$ on $C([0, T] : X)$, as $n \to \infty$. □

4. Nonlocal Controllability

In this section, we will show the controllability of neutral functional differential evolution equation with nonlocal initial condition using Sadovskii's fixed point theorem. That is, we will find the condition that the state of (2) can be steered from initial value x_0 to target $x(x_0 : u)(T) = x^1$ in time interval $[0, T]$. Also, we will show the representation of control function u as $u \in L^2([0, T] : U)$. If we define the linear operator $W : U \to X$ by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

for $T \in I$, then $\widetilde{W}$ exist on $L^2([0, T] : U)/kerW$. Also, we define the following the control function for arbitrary function $x(\cdot)$.

$$u(t) = \widetilde{W}^{-1} \Big[x^1 - S(T)\{x_0 + F(0, p(0)) - g(x)\} + F(T, p(T)) - \int_0^T AS(T-s)F(s, p(s))ds - \int_0^T S(T-s)G(s, q(s))ds \Big](t)$$

Also, we define the operator P using control function.

$$\begin{aligned}(Px)(t) &= S(t)[x_0 + F(0, p(0)) - g(x)] - F(t, p(t)) \\ &+ \int_0^t AS(t-s)F(s, p(s))ds + \int_0^t S(t-s)G(s, q(s))ds \\ &+ \int_0^t S(t-s)B\widetilde{W}^{-1} \Big[x^1 - S(T)\{x_0 + F(0, p(0)) - g(x)\} + F(T, p(T)) \\ &- \int_0^T AS(T-\tau)F(\tau, p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau, q(\tau))d\tau \Big](s)ds\end{aligned}$$

Theorem 4.1. *Suppose that (H1)-(H3)are satisfied, for $x_0 \in X$ and $u \in L^2([0, T] : U)$, if*

$$\Big\{ M(M_0L_1 + L_4 + L_3T) + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1 \Big\} \big(1 + M_B \|\widetilde{W}^{-1}\| T\big) < 1, \tag{4}$$

then the nonlocal Cauchy problem (1) *is controllable on* $[0, T]$.

Proof. Let

$$B_k = \{x \in E : \|x(t)\| \leq k, \quad 0 \leq t \leq T\}$$

for positive integer$k$, then clearly, B_k is bounded closed convex set. By (H1) and properties of semigroup,

$$\begin{aligned}\|AS(t-s)F(s,p(s))\| &= \|A^{1-\beta}S(t-s)A^{\beta}F(s,p(s))\| \\ &\leqslant \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(k+1),\end{aligned}$$

and $AS(t-s)F(s,p(s))$ is integrable on $[0,T]$ from Bochner's theorem. Hence P is well define on B_k.

Now we will show that the some positive integer k exist such that $PB_k \subseteq B_k$. Suppose that the function $x_k(\cdot) \in B_k$ exist such that $Px_k \notin B_k$ for some positive integer k. Then since $\|Px_k(t)\| > k$ for t satisfying B_k,

$$\begin{aligned}
k &< \|P(x_k)(t)\| \\
&= \Bigg\| S(t)[x_0 + F(0,p_k(0)) - g(x_k)] - F(t,p_k(t)) \\
&\quad + \int_0^t AS(t-s)F(s,p_k(s))ds + \int_0^t S(t-s)G(s,q_k(s))ds \\
&\quad + \int_0^t S(t-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p_k(0)) - g(x_k)\} + F(T,p_k(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p_k(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q_k(\tau))d\tau\Big](s)ds\Bigg\| \\
&\leq \|S(t)[x_0 + F(0,p_k(0)) - g(x_k)]\| + \|A^{-\beta}A^{\beta}F(t,p_k(t))\| \\
&\quad + \left\|\int_0^t A^{1-\beta}S(t-s)A^{\beta}F(s,p_k(s))ds\right\| + \left\|\int_0^t S(t-s)G(s,q_k(s))ds\right\| \\
&\quad + \Bigg\|\int_0^t S(t-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p_k(0)) - g(x_k)\} + F(T,p_k(T)) \\
&\qquad - \int_0^T AS(T-\tau)F(\tau,p_k(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q_k(\tau))d\tau\Big](s)ds\Bigg\| \\
&\leq M\{\|x_0\| + M_0L_1(k+1) + L_4k + L_4'\} + M_0L_1(k+1) \\
&\quad + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(k+1)ds + M\int_0^t L_3kds \\
&\quad + M_B\int_0^t \|\widetilde{W}^{-1}\|\big[\|x^1\| + M(\|x_0\| + M_0L_1(k+1) + L_4k + L_4') \\
&\quad + M_0L_1(k+1) + \int_0^T \frac{C_{1-\beta}}{(T-\tau)^{1-\beta}}L_1(k+1)d\tau + \int_0^T ML_3kd\tau\big](s)ds
\end{aligned}$$

$$\begin{aligned}
&\leq M\{\|x_0\| + M_0L_1(k+1) + L_4k + L_4'\} + M_0L_1(k+1) \\
&\quad + \frac{1}{\beta}T^\beta C_{1-\beta}L_1(k+1) + ML_3kT \\
&\quad + M_B\|\widetilde{W}^{-1}\|\big[\|x^1\| + M(\|x_0\| + M_0L_1(k+1) + L_4k + L_4') + M_0L_1(k+1) \\
&\quad + \Big[\frac{1}{\beta}T^\beta C_{1-\beta}L_1(k+1) + ML_3kT\Big]T.
\end{aligned}$$

