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SPHERES IN THE SHILOV BOUNDARIES OF BOUNDED SYMMETRIC DOMAINS

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Abstract. In this paper, we classify all nonconstant smooth CR maps from a sphere $S_{n,1} \subset \mathbb{C}^n$ with $n > 3$ to the Shilov boundary $S_{p,q} \subset \mathbb{C}^{p\times q}$ of a bounded symmetric domain of Cartan type I under the condition that $p - q < 3n - 4$. We show that they are either linear maps up to automorphisms of $S_{n,1}$ and $S_{p,q}$ or D'Angelo maps. This is the first classification of CR maps into the Shilov boundary of bounded symmetric domains other than sphere that includes nonlinear maps.

1. Introduction

The *rigidity of holomorphic maps* between open pieces of a sphere was first studied by Poincaré [13] in 2-dimensional case and later by Alexander [1] and Chern and Moser [2] for general dimensions. Then Webster [16] obtained rigidity for holomorphic maps between open pieces of spheres of different dimension, proving that any such map between spheres in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$ extends as a totally geodesic map between balls with respect to the Bergman metric. Later, Huang [6] generalized Webster's result for CR maps between open pieces of spheres in $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ under the assumption $n' - 1 < 2(n-1)$. Beyond this bound, the rigidity fails as illustrated by the Whitney map.

Unit ball is a bounded symmetric domain of Cartan type I with rank 1 and sphere is its Shilov boundary. However, comparing with rigidity of holmorphic maps between spheres mentioned above, holomorphic rigidity for maps between bounded symmetric domains D and D' of higher rank remains much less understood. If the rank r' of D' does not exceed the rank r of D and both ranks $r, r' \geq 2$, the

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rigidity of proper holomorphic maps $f\colon D \to D'$ was conjectured by Mok [12] and proved by Tsai [15], showing that f is necessarily totally geodesic (with respect to the Bergmann metric).

For the case $r < r'$, in [11], Zaitsev and author showed the *rigidity of CR maps* $f : S_{p,q} \to S_{p',q'}$ under the assumption that $q \geq 2$ and $(p' - q') < 2(p - q)$. Here, $S_{p,q}$ and $S_{p',q'}$ are the Shilov boundaries of a bounded symmetric domains of Cartan type I(See §1 for definition) and q and q' are the ranks of $S_{p,q}$ and $S_{p',q'}$, respectively. When $(p' - q') = 2(p - q)$, then the rigidity fails to hold, as authors introduced the generalized Whitney map as a counterexample in the same paper.

Recently, in [14], A. Seo introduced a nonlinearizable proper holomorphic maps between $S_{p,q}$ and $S_{2p-1,2q-1}$. Therefore, to classify all CR maps between $S_{p,q}$ and $S_{p',q'}$ when $p' - q' \geq 2(p - q)$, one should consider nonlinear maps. In [9], Huang, Ji and Xu classified all locally defined CR maps between $S_{n,1}$ and $S_{n',1}$ under the assumption that $3 < n \leq n' < 3n - 3$. It is proved that such map is either a linear map or a D'Angelo map.

In this paper, we generalize the result of Huang, Ji and Xu. We define D'Angelo map from a sphere into the Shilov boundary of bounded symmetric domains of type I as follows:

Definition 1.1. Let $\mathbb{C}^{p\times q}$ be the set of all complex $p \times q$ matrices. A map f_θ : $S_{n,1} \to S_{p,q}$ for a fixed $0 < \theta \leq \pi/2$, is called a *D'Angelo map* if f_θ is equivalent to the following map

$$z \in \mathbb{C}^n \mapsto \begin{pmatrix} W_\theta(z) & 0 \\ 0 & I_{q-1} \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{p\times q}.$$

up to automorphisms of $S_{n,1}$ and $S_{p,q}$, where $W_\theta(z)$ is a map from $S_{n,1}$ to $S_{3n-3,1}$ defined by

$$(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \to (z', \cos(\theta)w, \sin(\theta)z'w, \sin(\theta)w^2) \in \mathbb{C}^{2n}$$

and I_{q-1} is the identity matrix of size $(q - 1)$.

