J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2015.22.1.75 ISSN(Online) 2287-6081
Volume 22, Number 1 (February 2015), Pages 75–90

GLOBAL SOLUTIONS OF THE COOPERATIVE CROSS-DIFFUSION SYSTEMS

Seong-A Shim

Abstract. In this paper the existence of global solutions of the parabolic cross-diffusion systems with cooperative reactions is obtained under certain conditions. The uniform boundedness of $W_{1,2}$ norms of the local maximal solution is obtained by using interpolation inequalities and comparison results on differential inequalities.

1. Introduction

This article deals with the following quasilinear parabolic system in population dynamics which is called cooperative cross-diffusion system.

$$(1.1)\quad \begin{cases} u_t = (d_1\, u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} + u(a_1 - b_1 u + c_1 v) & \text{in } [0,1]\times(0,\infty), \\ v_t = (d_2\, v + \alpha_{21}uv + \alpha_{22}v^2)_{xx} + v(a_2 + b_2 u - c_2 v) & \text{in } [0,1]\times(0,\infty), \\ u_x(x,t) = v_x(x,t) = 0 & \text{at } x = 0,1, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } [0,1], \end{cases}$$

where α_{12}, α_{21}, d, a_i, b_i, c_i are positive constants for $i = 1, 2$. The initial functions u_0, v_0 are not constantly zero. In the system (1.1) u and v are nonnegative functions which represent the population densities of two species in a cooperative relationship. d_1 and d_2 are the *diffusion* rates of the two species, respectively. a_1 and a_2 denote the intrinsic growth rates, b_1 and c_2 account for intra-specific cooperative pressures, b_2 and c_1 are the coefficients for inter-specific competitions. α_{11} and α_{22} are usually referred as *self-diffusion*, and α_{12}, α_{21} are *cross-diffusion* pressures. By adopting the coefficients α_{ij} $(i, j = 1, 2)$ the system (1.1) takes into account the pressures created by mutually interacting species. For more details on the backgrounds of this model, the readers are refered to Okubo and Levin[7].

Pao[8] in 2005, and Delgado et al.[4] in 2008 have obtained some results on the existence of global solutions of the elliptic cross-diffusion systems with cooperative

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Received by the editors December 13, Accepted January 27, 2015.
2010 *Mathematics Subject Classification.* 35B40, 35K55.
Key words and phrases. cross-diffusion systems, cooperative dynamics, existence of global solutions, uniform bound.

reactions. In this paper the existence of global solutions of the parabolic cross-diffusion systems with cooperative reactions is obtained under certain conditions. To state results on the system (1.1) we use the following notation throughout this paper.

Notations. Let Ω be a region in $\mathbb{R}^n$. The norm in $L_p(\Omega)$ is denoted by $|\cdot|_{L_p(\Omega)}$, $1 \le p \le \infty$, where $|f|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}$, if $1 \le p < \infty$, and $|f|_{L_\infty(\Omega)} = \sup\{|f(x)| : x \in \Omega\}$. The usual Sobolev spaces of real valued functions in Ω with exponent $k \ge 0$ are denoted by $W_p^k(\Omega)$, $1 \le p < \infty$. And $\|\cdot\|_{W_p^k(\Omega)}$ represents the norm in the Sobolev space $W_p^k(\Omega)$. For $\Omega = [0,1] \subset \mathbb{R}^1$ we shall use the simplified notation $\|\cdot\|_{k,p}$ for $\|\cdot\|_{W_p^k(\Omega)}$ and $|\cdot|_p$ for $|\cdot|_{L_p(\Omega)}$.

The local existence of solutions to (1.1) was established by Amann [1], [2], [3]. According to his results the system (1.1) has a unique nonnegative solution $u(\cdot,t)$, $v(\cdot,t)$ in $C([0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\Omega))$, where $T \in (0,\infty]$ is the maximal existence time for the solution u, v. The following result is also due to Amann [2].

Theorem 1.1. *Let u_0 and v_0 be in $W_p^1(\Omega)$. The system* (1.1) *possesses a unique nonnegative maximal smooth solution $u(x,t), v(x,t) \in C([0,T), W_p^1(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,T))$ for $0 \le t < T$, where $p > n$ and $0 < T \le \infty$. If the solution satisfies the estimates $\sup\limits_{0<t<T} \|u(\cdot,t)\|_{W_p^1(\Omega)} < \infty$, $\sup\limits_{0<t<T} \|v(\cdot,t)\|_{W_p^1(\Omega)} < \infty$, then $T = +\infty$. If, in addition, u_0 and v_0 are in $W_p^2(\Omega)$ then $u(x,t), v(x,t) \in C([0,\infty), W_p^2(\Omega))$, and $\sup\limits_{0\le t<\infty} \|u(\cdot,t)\|_{W_p^2(\Omega)} < \infty$, $\sup\limits_{0\le t<\infty} \|v(\cdot,t)\|_{W_p^2(\Omega)} < \infty$.*

Here we state the main results of this paper. Throughout this this paper we assume the condition

$$b_1 c_2 > b_2 c_1 \tag{1.2}$$

which means the inter-specific competition pressures are greater than the intra-specific cooperative pressures.

