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ELEMENTARY DISKS IN TRUNCATED TRIANGULATIONS

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Abstract. A normal surface is determined by how the surface under consideration meets each tetrahedron in a given triangulation. We call such a nice embedded disk, which is a component of the intersection of the surface with a tetrahedron, an *elementary disk*. We classify all elementary disk types in a truncated ideal triangulation.

1. Introduction

Normal surface theory is a powerful tool to study 3-manifolds in combinatorial method. The ordinary normal surface theory was introduced by Haken considering compact 3-manifolds with a triangulation [1]. But by being developed the normal surface Q-thoery by J.L. Tollefson [5], it could be extended to non-compact 3-manifolds with an ideal triangulation [3] including hyperbolic 3-manifolds. To obtain desired results, we often need to work on a "good triangulation" like 0- or 1-efficient triangulations [2] or reconstruct triangulation by using phachner moves, by crushing or by subdividing, etc. In [4], the normal surface theory is extended to truncated triangulations which is obtained by cutting off ideal vertices from ideal triangulation. In this paper, we classify all elementary disk types for each truncated tetrahedron to support the theory on truncated triangulation in [4]. Even a huge number of elementary disk types it is necessary to work on truncated tetrahedra rather than subdivision into ordinary tetrahedra since the truncated triangulation preserves the structure of original ideal triangulation so that we can pull back the results on the truncated triangulation to the original ideal triangulation.

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2. Elementary Disk Types in Truncated Tetrahedra

Let M be a hyperbolic 3-manifold with an ideal triangulation $\Im$ and $\hat{M}$ a compact 3-manifold obtained by removing regular neighborhoods of ideal vertices so that the resulting triangulation is a truncated triangulation $\hat{M}$ with truncated tetrahedra. We will describe all simple closed normal curves, up to normal isotopies, on the boundary of a truncated tetrahedron, since each elementary disk is determined by its boundary curve which is a simple closed normal curve on the boundary of the truncated tetrahdron.

Let τ be a truncated tetrahedron which has four triangular faces and four hexagonal faces on the boundary. Denote the triangles on $\partial\tau$ by T_i and the vertices of T_i by $\{a_{ii_1}, a_{ii_2}, a_{ii_3}\}$, for $i, i_j = 1, 2, 3, 4$ $(j = 1, 2, 3)$ and $i \neq i_j$, where a_{ii_j} and $a_{i_j i}$ are on the same edge of τ (see Figure 1).

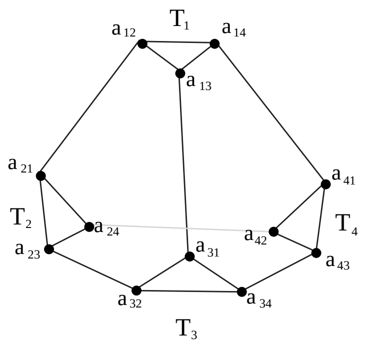

Figure 1. Truncated tetrahedron τ

Each simple closed normal curve devides $\partial\tau$ into two regions, say R_1 and R_2 so that the vertices of τ split into two. Denote the number of vertices of τ belonging to the region R_i by n_i, $i = 1, 2$. We have 6 ordered pairs for (n_1, n_2) up to symmetry; $(1, 11)$, $(2, 10)$, $(3, 9)$, $(4, 8)$, $(5, 7)$, $(6, 6)$. We will describe the number n_1 as the sum of the number of vertices in each triangle T_i, say t_i, $i = 1, 2, 3, 4$, and classfy all simple closed normal curve types according to the following 4-tuple (t_1, t_2, t_3, t_4), where t_i is the number of vertices in $T_i \cap R_1$, $i = 1, 2, 3, 4$, and $t_1 + t_2 + t_3 + t_4 = n_1$.

Case 1. $(n_1, n_2) = (1, 11)$

In this case, we will have a triangular disk type in every vertex of the truncated tetrahedron τ so that there are 12 types of triangles in total.