If two side is divided by k, and choosing the lower limit as $k \to \infty$, then

$$\begin{aligned}
1 &\leq \liminf_{k\to\infty} \frac{M}{k}\{\|x_0\| + M_0L_1(k+1) + L_4k + L_4'\} + \frac{1}{k}M_0L_1(k+1) \\
&\quad + \frac{1}{k}\Big\{\frac{1}{\beta}T^\beta C_{1-\beta}L_1(k+1) + \frac{1}{k}ML_3kT + \Big[\frac{1}{\beta}T^\beta C_{1-\beta}L_1(k+1) + ML_3kT\Big]T\Big\} \\
&\quad + \frac{1}{k}M_B\|\widetilde{W}^{-1}\|\big[\|x^1\| + M(\|x_0\| + M_0L_1(k+1) + L_4k + L_4') + M_0L_1(k+1) \\
&\leq M(M_0L_1 + L_4) + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1 + ML_3T \\
&\quad + M_B\|\widetilde{W}^{-1}\|T\Big\{MM_0L_1 + ML_4 + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1 + ML_3T\Big\} \\
&= M(M_0L_1 + L_4 + L_3T) + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1 \\
&\quad + M_B\|\widetilde{W}^{-1}\|T\Big\{M(M_0L_1 + L_4 + L_3T) + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1\Big\} \\
&= \Big\{M(M_0L_1 + L_4 + L_3T) + M_0L_1 + \frac{1}{\beta}T^\beta C_{1-\beta}L_1\Big\}(1 + M_B\|\widetilde{W}^{-1}\|T).
\end{aligned}$$

But this is a contrdiction from (4). Therefore $PB_k \subseteq B_k$ for the positive integer k.

Next, we will prove that the operator P has fixed point on B_k as the nonlocal Cauchy problem (1) have the mild solution. Let $P = P_1 + P_2$ such that P_1 , P_2 are the operators on B_k, and defined by

$$\begin{aligned}
(P_1x)(t) &= S(t)[x_0 + F(0,p(0)) - g(x)] - F(t,p(t)) \\
&\quad + \int_0^t AS(t-s)F(s,p(s))ds + \int_0^t S(t-s)G(s,q(s))ds
\end{aligned}$$

$$\begin{aligned}
(P_2x)(t) &= \int_0^t S(t-s)B\widetilde{W}^{-1}\big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\big](s)ds.
\end{aligned}$$

We will prove that if $0 \leq t \leq T$, then P_1 is a contraction function, and P_2 is a compact operator.

Now, we show that P_1 is a contraction function for $x, y \in B_k$. By assumption (H1)and (2) for each $t \in [0, T]$,

$$\begin{aligned}
&\|(P_1x)(t) - (P_1y)(t)\| \\
&\leq \|S(t)[F(0,p(0)) - F(0,v(0))]\| + \|F(t,p(t)) - F(t,v(t))\| \\
&\quad + \left\| \int_0^t AS(t-s)[F(s,p(s)) - F(s,v(s))]ds \right\| \\
&\quad + \left\| \int_0^t S(t-s)[G(s,q(s)) - G(s,r(s))]ds \right\| \\
&\leq MM_0L \sup_{0\leq s\leq T} \|x(s) - y(s)\| + M_0L \sup_{0\leq s\leq T} \|x(s) - y(s)\| \\
&\quad + \frac{1}{\beta}T^\beta C_{1-\beta}L \sup_{0\leq s\leq T} \|x(s) - y(s)\| + ML_2 \sup_{0\leq s\leq T} \|x(s) - y(s)\| \\
&= \left\{(M+1)M_0L + \frac{1}{\beta}T^\beta C_{1-\beta}L + ML_2\right\} \cdot \sup_{0\leq s\leq T} \|x(s) - y(s)\| \\
&= L_0 \sup_{0\leq s\leq T} \|x(s) - y(s)\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|P_1x - P_1y\|_C &= \sup_{0\leq t\leq T} \|(P_1x)(t) - (P_1y)(t)\| \\
&\leq L_0\|x - y\|_C.
\end{aligned}$$

Since $L_0 < 1$, P_1 is a contraction function.