Theorem 1.2. *Suppose that the initial functions u_0, v_0 are in $W_2^2([0,1])$. Also assume the condition* (1.2)*. Let $(u(x,t), v(x,t))$ be the maximal solution to the system* (1.1) *as in Theorem* 1.1*. Then there exist positive constant*

$$M_0 = M_0(\|u_0\|_1, \|v_0\|_1, a_1, a_2, b_1, b_2, c_1, c_2)$$

such that

$$\sup\{\|u(\cdot,t)\|_1,\ \|v(\cdot,t)\|_1 : t \in [0,T)\} \le M_0$$

For the boundedness results of L_2 and $W_{1,2}$ norms of the maximal solution to the system (1.1) we assume the following condition in Theorem 1.3, Theorem 1.4

$$\alpha_{12}^2 < 8\alpha_{11}\alpha_{21} \quad \text{and} \quad \alpha_{21}^2 < 8\alpha_{12}\alpha_{22}. \tag{1.3}$$

Theorem 1.3. *Suppose that the initial functions u_0, v_0 are in $W_2^2([0,1])$. Also assume the conditions* (1.2) *and* (1.3). *Let $(u(x,t), v(x,t))$ be the maximal solution to the system* (1.1) *as in Theorem* 1.1. *Then there exists a positive constant $M_1 = M_1(\|u_0\|_1, \|v_0\|_1, d_i, a_i, b_i, c_i, i = 1,2)$ such that*

$$\sup\{\|u(\cdot,t)\|_2, \|v(\cdot,t)\|_2 : t \in [0,T)\} \leq M_1.$$

Theorem 1.4. *Suppose that the initial functions u_0, v_0 are in $W_2^2([0,1])$. Also assume the conditions* (1.2) *and* (1.3). *Let $(u(x,t), v(x,t))$ be the maximal solution to the system* (1.1) *as in Theorem* 1.1. *Then there exists a positive constant $M_2 = M_2(\|u_0\|_1, \|v_0\|_1, d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1,2)$ such that*

$$\sup\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in [0,T)\} \leq M_2.$$

From the results of Theorems 1.2, 1.3 and 1.4 and the Sobolev embedding inequality we have positive constants $M' = M'(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1,2)$ and $M = M(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1,2)$ such that for the maximal solution (u,v) of (1.1) with the conditions (1.2), (1.3)

$$\begin{aligned} &\sup\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in [0,T)\} \leq M', \\ &\sup\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times [0,T)\} \leq M. \end{aligned} \tag{1.4}$$

We also conclude that $T = +\infty$ from Theorem 1.1.

This paper is organized as follows. Section 2 provides preliminaries on differential equations and a few consequences of Gagliardo-Nirenberg interpolation inequality which are necessary for the proofs of Theorems 1.2, 1.3, and 1.4. And Sections 3, 4, and 5 present the proofs of Theorems 1.2, 1.3, and 1.4, respectively.

2. Preliminaries

This section introduce the Gagliardo-Nirenberg interpolation inequality and its consequences. Also some preliminary results on the bounds and comparisons of differential equations and inequalities are provided.

Theorem 2.1 (Gagliardo-Nirenberg interpolation inequality)**.** *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ in C^m. For every function u in $W^{m,r}(\Omega)$, $1 \leq q, r \leq \infty$*

the derivative $D^j u$, $0 \le j < m$, *satisfies the inequality*

$$|D^j u|_p \le C(|D^m u|_r^a |u|_q^{1-a} + |u|_q), \tag{2.1}$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$ *for all* a *in the interval* $\frac{j}{m} \le a < 1$, *provided one of the following three conditions* :

(i) $r \le q$,

(ii) $0 < \frac{n(r-q)}{mrq} < 1$, *or*

(iii) $\frac{n(r-q)}{mrq} = 1$ *and* $m - \frac{n}{q}$ *is not a nonnegative integer.*

(*The positive constant* C *depends only on* n, m, j, q, r, a.)

Proof. We refer the reader to A. Friedman [5] or L. Nirenberg [6] for the proof of this well-known calculus inequality. □

Corollary 2.1. *There exist positive constants* C, $\tilde{C}$, *and* $\hat{C}$ *such that for every function* u *in* $W_2^1([0,1])$

$$|u|_4 \le C(|u_x|_2^{\frac{1}{2}} |u|_1^{\frac{1}{2}} + |u|_1). \tag{2.2}$$

$$|u|_{\frac{5}{2}} \le \tilde{C}(|u_x|_2^{\frac{2}{5}} |u|_1^{\frac{3}{5}} + |u|_1). \tag{2.3}$$

$$|u|_2 \le \hat{C}(|u_x|_2^{\frac{1}{3}} |u|_1^{\frac{2}{3}} + |u|_1), \tag{2.4}$$

Proof. $n = 1$, $m = 1$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2.1. Letting $j = 0$ in this case the necessary condition on p, a for inequality (2.1) becomes

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q} = 1 - \frac{2}{3}a \tag{2.5}$$

From equation (2.5) if $p = 4$, then $a = \frac{1}{2}$, if $p = \frac{5}{2}$, then $a = \frac{2}{5}$, and if $p = 2$, then $a = 13$. Therefore we have inequalities (2.2), (2.3), (2.4). □

Corollary 2.2. *For every function* u *in* $W_2^2([0,1])$

$$|u_x|_2 \le C(|u_{xx}|_2^{\frac{3}{5}} |u|_1^{\frac{2}{5}} + |u|_1). \tag{2.6}$$

Proof. $m = 2$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2.1. □

Theorem 2.2 (Young's Inequality)**.** *If* a *and* b *are nonnegative real numbers and* p *and* q *are positive real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

The equality hold if and only if $a^p = b^q$.

Theorem 2.3 (Hölder's Inequality)**.** *If* $f, g : \mathbb{R}^n \to \mathbb{R}$ *are Lebesgue measurable and* $p, q \in [1, \infty]$ *are real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$|fg|_1 \leq |f|_p |g|_q.$$

Lemma 2.1 below presents a few basic inequalities that will be used for the computations in this paper.