Case 2. $(n_1, n_2) = (2, 10)$

Here, we have 3 types of quadrilaterals in each class of $(t_1, t_2, t_3, t_4) = (2, 0, 0, 0)$,

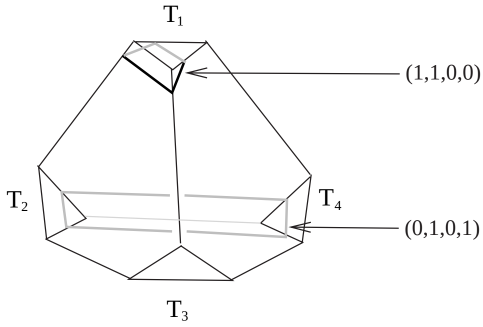

Figure 2. Quadrilaterals for $(2,0,0,0)$ and $(0,1,0,1)$

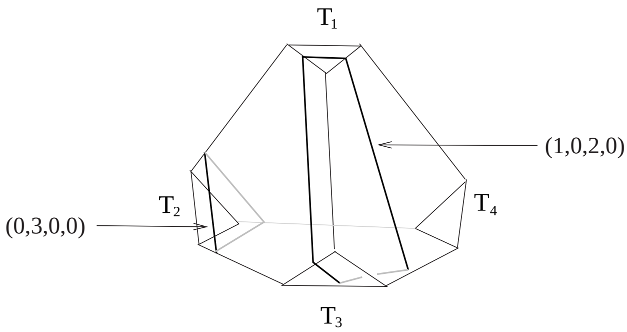

Figure 3. A triangle for $(3,0,0,0)$ and pentagons for $(2,1,0,0)$

$(0,2,0,0)$, $(0,0,2,0)$, $(0,0,0,2)$ and 1 type of quadrilateral in each class of $(t_1,t_2,\ t_3,t_4) = (1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, $(0,0,1,1)$. Hence, there are total 18 types of quadrilaterals in this case.

Case 3. $(n_1,n_2) = (3,9)$

In this case, we have triangles and pentagons. There is 1 type of triangle in each class of $(t_1,t_2,t_3,t_4) = (3,0,0,0)$, $(0,3,0,0)$, $(0,0,3,0)$, $(0,0,0,3)$ and 2 types of pentagons in each class of $(t_1,t_2,t_3,t_4) = (2,1,0,0)$, $(2,0,1,0)$, $(2,0,0,1)$, $(1,2,0,0)$, $(0,2,1,0)$, $(0,2,0,1)$, $(1,0,2,0)$, $(0,1,2,0)$, $(0,0,2,1)$, $(1,0,0,2)$, $(0,1,0,2)$, $(0,0,1,2)$. Note that there are no type of elementary disks in the class of $(t_1,t_2,t_3,t_4) = (1,1,1,0)$, $(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$. Hence, there are total 4 types of triangles and 24 types of pentagons in the case of $(n_1,n_2) = (3,9)$.

It is not so easy to figure out all types of disks directly for the following three cases. So we will describe each type of disks by classifying all the vertices in the region R_1.

Case 4. $(n_1,n_2) = (4,8)$

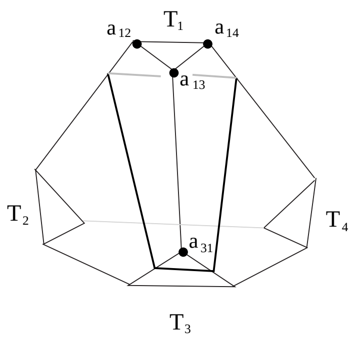

Figure 4. A quadrilatral for the class $(3,0,1,0)$ with vertices $\{a_{12}, a_{13}, a_{14}\} \cup \{a_{31}\}$ in the region R_1

In this case, we have 4 subcases in the way of distributing 4 vertices into 4 triangles T_1, T_2, T_3, T_4 as follows; $\{3,1,0,0\}$, $\{2,2,0,0\}$, $\{2,1,1,0\}$ and $\{1,1,1,1\}$ with no order.

(1) Subcase of $\{3,1,0,0\}$: This is the case that 3 vertices in T_i and 1 vertex in T_j for $i \neq j$ and $i, j = 1,2,3,4$. We have 12 different classes for this case and only one choice of 4 vertices in R_1, say $\{a_{ii_1}, a_{ii_2}, a_{ii_3}\} \cup \{a_{ji}\}$, for each $i, j = 1,2,3,4$ with $i \neq j$. Since the curve bounding the vertices is of lengh 4, there are 12 types of quadrilaterals in total for this case.