This map is not linear after composing with any automorphisms of $S_{n,1}$ and $S_{p,q}$. For $q = 1$ and $\theta = \pi/2$, this is the classical Whitney map between unit balls in $\mathbb{C}^n$ and $\mathbb{C}^{2n-1}$ respectively. In this paper, we classify all *locally defined CR maps* from a sphere $S_{n,1}$ with $n > 3$ into the Shilov boundary $S_{p,q}$ of a general Cartan type I bounded symmetric domain of higher rank. We showed

Theorem 1.2. *Let f be a nonconstant smooth CR map from an open piece of $S_{n,1}$ into $S_{p,q}$. Assume that $n > 3$ and $p-q < 3n-4$. Then after composing with suitable automorphisms of $S_{n,1}$ and $S_{p,q}$, f is either a linear embedding or D'Angelo map.*

Note that our basic assumption $p-q < 3n-4$ corresponds precisely to the *optimal bound* $n'-1 < 3(n-1)$ in the rank 1 case ($q=1$) of maps between spheres, where $n-1$ and $n'-1$ are the CR dimensions of the spheres.

Throughout this paper we adopt the Einstein summation convention unless mentioned otherwise.

2. Preliminaries

In this section, we review CR structure and Grassmannian frames adapted to $S_{p,q}$. For details, we refer [2] and [11] as references. In this section, we let Greek indices $\alpha, \beta, \gamma, \ldots$ and Latin indices $j, k, \ell, \ldots$ run over $\{1, \ldots, q\}$ and $\{1, \ldots, p-q\}$, respectively. For $q = 1$, i.e., sphere case, we omit Greek indices.

A Hermitian symmetric domain $D_{p,q}$ of Cartan type I has a standard realization in the space $\mathbb{C}^{p\times q}$ of $p \times q$ matrices, given by

$$D_{p,q} := \{z \in \mathbb{C}^{p\times q} : I_q - z^*z \text{ is positive definite}\},$$

where I_q is the $q \times q$ identity matrix and $z^* = \bar{z}^t$. The *Shilov boundary* of $D_{p,q}$ is given by

$$S_{p,q} = \{z \in \mathbb{C}^{p\times q} : I_q - z^*z = 0\}.$$

In particular, $S_{p,q}$ is a CR manifold of CR dimension $(p-q) \times q$. For $q = 1$, $S_{p,1}$ is the unit sphere in $\mathbb{C}^p$. We shall always assume $p > q$ so that $S_{p,q}$ has positive CR dimension.

Let $\mathsf{Aut}\,(S_{p,q})$ be the Lie group of all CR automorphisms of $S_{p,q}$. By [10, Theorem 8.5], every $f \in \mathsf{Aut}\,(S_{p,q})$ extends to a biholomorphic automorphism of the bounded symmetric domain $D_{p,q}$. Consider the standard linear inclusion

$$z \mapsto \begin{pmatrix} I_q \\ z \end{pmatrix}, \; z \in S_{p,q}.$$

Then we may regard $S_{p,q}$ as a real submanifold in the Grassmanian $Gr(q, p+q)$ of all q-planes in $\mathbb{C}^{p+q}$ and $\mathsf{Aut}\,(S_{p,q})(= \mathsf{Aut}\,(D_{p,q}))$ becomes a subgroup of the automorphism group of $Gr(q, p+q)$.

For column vectors $u = (u_1, \ldots, u_{p+q})^t$ and $v = (v_1, \ldots, v_{p+q})^t$ in $\mathbb{C}^{p+q}$, define a Hermitian inner product by

$$\langle u, v\rangle := -(u_1\bar{v}_1 + \cdots + u_q\bar{v}_q) + (u_{q+1}\bar{v}_{q+1} + \cdots + u_{p+q}\bar{v}_{p+q}).$$

A *Grassmannian frame adapted to* $S_{p,q}$, or simply $S_{p,q}$*-frame* is a frame $\{Z_1, \ldots, Z_{p+q}\}$ of $\mathbb{C}^{p+q}$ with $\det(Z_1, \ldots, Z_{p+q}) = 1$ such that scalar product $\langle\cdot,\cdot\rangle$ in basis $(Z_1, \ldots, Z_{p+q})$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & I_q \\ 0 & I_{p-q} & 0 \\ I_q & 0 & 0 \end{pmatrix}.$$

Now let $\mathcal{B}_{p,q}$ be the set of all $S_{p,q}$-frames. Then $\mathcal{B}_{p,q}$ is identified with $SU(p,q)$ by the left action. The Maurer-Cartan form $\pi = (\pi_\Lambda{}^\Gamma)$ on $\mathcal{B}_{p,q}$ is given by the equation

$$dZ_\Lambda = \pi_\Lambda{}^\Gamma Z_\Gamma, \tag{2.1}$$

where π satisfies the trace-free condition

$$\sum_\Lambda \pi_\Lambda{}^\Lambda = 0$$

and the structure equation

$$d\pi_\Lambda{}^\Gamma = \pi_\Lambda{}^\Omega \wedge \pi_\Omega{}^\Gamma,$$

where the capital Greek indices Λ, Γ, Ω etc. run from 1 to $p+q$.