Lemma 2.1. *Let* $x \geq 0$*,* $y \geq 0$*. Then*

$$(x+y)^2 \geq \tfrac{1}{2}x^2 - y^2 \tag{2.7}$$

$$x^k \leq x^s + 1, \qquad \text{if } 0 < k \leq s \tag{2.8}$$

$$x^k \leq x^s + x^t, \qquad \text{if } 0 < t \leq k \leq s \tag{2.9}$$

$$(x+y)^k \leq 2^{k-1}(x^k + y^k), \qquad \text{if } k \geq 1 \tag{2.10}$$

$$x^k + y^k \leq 2^{1-k}(x+y)^k, \qquad \text{if } 0 < k \leq 1 \tag{2.11}$$

Proof. Inequalities (2.7), (2.8), (2.9) are simply proved.
To show inequalities (2.10), (2.11), let $g : [0, \infty) \to \mathbb{R}$, $g(x) = 2^{k-1}(x^k + y^k) - (x + y)^k$. Then

$$g'(x) = k2^{k-1}x^{k-1} - k(x+y)^{k-1}.$$

Hence the function $g(x)$ has the critical value 0 at $x = y$ which is the minimum value if $k \geq 1$, and the maximum value if $0 < k \leq 1$. Thus we obtain inequalities (2.10) and (2.11). □

Theorem 2.4 (Picard's local existence and uniqueness theorem)**.** *If* $f(x, t)$ *is a continuous real-valued function that satisfies the Lipschitz condition*

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

in some open rectangle $R = \{(x, t) \mid a < x < b, c < t < d\}$ *that contains the point* (x_0, t_0)*, then the initial value problem*

$$x' = f(x,t), \qquad x(t_0) = x_0$$

has a unique solution in some closed interval $I = [t_0 - \epsilon, t_0 + \epsilon]$*, where* $\epsilon > 0$*.*

Theorem 2.5. *Let* $f(x)$ *be a real-valued differentiable functions defined on an open interval* (a, b)*. Then for every initial point* x_0 *in* (a, b) *a solution of the initial value problem*

$$x' = f(x), \qquad x(0) = x_0$$

is either constant or strictly monotone.

Proof. The conclusion follow from the fact that $f(x(t))$ never changes sign for the solution $x(t)$ of the given initial value problem. To see why this is so, suppose that $x(t)$ is not a constant solution, and $f(x(t))$ changes sign. Then it would have to be $f(x(t_1)) = 0$ at some $t_1 > 0$ and $f(x(t)) \neq 0$ for t in the left of t_1 or right of t_1. But it contradict the fact that from Theorem 2.4 the constant solution $y(t) \equiv x(t_1)$ is a unique solution in some closed interval $[t_1 - \epsilon, t_1 + \epsilon]$, where $\epsilon > 0$. □

Corollary 2.3. *Let $c_1 > 0$, $p > 1$, and c_2, c_3 be any real numbers. Then there exists a positive constant $M = M(x_0, p, c_1, c_2, c_3)$ such that the solution of the initial value problem*

$$x' = -c_1 x^p + c_2 x + c_3, \qquad x(0) = x_0 \geq 0$$

satisfies that

$$x(t) \leq M \quad \text{for all } t \geq 0.$$

Proof. The function $f(x) = -c_1 x^p + c_2 x + c_3$ is differentiable functions on $\mathbb{R}$ and falls in either of the two cases:

case(a) $f(x) \leq 0$ for all $x \geq 0$

case(b) there exist a positive constant $m = m(p, c_1, c_2, c_3)$ such that $f(m) = 0$, $f(x) > 0$ for x in some interval on the left of m, and $f(x) < 0$ for all $x > M$.

In case (a) $x'(0) = f(x_0) \leq 0$, and thus by Theorem 2.5 $x'(t) \leq 0$ for all $t \geq 0$. Hence $x(t) \leq x_0$ for all $t \geq 0$. In case (b) if $0 < x_0 < m$ then the solution $x(t)$ cannot cross the constant solution $y(t) \equiv m$ by Theorem 2.5, and thus $x(t) \leq m$ for all $t \geq 0$. If $x_0 \geq m$ then $x'(0) = f(x_0) \leq 0$, and thus by Theorem 2.5 $x'(t) \leq 0$ for all $t \geq 0$. Hence $x(t) \leq x_0$ for all $t \geq 0$. Therefore in any case there exists a positive constant $M = M(x_0, p, c_1, c_2, c_3)$ such that $x(t) \leq M$ for all $t \geq 0$. □

Lemma 2.2 (Gronwall's inequality and the Comparison Principle for differential equations)**.** *Let $a < b \leq \infty$, and $\xi(t)$ and $\beta(t)$ be real-valued continuous functions defined on the interval $[a, b]$. If $\xi(t)$ is differentiable in (a, b) and satisfies the differential inequality*

$$\xi'(t) \leq \beta(t)\xi(t), \qquad t \in (a, b),$$

then $\xi(t)$ is bounded by the solution of the corresponding differential equation $y'(t) = \beta(t)y(t)$, $y(a) = \xi(a)$, that is,

$$\xi(t) \leq \xi(a) \exp\left(\int_a^t \beta(s)\, ds\right)$$

for all $t \in [a, b]$*. And it follows that if in addition* $\xi(a) \leq 0$*, then* $\xi(t) \leq 0$ *for all* $t \in [a, b]$*.*

Proof. We refer the reader to [2]. □

Lemma 2.3. *Let* $c_1 > 0$*,* $p > 1$*, and* c_2*,* c_3 *be any real numbers. Suppose that two differentiable functions* $\phi(t)$ *and* $x(t)$ *satisfy*

$$\begin{array}{ll} \phi' \leq -c_1\phi^p + c_2\phi + c_3, & \phi(0) = \phi_0 \\ x' = -c_1 x^p + c_2 x + c_3, & x(0) = \phi_0. \end{array}$$