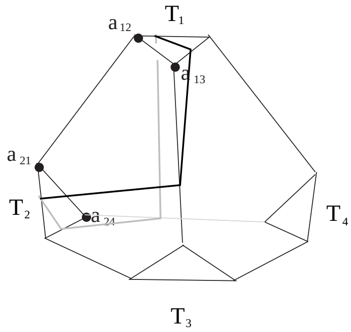

Figure 5. A hexagonal disk for the class $(2,2,0,0)$ with 4 vertices $\{a_{12}, a_{13}\} \cup \{a_{21}, a_{24}\}$

(2) Subcase of $\{2,2,0,0\}$: In this case, we have 4 different classes $(2,2,0,0)$, $(2,0,2,0)$, $(2,0,0,2)$, $(0,2,2,0)$, $(0,2,0,2)$, $(0,0,2,2)$. For each class, we have 4 types of hexagonal disks (see Figure 5). Note that if we have 2 vertices $\{a_{ii_1}, a_{ii_2}\}$ from a triangle T_i, then the remaining two vertices must be either $\{a_{i_1 i}, a_{i_1 *}\}$ from

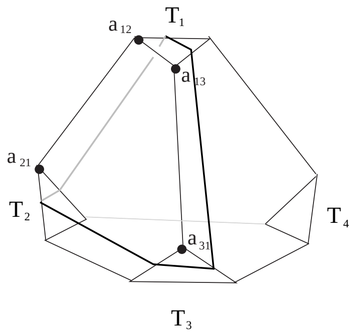

Figure 6. A hexagonal disk for the class $(2,1,1,0)$

T_{i_1} or $\{a_{i_2 i}, a_{i_2 *}\}$ from T_{i_2} for 2 choices of $*$. After counting overlaping cases, we have 24 types of hexagonal disks in total.

(3) Subcase of $\{2,1,1,0\}$: If we choose two vertices $\{a_{ii_1}, a_{ii_2}\}$ from a triangle T_i, then the remaning two vertices are automatically determined as $\{a_{i_1 i}\}$ from T_{i_1} and $\{a_{i_2 i}\}$ from T_{i_2} (see Figure 6). Since we have 3 choices for two vertices from a triangle, there are 12 hexagonal disk types in this case.

Case 5. $(n_1, n_2) = (5, 7)$

For this pair, we have 5 vertices in the region R_1. We distribute these 5 vertices into 4 triangles in 4 different types; $\{3,2,0,0\}$, $\{3,1,1,0\}$, $\{2,2,1,0\}$ and $\{2,1,1,1\}$ with no order.

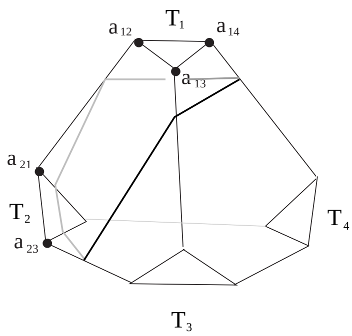

Figure 7. A pentagon for the class $(3,2,0,0)$ with vertices $\{a_{12}, a_{13}, a_{14}\} \cup \{a_{21}, a_{23}\}$

(1) Subcase of $\{3,2,0,0\}$: For a chosen 3 vertices $\{a_{ii_1}, a_{ii_2}, a_{ii_3}\}$ from a triangle T_i, we have 6 choices for the remaining 2 vertices; two of $\{a_{i_1 i}, a_{i_1 *}\}$ from T_{i_1}, two of $\{a_{i_2 i}, a_{i_2 *}\}$ from T_{i_2} and $\{a_{i_3 i}, a_{i_3 *}\}$ from T_{i_3}. We can easily see that the curve

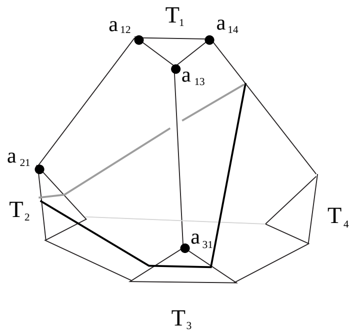

Figure 8. A pentagon for the class $(3,1,1,0)$ with vertices $\{a_{12}, a_{13}, a_{14}\} \cup \{a_{21}\} \cup \{a_{31}\}$