From now, we will use the notation

$$Z := (Z_1, \ldots, Z_q), \quad X = (X_1, \ldots, X_{p-q}) := (Z_{q+1}, \ldots, Z_p), \quad Y = (Y_1, \ldots, Y_q) := (Z_{p+1} \ldots, Z_{p+q})$$

so that the Maurer-Cartan form with respect to the basis (Z, X, Y) can be written as

$$\pi = \begin{pmatrix} \pi_\alpha{}^\beta & \pi_\alpha{}^{q+j} & \pi_\alpha{}^{p+\beta} \\ \pi_{q+k}{}^\beta & \pi_{q+k}{}^{q+j} & \pi_{q+k}{}^{p+\beta} \\ \pi_{p+\alpha}{}^\beta & \pi_{p+\alpha}{}^{q+j} & \pi_{p+\alpha}{}^{p+\beta} \end{pmatrix} =: \begin{pmatrix} \psi_\alpha{}^\beta & \theta_\alpha{}^j & \varphi_\alpha{}^\beta \\ \sigma_k{}^\beta & \omega_k{}^j & \theta_k{}^\beta \\ \xi_\alpha{}^\beta & \sigma_\alpha{}^j & \widehat{\psi}_\alpha{}^\beta \end{pmatrix}$$

with the symmetry relations

$$\begin{pmatrix} \psi_\alpha{}^\beta & \theta_\alpha{}^j & \varphi_\alpha{}^\beta \\ \sigma_k{}^\beta & \omega_k{}^j & \theta_k{}^\beta \\ \xi_\alpha{}^\beta & \sigma_\alpha{}^j & \widehat{\psi}_\alpha{}^\beta \end{pmatrix} = -\begin{pmatrix} \widehat{\psi}_{\bar\beta}{}^{\bar\alpha} & \theta_{\bar j}{}^{\bar\alpha} & \varphi_{\bar\beta}{}^{\bar\alpha} \\ \sigma_{\bar\beta}{}^{\bar k} & \omega_{\bar j}{}^{\bar k} & \theta_{\bar\beta}{}^{\bar k} \\ \xi_{\bar\beta}{}^{\bar\alpha} & \sigma_{\bar j}{}^{\bar\alpha} & \psi_{\bar\beta}{}^{\bar\alpha} \end{pmatrix}. \tag{2.2}$$

By abuse of notation, we also denote by Z the q-dimensional subspace of $\mathbb{C}^{p+q}$ spanned by $Z_1, \ldots, Z_q$. Then the defining equations of $S_{p,q}$ can be written as

$$S_{p,q} = \{Z \in Gr(q, p+q) : \langle\cdot,\cdot\rangle|_Z = 0\}$$

and hence their differentiation yields

$$\langle dZ_\alpha, Z_\beta\rangle + \langle Z_\alpha, dZ_\beta\rangle = 0. \tag{2.3}$$

By substituting $dZ_\Lambda = \pi_\Lambda{}^\Gamma Z_\Gamma$ into $(1,0)$ component of (2.3) we obtain, in particular,

$$\varphi_\alpha{}^\gamma \langle Y_\gamma, Z_\beta\rangle = \varphi_\alpha{}^\beta = 0$$

when restricted to the $(1,0)$ tangent space. Comparing the dimensions, we conclude that $\varphi = (\varphi_\alpha{}^\beta)$ span the space of contact forms on $S_{p,q}$, i.e.,

$$T^c S_{p,q} := ker(\varphi_\alpha{}^\beta) \subset TS_{p,q}$$

is the complex tangent space of $S_{p,q}$. The structure equation is given by

$$d\varphi_\alpha{}^\beta = \theta_\alpha{}^j \wedge \theta_j{}^\beta \mod \varphi. \tag{2.4}$$

Moreover, since

$$dZ_\alpha = \psi_\alpha{}^\beta Z_\beta + \theta_\alpha{}^j X_j + \varphi_\alpha{}^\beta Y_\beta,$$

we conclude that $\theta_\alpha{}^j$ form a basis in the space of $(1,0)$ forms.