Then

$$\phi(t) \leq x(t) \quad \textit{for all } t \geq 0.$$

And especially, if $\phi_0 \geq 0$ *then there exists a positive constant* $M = M(\phi_0, p, c_1, c_2, c_3)$ *such that*

$$\phi(t) \leq M \quad \textit{for all } t \geq 0.$$

Proof. Let $\xi = \phi - x$. Then

$$\begin{array}{rcl} \xi' & = & \phi' - x' \\ & \leq & -c_1(\phi^p - x^p) + c_2(\phi - x) \\ & = & \xi(-c_1\eta + c_2), \end{array}$$

where

$$\eta(t) = \begin{cases} \frac{\phi(t)^p - x(t)^p}{\phi - x}, & \text{if } \phi(t) \neq x(t) \\ p\phi(t)^{p-1}, & \text{if } \phi(t) = x(t) \end{cases}$$

Here notice that $\eta(t)$ is a continuous function using the mean value theorem and the continuities of $\phi(t)$ and $x(t)$. Now, since $\xi(0) = 0$ we conclude that $\xi(t) = \phi(t) - x(t) \leq 0$ for all $t \geq 0$ from Lemma 2.2. And if $\phi_0 \geq 0$, from Corollary 2.3 there exists a positive constant $M = M(\phi_0, p, c_1, c_2, c_3)$ such that $x(t) \leq M$ for all $t \geq 0$. Thus $\phi(t) \leq M$ for all $t \geq 0$. □

3. L_1-bound of Solutions to (1.1)

Proof of Theorem 1.2. By taking integration over the interval $[0, 1]$ for the first and second equations in (1.1) we have that

$$\begin{array}{rcl} \frac{d}{dt}\int_0^1 u(x,t)\, dx & = & \int_0^1 (a_1 u - b_1 u^2 + c_1 uv)\, dx \\ \frac{d}{dt}\int_0^1 v(x,t)\, dx & = & \int_0^1 (a_2 v - b_2 uv + c_2 v^2)\, dx, \end{array}$$

$$\frac{d}{dt}\int_0^1 (b_2 u + c_1 v)\ dx = \int_0^1 (a_1 b_2 u + a_2 c_1 v)\, dx - \int_0^1 \left(b_1 b_2 u^2 - 2 b_2 c_1 uv + c_1 c_2 v^2\right)\ dx.$$

Let $\delta = \dfrac{b_2 c_1 (b_1 c_2 - b_2 c_1)}{b_1 b_2 + c_1 c_2}$. The condition $b_1 c_2 > b_2 c_1$ implies $\delta > 0$. It also holds that

$$\delta = \frac{b_2 c_1 (b_1 c_2 - b_2 c_1)}{b_1 b_2 + c_1 c_2} < \min\{b_1 b_2,\ c_1 c_2\}.$$

Thus it is shown that

$$\int_0^1 \left(b_1 b_2 u^2 - 2 b_2 c_1 uv + c_1 c_2 v^2\right)\ dx - \delta \int_0^1 \left(u^2 + v^2\right)\ dx \geq 0$$

from the facts

$$(3.1)\quad (b_2 c_1)^2 - (b_1 b_2 - \delta)(c_1 c_2 - \delta) = -\delta^2 + (b_1 b_2 + c_1 c_2)\,\delta - b_2 c_1 (b_1 c_2 - b_2 c_1) < 0.$$

Using (3.1) we have

$$\frac{d}{dt}\int_0^1 (b_2 u + c_1 v)\ dx \leq \int_0^1 (a_1 b_2 u + a_2 c_1 v)\, dx - \delta \int_0^1 \left(u^2 + v^2\right)\ dx,$$

and thus

$$\frac{d}{dt}\int_0^1 (u + v)\ dx \leq C_1 \int_0^1 (u+v)\, dx - C_0' \int_0^1 \left(u^2 + v^2\right)\ dx,$$

where $C_1 = \dfrac{\max\{a_1 b_2,\ a_2 c_1\}}{\min\{b_2,\ c_1\}}$, $C_0' = \dfrac{\delta}{\min\{b_2,\ c_1\}}$. From Hölder's inequality

$$\int_0^1 u\, dx \leq \left(\int_0^1 u^2\, dx\right)^{\frac{1}{2}}, \qquad \int_0^1 v\, dx \leq \left(\int_0^1 v^2\, dx\right)^{\frac{1}{2}},$$

it follows that

$$\frac{d}{dt}\int_0^1 (u + v))\ dx \leq C_1 \int_0^1 (u+v)\, dx - C_0' \left\{\left(\int_0^1 u\, dx\right)^2 + \left(\int_0^1 v\, dx\right)^2\right\},$$

and thus

$$(3.2)\qquad \frac{d}{dt}\int_0^1 (u + v)\ dx \leq C_1 \int_0^1 (u+v)\, dx - C_0 \left\{\int_0^1 (u+v)\, dx\right\}^2,$$

where $C_0 = \dfrac{C_0'}{2}$. Hence by the Gronwall's type inequailty in Lemma 2.3 there exists positive constant $M_0 = M_0\left(\|u_0\|_1, \|v_0\|_1, a_1, a_2, b_1, b_2, c_1, c_2\right)$ satisfying

$$(3.3)\qquad \int_0^1 u(x,t)\, dx \leq M_0, \qquad \int_0^1 v(x,t)\, dx \leq M_0$$

for all $t \geq 0$. □

4. L_2-bound of Solutions to (1.1)

Proof of Theorem 1.3. Multiplying the first and second equations in (1.1) by $u = u(x,t)$ and $v = v(x,t)$, respectively, and taking integrations over $[0,1]$ we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx &= \int_0^1 u(d_1\,u+\alpha_{11}u^2+\alpha_{12}uv)_{xx}\,dx+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx\\ \frac{1}{2}\frac{d}{dt}\int_0^1 v^2\,dx &= \int_0^1 v(d_2\,v+\alpha_{21}uv+\alpha_{22}v^2)_{xx}\,dx+\int_0^1 v^2(a_2-b_2u+c_2v)\,dx.\end{aligned}$$