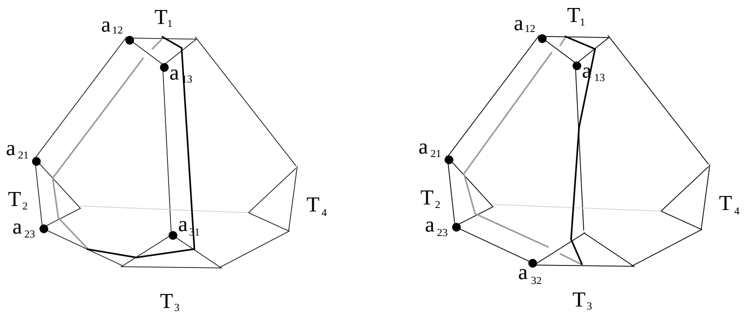

Figure 9. Heptagons for the class $\{2,2,1,0\}$ with vertices $\{a_{12}, a_{13}\} \cup \{a_{21}, a_{23}\} \cup \{*\}$

bounding these 5 vertices forms a pentagon (see Figure 7). Hence, we have 24 types of pentagons for this subcase.

(2) Subcase of $\{3,1,1,0\}$: In this case, for a fixed bunch of 3 vertices $\{a_{ii_1}, a_{ii_2}, a_{ii_3}\}$ from a triangle T_i, we can pick up the remaining two vertices from $\{a_{i_1i}, a_{i_2i}, a_{i_3i}\}$. This arranges 12 types of pentagons.

(3) Subcase of $\{2,2,1,0\}$: Choosing 5 vertives for this case can only happen by adding 1 vertex to a bunch of 4 vertices from the subcase $\{2,2,0,0\}$ of Case 4. We have different patterns of choosing the remaining 1 vertex in each case of $\{2,2,0,0\}$. If 4 vertices are $\{a_{ii_1}, a_{ii_2}\} \cup \{a_{i_1i}, a_{i_1i_2}\}$ so that 4 vertices are on the same face of tetrahedron, then we have 2 choices for the remaining 1 vertex, either $\{a_{i_2i}\}$ or $\{a_{i_2i_1}\}$ (see Figure 9). If 4 vertices are $\{a_{ii_1}, a_{ii_2}\} \cup \{a_{i_1i}, a_{i_1j}\}$ for $j \neq i, i_2$, then we have also 2 choices for the remaining 1 vertex, either $\{a_{ji_1}\}$ or $\{a_{i_2i}\}$ (see Figure 10). Therefore, we have total 48 types of heptagons.

(4) Subcase of $\{2,1,1,1\}$: In this case, no elementary disks appear.

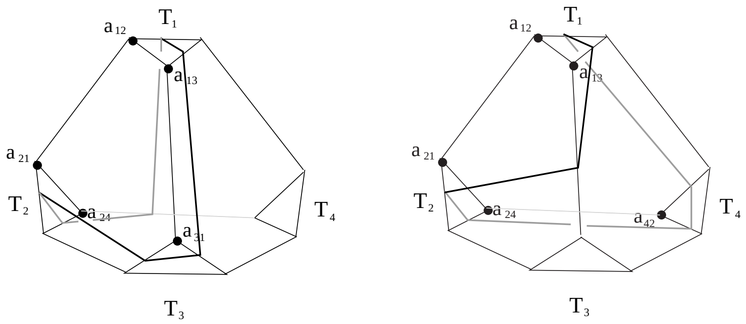

Figure 10. Heptagons for the class $\{2, 2, 1, 0\}$ with vertices $\{a_{12}, a_{13}\} \cup \{a_{21}, a_{24}\} \cup \{*\}$

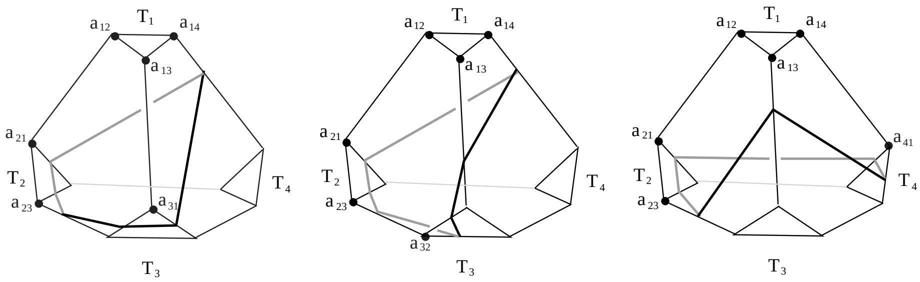

Figure 11. Hexagons for the class $\{3, 2, 1, 0\}$ with vertices $\{a_{12}, a_{13}, a_{14}\} \cup \{a_{21}, a_{23}\} \cup \{*\}$