There are several types of frame changes.

Definition 2.1. We call a change of frame

i) *change of position* if

$$\widetilde{Z}_\alpha = W_\alpha{}^\beta Z_\beta, \quad \widetilde{Y}_\alpha = V_\alpha{}^\beta Y_\beta, \quad \widetilde{X}_j = X_j,$$

where $W = (W_\alpha{}^\beta)$ and $V = (V_\alpha{}^\beta)$ are $q \times q$ matrices satisfying $V^* W = I_q$;

ii) *change of real vectors* if

$$\widetilde{Z}_\alpha = Z_\alpha, \quad \widetilde{X}_j = X_j, \quad \widetilde{Y}_\alpha = Y_\alpha + H_\alpha{}^\beta Z_\beta,$$

where $H = (H_\alpha{}^\beta)$ is a hermitian matrix;

iii) *dilation* if

$$\widetilde{Z}_\alpha = \lambda_\alpha^{-1} Z_\alpha, \quad \widetilde{Y}_\alpha = \lambda_\alpha Y_\alpha, \quad \widetilde{X}_j = X_j,$$

where $\lambda_\alpha > 0$;

iv) *rotation* if

$$\widetilde{Z}_\alpha = Z_\alpha, \quad \widetilde{Y}_\alpha = Y_\alpha, \quad \widetilde{X}_j = U_j{}^k X_k,$$

where $(U_j{}^k)$ is a unitary matrix.

Finally, we shall use the change of frame given by

$$\widetilde{Z}_\alpha = Z_\alpha, \quad \widetilde{X}_j = X_j + C_j{}^\beta Z_\beta, \quad \widetilde{Y}_\alpha = Y_\alpha + A_\alpha{}^\beta Z_\beta + B_\alpha{}^j X_j,$$

such that

$$C_j{}^\alpha + B_j{}^\alpha = 0$$

and

$$(A_\alpha{}^\beta + \overline{A_\beta{}^\alpha}) + B_\alpha{}^j B_j{}^\beta = 0,$$

where

$$B_j{}^\alpha := \overline{B_\alpha{}^j}.$$

The new frame $(\widetilde{Z}, \widetilde{Y}, \widetilde{X})$ is an $S_{p,q}$-frame and the related 1-forms $\widetilde{\varphi}_\alpha{}^\beta$ remain the same, while $\widetilde{\theta}_\alpha{}^j$ change to

$$\widetilde{\theta}_\alpha{}^j = \theta_\alpha{}^j - \varphi_\alpha{}^\beta B_\beta{}^j.$$

3. $S_{p,q}$-frames Adapted to CR Mappings

Let $f : S_{n,1} \to S_{p,q}$ be a (germ of a) smooth CR mapping. We shall identify $S_{n,1}$ and its image $f(S_{n,1}) \subset S_{p,q}$. We consider the connection forms φ, θ^j, ψ, $\omega_j{}^k$, σ_j, ξ with $j, k = 1, \dots, n-1$ on $S_{n,1}$ and denote by capital letters $\Phi_\alpha{}^\beta$, $\Theta_\alpha{}^J$, $\Psi_\alpha{}^\beta$, $\Omega_J{}^K$, $\Sigma_K{}^\beta$, $\Xi_\alpha{}^\beta$ with $\alpha, \beta = 1, \dots, q$ and $J, K = 1, \dots, p-q$, their corresponding counterparts on $S_{p,q}$. We also define one forms $\varphi_\alpha{}^\beta$, $\theta_\alpha{}^J$ adapted to f as follows:

Definition 3.1. We say that f is of *contact rank* r if f sends any nonzero vector in $TS_{n,1}/T^cS_{n,1}$ to a rank r vector in $TS_{p,q}/T^cS_{p,q}$.

For a map f of contact rank r, we define $\varphi_\alpha{}^\beta$, $\theta_\alpha{}^J$ for $\alpha = 1, \dots, q$ and $J = 1, \dots, p-q$ *adapted to* f by

$$\begin{aligned}&\varphi_1{}^1 = \cdots = \varphi_r{}^r = \varphi,\\ &\theta_1{}^j = \cdots = \theta_r{}^{(r-1)(n-1)+j} = \theta^j, \quad j = 1, \dots, n-1\end{aligned}$$

and 0 otherwise.