Using Neumann boundary conditions

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx &= \int_0^1 u(d_1\,u+\alpha_{11}u^2+\alpha_{12}uv)_{xx}\,dx+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx\\ &= -\int_0^1 u_x(d_1\,u+\alpha_{11}u^2+\alpha_{12}uv)_x\,dx+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx\\ &= -\int_0^1 u_x(d_1\,u_x+2\alpha_{11}uu_x+\alpha_{12}vu_x+\alpha_{12}uv_x)\,dx\\ &\quad+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx\\ &= -\int_0^1 (d_1+2\alpha_{11}u+\alpha_{12}v)u_x^2\,dx-\int_0^1 \alpha_{12}uu_xv_x\,dx\\ &\quad+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx,\end{aligned}$$

and similarly

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v^2\,dx = -\int_0^1 (d_2+2\alpha_{21}u+\alpha_{22}v)v_x^2\,dx-\int_0^1 \alpha_{21}vu_xv_x\,dx\\ +\int_0^1 v^2(a_2+b_2u-c_2v)\,dx.\end{aligned}$$

(4.1)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx\\ &= -d_1\int_0^1 u_x^2\,dx-d_2\int_0^1 v_x^2\,dx\\ &\quad-\int_0^1 \left((2\alpha_{11}u+\alpha_{12}v)u_x^2+(\alpha_{12}u+\alpha_{21}v)u_xv_x+(2\alpha_{21}u+\alpha_{22}v)v_x^2\right)\,dx\\ &\quad+\int_0^1 u^2(a_1-b_1u+c_1v)\,dx+\int_0^1 v^2(a_2+b_2u-c_2v)\,dx,\end{aligned}$$

Using condition (1.3) that $\alpha_{12}^2 < 8\alpha_{11}\alpha_{21}$ and $\alpha_{21}^2 < 8\alpha_{12}\alpha_{22}$, we have

$$\begin{aligned}&(\alpha_{12}u + \alpha_{21}v)^2 - 4(2\alpha_{11}u + \alpha_{12}v)(2\alpha_{21}u + \alpha_{22}v)\\ &\quad = (\alpha_{12}^2 - 8\alpha_{11}\alpha_{21})u^2 - 2(\alpha_{12}\alpha_{21} + 8\alpha_{11}\alpha_{22})uv + (\alpha_{21}^2 - 8\alpha_{12}\alpha_{22})v^2 \le 0.\end{aligned}$$

Thus it follows from (4.1) that

(4.2)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx \;&\le\; -d_1\int_0^1 u_x^2\,dx - d_2\int_0^1 v_x^2\,dx\\ &\quad + \int_0^1 u^2(a_1 - b_1u + c_1v)\,dx + \int_0^1 v^2(a_2 + b_2u - c_2v)\,dx,\\ &\le\; -d_1\int_0^1 u_x^2\,dx - d_2\int_0^1 v_x^2\,dx\\ &\quad + a_1\int_0^1 u^2\,dx + a_2\int_0^1 v^2\,dx + c_1\int_0^1 u^2v\,dx + b_2\int_0^1 uv^2\,dx.\end{aligned}$$

By Young's inequality

$$\int_0^1 u^2v\,dx \le \int_0^1 \frac{1}{2}\left(\epsilon u^4 + \frac{1}{\epsilon}v^2\right)dx, \qquad \int_0^1 uv^2\,dx \le \int_0^1 \frac{1}{2}\left(\epsilon v^4 + \frac{1}{\epsilon}u^2\right)dx$$

holds for any $\epsilon > 0$. And by applying Lemma 2.1 to inequality (2.2) and using the uniform L_1-boundedness of u and v from Step 1, we have

$$\int_0^1 u^4\,dx \le C\left(\int_0^2 u_x^2\,dx + 1\right), \qquad \int_0^1 v^4\,dx \le C\left(\int_0^2 v_x^2\,dx + 1\right),$$

where C is a positive constant depending only on a_i, b_i, c_i $(i,j = 1,2)$. Thus (4.2) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx\\ &\quad\le -d_1\int_0^1 u_x^2\,dx - d_2\int_0^1 v_x^2\,dx + a_1\int_0^1 u^2\,dx + a_2\int_0^1 v^2\,dx\\ &\qquad + c_1\epsilon C\left(\int_0^1 u_x^2\,dx + 1\right) + \frac{c_1}{2\epsilon}\int_0^1 v^2\,dx\\ &\qquad + b_2\epsilon C\left(\int_0^1 v_x^2\,dx + 1\right) + \frac{b_2}{2\epsilon}\int_0^1 u^2\,dx\\ &\quad\le -\frac{d_1}{2}\int_0^1 u_x^2\,dx - \frac{d_2}{2}\int_0^1 v_x^2\,dx + C_1'\int_0^1 \left(u^2+v^2\right)\,dx + C_0',\end{aligned}$$

where $\epsilon = \frac{1}{C}\min\{\frac{d_1}{2c_1}, \frac{d_2}{2b_2}\}$ and the constants C_0' and C_1' are depending on d_i, a_i, b_i, c_i $(i,j = 1,2)$. And by applying Lemma 2.1 to inequality (2.4) and using the uniform L_1-boundedness of u and v from Step 1, we have

$$\left(\int_0^1 u^2\,dx\right)^3 \le \tilde{C}\left(\int_0^2 u_x^2\,dx+1\right), \qquad \left(\int_0^1 v^2\,dx\right)^3 \le \tilde{C}\left(\int_0^2 v_x^2\,dx+1\right),$$