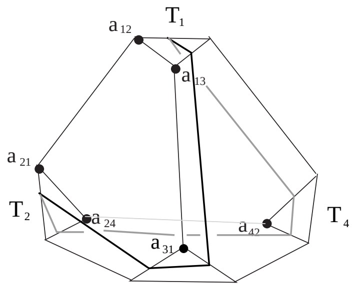

Figure 12. A octagon for the class $(2, 2, 1, 1)$ with vertices $\{a_{12}, a_{13}\} \cup \{a_{21}, a_{24}\} \cup \{a_{31}\} \cup \{a_{42}\}$

Case 6. $(n_1, n_2) = (6, 6)$

This is the last case to consider. Since the numbers of vertices in the region R_1 and R_2 are same as 6, there are symmetries between some classes. There are 5 cases distributing 6 vertices into 4 triangles; $\{3, 3, 0, 0\}$, $\{3, 2, 1, 0\}$, $\{3, 1, 1, 1\}$, $\{(2, 2, 2, 0)\}$

and $\{2,2,1,1\}$. But due to the symmetries between the classes of $\{3,1,1,1\}$ and the classes of $\{(2,2,2,0)\}$, we only consider the following 4 subcases; $\{3,3,0,0\}$, $\{3,2,1,0\}$, $\{3,1,1,1\}$ and $\{2,2,1,1\}$.

(1) Subcase of $\{3,3,0,0\}$: This just brings quadrilaterals in the ordinary tetrahedron. Hence we have 3 types of quadrilaterals for this case.

(2) Subcase of $\{3,2,1,0\}$: Each class of this case is obtained by adding 1 vertex to a class of $\{3,2,0,0\}$ in Case 5. If we have a class with 5 vertices $\{a_{ii_1}, a_{ii_2}, a_{ii_3}\} \cup \{a_{i_1i}, a_{i_1j}\}$, then there are 3 choices for the last vertex; either $\{a_{ji}\}$ or $\{a_{ji_1}\}$ or $\{a_{ki}\}$ for $k \neq i, i_1, j$ (see Figure 11). In tatal we have 72 cases. But there are symmetries between $(3,2,1,0)$ and $(0,1,2,3)$, $(3,2,0,1)$ and $(0,1,3,2)$, $(3,1,2,0)$ and $(0,2,1,3)$, etc. Therefore we have only 36 types of hexagons for this case.

(3) Subcase of $\{3,1,1,1\}$: We can easly see that there is only one choice of $\{*,1,1,1\}$ for a fixed case of $\{3,*,*,*\}$ which is determined by a hexagon. Hence there are 4 types of hexagons for this case.

(4) Subcase of $\{2,2,1,1\}$: To classify all the vertices in the region R_1 for this case, we will add one more vertex to a class of $\{2,2,1,0\}$. But this setup is not applicable to all the class of $\{2,2,1,0\}$. If the vertices of a class of $\{2,2,1,0\}$ are all on the same face of tetrahedron, then we cannot add the last vertex from the remaining empty triangle to build a simple closed normal curve. Hence we only consider the classes with vertices $\{a_{ii_1}, a_{ii_2}\} \cup \{a_{i_1i}, a_{i_1j}\} \cup \{*\}$ for $j \neq i, i_2$. There were two choices for $*$, either $\{a_{i_2i}\}$ from T_{i_1} or $\{a_{ji_1}\}$ from T_j in the case of $\{2,2,1,0\}$. Here, we will combine the two cases to get a class of $\{2,2,1,1\}$, just the class with vertices $\{a_{ii_1}, a_{ii_2}\} \cup \{a_{i_1i}, a_{i_1j}\} \cup \{a_{i_2i}\} \cup \{a_{ji_1}\}$ (see Figure 12). This gives an octagon splitting the boundary of tetrahedron into two regions each having 6 vertices. We have all 12 types of octagons in this case.

To conclude, there are 16 types of triangles, 33 types of quadrilaterals, 60 types of pentagons, 76 types of hexagons, 48 types of heptagons and 12 types of octagons in a truncated tetrahedron. In total, there are 245 types of elementary disks in a truncated tetrahedron and $245t$ types of elementaty disks in a truncated triangulation $\hat{\Im}$ with t truncated tetrahedra.

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