In this section we show the following lemma.

Lemma 3.2. *For any nonconstant local CR map $f : S_{n,1} \to S_{p,q}$ with $p-q < 3(n-1)$, there exist $r \in \{1, 2\}$ and a choice of $S_{p,q}$-frames such that f is of contact rank r and the forms $\varphi_\alpha{}^\beta$, $\theta_\alpha{}^J$ adapted to f satisfy*

$$\begin{aligned}&\Phi_\alpha{}^\beta - \varphi_\alpha{}^\beta = 0,\\ &\Theta_\alpha{}^J - \theta_\alpha{}^J = 0.\end{aligned} \tag{3.1}$$

Proof is a slight modification of the proof of Lemma 4.2 and argument in §.5 of [11]. We refer [11] for details.

Proof. Since φ and $\Phi = (\Phi_\alpha{}^\beta)$ are contact forms on $S_{n,1}$ and $S_{p,q}$, respectively, the pull back of Φ via f is a span of φ. Choose a diagonal contact form of $S_{p,q}$ and say $\Phi_1{}^1$. Then we can write

$$\Phi_1{}^1 = \lambda\varphi \tag{3.2}$$

for some smooth function λ. At generic points, we may assume that either $\lambda \equiv 0$ or λ never vanishes. By differentiating (3.2) and using (2.4) we obtain

$$\Theta_1{}^J \wedge \Theta_J{}^1 = \lambda(\theta^j \wedge \theta_j) \mod \varphi. \tag{3.3}$$

Arguing similar to [11] we conclude $\lambda \geq 0$ and, after dilation of $\Phi_1{}^1$, we may assume that $\lambda = 1$ if $\lambda \not\equiv 0$.

Suppose that $\Phi_\alpha{}^\alpha$ vanishes identically for all α. Then we obtain

$$d\Phi_\alpha{}^\alpha = -\sum_J \Theta_\alpha{}^J \wedge \overline{\Theta_\alpha{}^J} = 0 \mod \varphi.$$

Since each $\Theta_\alpha{}^J$ is a $(1,0)$ form, it follows that

$$\Theta_\alpha{}^J = 0 \mod \varphi,$$

i.e., $f(S_{n,1})$ is a totally real submanifold. Since $S_{n,1}$ is Levi-nondegenerate, this implies that f is a constant map, which contradicts our assumption. Hence there exists at least one diagonal term of Φ whose pullback does not vanish identically.

Choose such a diagonal term of Φ, say $\Phi_1{}^1$. Then (3.3) yields

$$\sum_J \Theta_1{}^J \wedge \overline{\Theta_1{}^J} = \sum_j \theta^j \wedge \overline{\theta^j} \mod \varphi.$$

Therefore after a suitable rotation of $S_{p,q}$, we may assume that

$$\Theta_1{}^j = \theta^j \mod \varphi, \quad j = 1, \ldots, n-1, \tag{3.4}$$

$$\Theta_1{}^J = 0 \mod \varphi, \quad J = n, \ldots, p-q. \tag{3.5}$$

Write

$$\Phi_\alpha{}^1 = \lambda_\alpha\varphi, \quad \alpha \geq 2, \tag{3.6}$$

for some smooth functions λ_α. Then by differentiating (3.6) and using (2.4) together with (3.4), (3.5), we obtain

$$\Theta_\alpha{}^j \wedge \theta_j = \lambda_\alpha \theta^j \wedge \theta_j \mod \varphi, \quad \alpha \geq 2. \tag{3.7}$$

Choose a suitable change of position that leaves $\Theta_1{}^J$ invariant and replaces $\Theta_\alpha{}^J$ with $\Theta_\alpha{}^J - \lambda_\alpha\Theta_1{}^J$ for $\alpha \geq 2$. This change of position leaves $\Phi_1{}^1$ invariant and transforms

${\Phi_\alpha}^1$ into ${\Phi_\alpha}^1 - \lambda_\alpha {\Phi_1}^1$ for $\alpha \geq 2$. After performing such change of position, (3.6) becomes

$$ {\Phi_\alpha}^1 = 0, \quad \alpha \geq 2, $$

and (3.7) becomes

$$ {\Theta_\alpha}^j \wedge {\theta_j}^1 = 0 \mod \varphi, \quad \alpha \geq 2. $$