where $\tilde{C}$ is a positive constant depending only on a_i, b_i, c_i $(i,j=1,2)$. And thus

$$-\int_0^1 u_x^2\,dx \le 1-C'\left(\int_0^1 u^2\,dx\right)^3, \qquad -\int_0^1 v_x^2\,dx \le 1-C'\left(\int_0^1 v^2\,dx\right)^3,$$

where C' is a positive constant depending only on a_i, b_i, c_i $(i,j=1,2)$. Thus we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx\\ &\le -C_2'\left(\int_0^1 u^2\,dx\right)^3 - C_2'\left(\int_0^1 v^2\,dx\right)^3 + C_1'\int_0^1 \left(u^2+v^2\right)\,dx + C_0'' \\ &\le -C_2\left(\int_0^1 (u^2+v^2)\,dx\right)^3 + C_1\int_0^1 \left(u^2+v^2\right)\,dx + C_0\end{aligned} \tag{4.3}$$

by Lemma 2.1, where C_0, C_1, C_2 are positive constants d_i, a_i, b_i, c_i $(i,j=1,2)$. Hence by the Gronwall's type inequailty in Lemma 2.3 we obtain the following L_2-bound of u and v such that

$$\int_0^1 (u^2+v^2)\,dx \le M_1 \qquad \text{for all}\ \ t\ge 0, \tag{4.4}$$

where M_1 is a positive constant depending on $\|u_0\|_2$, $\|v_0\|_2$, d_i, a_i, b_i, c_i $(i,j=1,2)$. □

5. $W_{1,2}$-bound of Solutions to (1.1)

Proof of Theorem 1.4. To obtain uniform bounds of $|u_x|_2$ and $|v_x|_2$ for the solution of (1.1) let us denote that

$$P=d_1u+\alpha_{11}u^2+\alpha_{12}uv, \qquad Q=d_2v+\alpha_{21}uv+\alpha_{22}v^2.$$

We would show the uniform boundedness of $|P_x|_2$ and $|P_x|_2$ and then obtain the uniform bounds of $|u_x|_2$ and $|v_x|_2$ from it. Here we note from Theorem 1.1 that $P,Q\in C([0,T),W_2^1([0,1]))\cap C^\infty([0,1]\times(0,T))$ for $0\le t<T$, and

$$\begin{aligned}\int_0^1 P_tu_t\,dx &= \int_0^1 (d_1u_t+2\alpha_{11}uu_t+\alpha_{12}u_tv+\alpha_{12}uv_t)u_t\,dx\\ &= \int_0^1 \left[(d_1+2\alpha_{11}u+\alpha_{12}v)u_t^2+\alpha_{12}uu_tv_t\right]\,dx\end{aligned}$$

$$\begin{aligned}\int_0^1 Q_t v_t\,dx &= \int_0^1 (d_2 v_t + \alpha_{21} u_t v + \alpha_{21} u v_t + 2\alpha_{22} v u_t) v_t\,dx \\ &= \int_0^1 \left[(d_2 + \alpha_{21} u + 2\alpha_{22} v) v_t^2 + \alpha_{21} v u_t v_t\right]\,dx,\end{aligned}$$

$$\begin{aligned}\int_0^1 P_t P_{xx}\,dx &= -\int_0^1 P_{xt} P_x\,dx &= -\frac{1}{2}\frac{d}{dt}\int_0^1 P_x^2\,dx, \\ \int_0^1 Q_t Q_{xx}\,dx &= -\int_0^1 Q_{xt} Q_x\,dx &= -\frac{1}{2}\frac{d}{dt}\int_0^1 Q_x^2\,dx\end{aligned}$$

from the Neumann boundary conditions on u, v. Now, multiplying the first equation in (1.1) by P_t and the second equation by Q_t, we have

$$\begin{aligned}&\int_0^1 \left[(d_1 + 2\alpha_{11} u + \alpha_{12} v) u_t^2 + \alpha_{12} u u_t v_t\right]\,dx \\ &\quad = -\frac{1}{2}\frac{d}{dt}\int_0^1 P_x^2\,dx \\ &\qquad + \int_0^1 \left[u(d_1 + 2\alpha_{11} u + \alpha_{12} v)(a_1 - b_1 u + c_1 v) u_t + \alpha_{12} u^2 (a_1 - b_1 u + c_1 v) v_t\right]\,dx,\end{aligned}$$

$$\begin{aligned}&\int_0^1 \left[(d_2 + \alpha_{21} u + 2\alpha_{22} v) v_t^2 + \alpha_{21} v u_t v_t\right]\,dx \\ &\quad = -\frac{1}{2}\frac{d}{dt}\int_0^1 Q_x^2\,dx \\ &\qquad + \int_0^1 \left[v(d_2 + \alpha_{21} u + 2\alpha_{22} v)(a_2 + b_2 u - c_2 v) v_t + \alpha_{21} v^2 (a_2 + b_2 u - c_2 v) u_t\right]\,dx,\end{aligned}$$

and thus

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (P_x^2 + Q_x^2)\,dx \\ &\quad \le -d_1 \int_0^1 u_t^2\,dx - d_2 \int_0^1 v_t^2\,dx \\ &\qquad - \int_0^1 \left[(2\alpha_{11} u + \alpha_{12} v) u_t^2 + (\alpha_{12} u + \alpha_{21} v) u_t v_t + (\alpha_{21} u + 2\alpha_{22} v) v_t^2\right]\,dx \\ &\qquad + C_1 \int_0^1 \left(u + v + u^2 + uv + v^2 + u^3 + u^2 v + u v^2 + v^3\right)(|u_t| + |v_t|)\,dx,\end{aligned}\tag{5.1}$$

where C_1 is a positive constant depending on d_i, α_{ij}, a_i, b_i, c_i $(i, j = 1, 2)$. Here we notice from the condition (1.3) that there exists a positive constant $\lambda = \lambda(\alpha_{i,j}, i, j = 1, 2)$ satisfying