Also, we will prove that P_2 is a compact operator. First, we show that P_2 is continuous on B_k. Let $\{x_n\}$ be a sequence on B_k with $\{x_n\} \to x$. Then since

$$\begin{aligned}
F(s,p_n(s)) &\longrightarrow F(s,p(s)) \\
G(s,q_n(s)) &\longrightarrow G(s,q(s))
\end{aligned}$$

as $n \to \infty$,

$$\begin{aligned}
&\|(P_2x_n)(t) - (P_2x)(t)\| \\
&= \Bigg\| \int_0^t S(t-s)B\widetilde{W}^{-1}\big[-S(T)\{F(0,p_n(0)) - F(0,p(0))\} \\
&\quad + \{F(T,p_n(T)) - F(T,p(T))\} - \int_0^T AS(T-\tau)\{F(\tau,p_n(\tau)) - F(\tau,p(\tau))\}d\tau \\
&\quad - \int_0^T S(T-\tau)\{G(\tau,q_n(\tau)) - G(\tau,q(\tau))\}d\tau\big](s)ds \Bigg\| \\
&\longrightarrow 0.
\end{aligned}$$

Hence P_2 is a continuous.

Second, we show that $\{P_2x : x \in B_k\}$ is a equicontinuous family. Choosing the fixed $t_1 > 0$ and $t_2 > t_1$, and sufficiently small $\epsilon > 0$. Then

$$\begin{aligned}
&\|(P_2x)(t_2) - (P_2x)(t_1)\| \\
&= \Big\| \int_0^{t_2} S(t_2 - s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)ds \\
&\quad - \int_0^{t_1} S(t_1 - s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)ds\Big\|,
\end{aligned}$$

and putting

$$\begin{aligned}
&B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big] \quad = \quad Y.
\end{aligned}$$

Then

$$\begin{aligned}
&\|(P_2x)(t_2) - (P_2x)(t_1)\| \\
&\le \Big\| \int_0^{t_2} S(t_2 - s)Yds - \int_0^{t_1} S(t_1 - s)Yds \Big\| \\
&\le \|S(t_2 - s) - S(t_1 - s)\| \int_{t_1}^{t_2} Yds.
\end{aligned}$$

Since $x \in B_k$ is independent as $t_2 - t_1 \to 0$,

$$\|(P_2x)(t_2) - (P_2x)(t_1)\| \longrightarrow 0$$

Thus operator $P_2x(x \in B_k)$ is an equicontinuous at $t = 0$. Hence P_2 have nonempty compact value on B_k.

Finally, we will prove that $V(t) = \{(P_2x)(t) : x \in B_k\}$ is a relatively compact on X. Since

$$\begin{aligned}
V(0) &= (P_2x)(0) \\
&= \int_0^0 S(0 - s)B\widetilde{W}^{-1}\big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) \\
&\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\big](s)ds \\
&= 0,
\end{aligned}$$

$V(0)$ is a relatively compact on X. Let $0 \leq t \leq T$ be fixed point and $0 < \epsilon < t$. Defined by

$$(P_{2,\epsilon}x)(t) = \int_0^{t-\epsilon} S(t-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T)) - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)ds$$

for $x \in B_k$, then

$$\begin{aligned}&(P_{2,\epsilon}x)(t)\\ &= \int_0^{t-\epsilon} S(t-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T.p(T))\\ &\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)ds\\ &= S(\epsilon)\int_0^{t-\epsilon} S(t-\epsilon-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T))\\ &\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)ds.\end{aligned}$$

By compact property of $S(\epsilon)$, $V_\epsilon(t) = \{(P_{2,\epsilon}x)(t) : x \in B_k\}$ relatively compact on X for all $0 < \epsilon < t$. Also, for all $x \in B_k$,

$$\begin{aligned}&\|(P_2x)(t) - (P_{2,\epsilon}x)(t)\|\\ &\leqslant \int_{t-\epsilon}^t \|S(t-s)B\widetilde{W}^{-1}\Big[x^1 - S(T)\{x_0 + F(0,p(0)) - g(x)\} + F(T,p(T))\\ &\quad - \int_0^T AS(T-\tau)F(\tau,p(\tau))d\tau - \int_0^T S(T-\tau)G(\tau,q(\tau))d\tau\Big](s)\|ds\\ &\leq \int_{t-\epsilon}^t M_B\|\widetilde{W}^{-1}\|\Big[\|x^1\| + M\{\|x_0\| + M_0L_1(k+1) + L_4k + L_4'\}\\ &\quad + M_0L_1(k+1) + \frac{1}{\beta}T^\beta C_{1-\beta}L_1(k+1) + ML_3(k+1)\Big]T(s)ds.\end{aligned}$$

Therefore the closed set $V(t)$ is relatively compact set. Consequently, since $V(t)$ is relatively compact on X, P_2 is a compact operator by Arzela-Ascoli theorem.

Hence, since there exist a fixed point of $x(\cdot)$ for $P = P_1 + P_2$ on B_k, the nonlocal Cauchy problem (1) has a mild solution from Sadovskii's fixed point theorem. □

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