$$(2\alpha_{11} u + \alpha_{12} v) u_t^2 + (\alpha_{12} u + \alpha_{21} v) u_t v_t + (\alpha_{21} u + 2\alpha_{22} v) v_t^2 \ \ge\ \lambda(u + v)(u_t^2 + v_t^2),$$

since

$$\begin{aligned}&(\alpha_{12}u+\alpha_{21}v)^2-4(2\alpha_{11}u+\alpha_{12}v-\lambda u-\lambda v)(\alpha_{21}u+2\alpha_{22}v-\lambda u-\lambda v)\\&\quad=(\alpha_{12}^2-8\alpha_{11}\alpha_{21})u^2-2(\alpha_{12}\alpha_{21}+8\alpha_{11}\alpha_{22})uv+(\alpha_{21}^2-8\alpha_{12}\alpha_{22})v^2\\&\qquad+4\lambda\left[(2\alpha_{11}+\alpha_{21})u^2+(2\alpha_{11}+\alpha_{12}+\alpha_{21}+2\alpha_{22})uv+(\alpha_{12}+2\alpha_{22})v^2\right]\\&\qquad-4\lambda^2(u+v)^2\le 0\end{aligned}$$

for all $u\ge 0$, $v\ge 0$, if $\lambda=\lambda(\alpha_{ij},i,j=1,2)>0$ is small enough.

The terms $\int_0^1 u_t^2\,dx$, $\int_0^1 v_t^2\,dx$ in (5.1) ae estimated in terms of P and Q from inequality (2.7) in lemma 2.1.

$$\begin{aligned}-\int_0^1 u_t^2\,dx &= -\int_0^1\left[P_{xx}+u(a_1-b_1+c_1v)\right]^2\,dx\\&\le -\frac{1}{2}\int_0^1 P_{xx}^2\,dx+\int_0^1 u^2(a_1-b_1+c_1v)^2\,dx,\\-\int_0^1 v_t^2\,dx &= -\int_0^1\left[Q_{xx}+v(a_2+b_2u-c_2v)\right]^2\,dx\\&\le -\frac{1}{2}\int_0^1 Q_{xx}^2\,dx+\int_0^1 v^2(a_2+b_2u-c_2v)^2\,dx.\end{aligned}$$

Now we observe using Young's inequality that

$$\left|\int_0^1 uu_t\,dx\right|=\left|\int_0^1 u^{\frac12}\left(u^{\frac12}u_t\right)\,dx\right|\le\frac{1}{2\epsilon}\int_0^1 u\,dx+\frac{\epsilon}{2}\int_0^1 uu_t^2\,dx$$

hold for any $\epsilon>0$. Similar estimates are applied to the terms $\int_0^1 uv_t\,dx$, $\int_0^1 vu_t\,dx$, $\int_0^1 vu_t\,dx$, $\int_0^1 u^2u_t\,dx$, $\int_0^1 v^2u_t\,dx$, and so on. Using these inequalities and inequalities (2.8), (2.9) in lemma 2.1 we obtain that

$$\begin{aligned}&C_1\int_0^1\left(u+v+u^2+uv+v^2+u^3+u^2v+uv^2+v^3\right)(|u_t|+|v_t|)\,dx\\&\qquad+d_1\int_0^1 u^2(a_1-b_1+c_1v)^2\,dx+d_2\int_0^1 v^2(a_2+b_2u-c_2v)^2\,dx\\&\quad\le C_2\int_0^1\left(u+v+u^3+v^3\right)(|u_t|+|v_t|)\,dx\\&\qquad+d_1\int_0^1 u^2(a_1-b_1+c_1v)^2\,dx+d_2\int_0^1 v^2(a_2+b_2u-c_2v)^2\,dx\\&\quad\le\frac{C_3}{2\epsilon}\int_0^1\left(u+v+u^5+u^4v+u^3v^2+u^2v^3+uv^4+v^5\right)\,dx\\&\qquad+\frac{\epsilon C_3}{2}\int_0^1(u+v)\left(u_t^2+v_t^2\right)\,dx+d_1\int_0^1 u^2(a_1-b_1+c_1v)^2\,dx\\&\qquad+d_2\int_0^1 v^2(a_2+b_2u-c_2v)^2\,dx\\&\quad\le\left(\frac{C_3}{2\epsilon}+C_4\right)\int_0^1\left(1+u^5+v^5\right)\,dx+\frac{\epsilon C_3}{2}\int_0^1(u+v)\left(u_t^2+v_t^2\right)\,dx,\end{aligned}$$

where C_2, C_3, C_4 are positive constant depending on d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. Thus we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (P_x^2+Q_x^2)\,dx \\
&\quad \le -\frac{d_1}{2}\int_0^1 P_{xx}^2\,dx - \frac{d_2}{2}\int_0^1 Q_{xx}^2\,dx - \lambda\int_0^1 (u+v)(u_t^2+v_t^2)\,dx \\
&\qquad + \left(\frac{C_3}{2\epsilon}+C_4\right)\int_0^1 \left(1+u^5+v^5\right)\,dx + \frac{\epsilon C_3}{2}\int_0^1 (u+v)\left(u_t^2+v_t^2\right)\,dx
\end{aligned}$$

for any $\epsilon>0$. Here we choose a small $\epsilon>0$ so that $\frac{\epsilon C_3}{2}\le\lambda$, and thus

(5.2)

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (P_x^2+Q_x^2)\,dx \le -\frac{d_1}{2}\int_0^1 P_{xx}^2\,dx - \frac{d_2}{2}\int_0^1 Q_{xx}^2\,dx + C_5\int_0^1 \left(1+u^5+v^5\right)\,dx$$

where C_5 is a positive constants depending on d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. Now we observe that

$$P = d_1u+\alpha_{11}u^2+\alpha_{12}uv \ge \alpha_{11}u^2, \quad Q = d_2v+\alpha_{21}uv+\alpha_{22}v^2 \ge \alpha_{22}v^2,$$

and thus

$$\int_0^1 (u^5+v^5)\,dx \le C_6\int_0^1 (P^{\frac{5}{2}}+Q^{\frac{5}{2}})\,dx,$$

where C_6 is a positive constant depending only on d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. Applying the inequalities (2.6) and (2.3) to the function $P=d_1u+\alpha_{11}u^2+\alpha_{12}uv$ we have

$$|P_x|_2 \le \tilde{C}\left(|P_{xx}|_2^{\frac{3}{5}}|P|_1^{\frac{2}{5}}+|P|_1\right) \le \tilde{C}|P|_1^{\frac{2}{5}}\left(|P_{xx}|_2^{\frac{3}{5}}+|P|_1^{\frac{3}{5}}\right),$$

$$|P|_{\frac{5}{2}} \le \hat{C}\left(|P_x|_2^{\frac{2}{5}}|P|_1^{\frac{3}{5}}+|P|_1\right).$$

Here using the uniform boundedness of the L_1 norm of P, we have

$$-\int_0^1 P_{xx}^2\,dx \le C_7 - C_8\left(\int_0^1 P_x^2\,dx\right)^{\frac{5}{3}}, \tag{5.3}$$

$$\int_0^1 P^{\frac{5}{2}}\,dx \le C_9\left(\int_0^1 P_x^2\,dx\right)^{\frac{1}{2}} + C_{10} \tag{5.4}$$

where C_7, C_8, C_9, C_{10} are positive constants depending on d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. Similar estimates are obtained also for Q. Hence we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (P_x^2+Q_x^2)\,dx \\ &\le -C_{11}\left(\int_0^1 (P_x^2+Q_x^2)\,dx\right)^{\frac{5}{3}} + C_{12}\left(\int_0^1 (P_x^2+Q_x^2)\,dx\right)^{\frac{1}{2}} + C_{13},\end{aligned} \tag{5.5}$$

where C_{11}, C_{12}, C_{13} are positive constants depending on d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. Hence by the Gronwall's type inequailty in Lemma 2.3 we obtain the following $W_{1,2}$-bound of P and Q such that

$$\int_0^1 P_x^2\,dx < \tilde{M}_2, \quad \int_0^1 Q_x^2\,dx < \tilde{M}_2 \qquad \text{for all} \ \ t \ge 0, \tag{5.6}$$

where $\tilde{M}_2$ is a positive constant depending on $\|u_0\|_2$, $\|v_0\|_2$, d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$.

In order to obtain estimates for u_x and v_x, we notice that

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} P_u & P_v \\ Q_u & Q_v \end{pmatrix}^{-1} \begin{pmatrix} P_x \\ Q_x \end{pmatrix} = A^{-1}\begin{pmatrix} P_x \\ Q_x \end{pmatrix},$$

where

$$A = \begin{pmatrix} d_1+2\alpha_{11}u+\alpha_{12}v & \alpha_{12}u \\ \alpha_{21}v & d_2+\alpha_{21}u+2\alpha_{22}v \end{pmatrix}.$$

Here we note that $|A|$, the determinant of A, is bounded below by the positive constant d_1d_2, and $|A|$ is of class $O(u^2+v^2)$ as $u\to\infty$ and $v\to\infty$, we have the inequality

$$|u_x|+|v_x| \le C_{14}\left(|P_x|+|Q_x|\right) \quad \text{for every } x\in[0,1]\times[0,\infty)$$

for some constant C_{14} depending only on d_i, α_{ij}, $(i,j=1,2)$. Therefore we obtain the following $W_{1,2}$-bound of u and v such that

$$\int_0^1 u_x^2\,dx < M_2, \quad \int_0^1 v_x^2\,dx < M_2 \qquad \text{for all} \ \ t\ge 0,$$

where M_2 is a positive constant depending on $\|u_0\|_2$, $\|v_0\|_2$, d_i, α_{ij}, a_i, b_i, c_i $(i,j=1,2)$. □

References

1. H. Amann: Dynamic theory of quasilinear parabolic equations, III. Global Existence. *Math Z.* **202** (1989), 219-250.
2. ______: Dynamic theory of quasilinear parabolic equations, II. Reaction-diffusion systems. *Differential and Integral Equations* **3** (1990), no. 1, 13-75.

3. ______: *Non-homogeneous linear and quasilinear elliptic and parabolic boundary value problems*. Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992), 9-126, Teubner-Texte Math., 133, Teubner, Stuttgart, 1993.
4. M. Delgado, M. Montenegro & A. Suárez: A Lotka-Volterra symbiotic model with cross-diffusion. *J. Differential Equations* **246** (2009), 2131-2149.
5. A. Friedman: *Partial differential equations*. Holt, Rinehart and Winston, New York, 1969.
6. L. Nirenberg: On ellipic partial differential equations. *Ann. Scuo. Norm. Sup. Pisa* **13** (1959), no. 3, 115-162.
7. A. Okubo & L.A. Levin: *Diffusion and Ecological Problems: Modem Perspective, 2nd Edition*. Interdisciplinary Applied Mathematics, Vol. 14, Springer-Verlag, New York, 2001.
8. C.V. Pao: Strongly coupled elliptic systems and applications to Lotka-Volterra models with cross-diffusion. *Nonlinear Anal.* **60** (2005) 1197-1217.

DEPARTMENT OF MATHEMATICS, SUNGSHIN WOMEN'S UNIVERSITY
Email address: shims@sungshin.ac.kr