Since ${\Theta_\alpha}^j$ are $(1,0)$ but θ_j are $(0,1)$ and linearly independent, it follows that

$$ {\Theta_\alpha}^j = 0 \mod \varphi, \quad \alpha \geq 2. \tag{3.8} $$

Next for each $\alpha \geq 2$, let

$$ {\Phi_\alpha}^\alpha = \lambda_\alpha \varphi \tag{3.9} $$

for another smooth function λ_α. If $\lambda_\alpha \equiv 0$ for all $\alpha \geq 2$, then by differentiation, we obtain

$$ d{\Phi_\alpha}^\alpha = -\sum_J {\Theta_\alpha}^J \wedge \overline{{\Theta_\alpha}^J} = 0 \mod \varphi, \quad \alpha \geq 2, $$

which yields

$$ {\Theta_\alpha}^J = 0 \mod \varphi, \quad \alpha \geq 2. \tag{3.10} $$

In this case, by considering the differentiation of

$$ {\Phi_\alpha}^\beta = {\lambda_\alpha}^\beta \varphi $$

and substituting (3.10), we conclude that

$$ {\Phi_\alpha}^\beta = 0, \quad (\alpha, \beta) \neq (1,1), $$

which implies that $df(T)$ modulo $T^c S_{p,q}$ is a rank 1 vector for any $T \in TS_{n,1}$ transversal to $T^c S_{n,1}$. That is to say, f is of contact rank 1 and the forms adapted to f satisfy

$$ \begin{aligned} {\Phi_\alpha}^\beta - {\varphi_\alpha}^\beta &= 0, \\ {\Theta_\alpha}^J - {\theta_\alpha}^J &= 0 \mod \varphi. \end{aligned} $$

Suppose there exists α such that $\lambda_\alpha \not\equiv 0$. We may assume $\alpha = 2$. After a dilation of ${\Phi_2}^2$, we may assume that at generic points, $\lambda_2 = 1$. By differentiating (3.9) for $\alpha = 2$ and substituting (3.8) we obtain

$$ \sum_{J > n-1} {\Theta_2}^J \wedge {\Theta_J}^2 = \theta^j \wedge \theta_j \mod \varphi. $$

Hence after a suitable rotation

$$ {\widetilde{\Theta}_\alpha}{}^J = {\Theta_\alpha}^K U_K^J, $$

where $(U_K{}^J)$ is unitary matrix leaving $\Theta_\alpha{}^j$, $j = 1, \ldots, n-1$, invariant, we may assume that

$$\Theta_2{}^{n-1+j} = \theta^j \mod \varphi, \quad j = 1, \ldots, n-1$$

and

$$\Theta_2{}^J = 0 \mod \varphi$$

otherwise. Write

$$\Phi_\alpha{}^2 = \lambda_\alpha \varphi, \quad \alpha > 2, \tag{3.11}$$

for some smooth function λ_α. Then as before, we can choose a suitable change of position that leaves $\Theta_1{}^J$ and $\Theta_2{}^J$ invariant and replaces $\Theta_\alpha{}^J$ with $\Theta_\alpha{}^J - \lambda_\alpha \Theta_2{}^J$ for $\alpha > 2$, which also leaves $\Phi_1{}^1$, $\Phi_2{}^1$ and $\Phi_2{}^2$ invariant and transforms $\Phi_\alpha{}^2$ into $\Phi_\alpha{}^2 - \lambda_\alpha \Phi_2{}^2$ for $\alpha > 2$. By (3.8), after performing such change of position, the following property

$$\Theta_\alpha{}^j = 0 \mod \varphi, \quad \alpha \geq 2$$

still holds and (3.11) becomes

$$\Phi_\alpha{}^2 = 0, \quad \alpha > 2.$$

By differentiating this we obtain

$$\Theta_\alpha{}^{n-1+j} \wedge \theta_j = 0 \mod \varphi, \quad \alpha > 2,$$

which yields

$$\Theta_\alpha{}^{n-1+j} = 0 \mod \varphi, \quad \alpha > 2. \tag{3.12}$$

Write

$$\Phi_\alpha{}^\alpha = \lambda_\alpha \varphi, \quad \alpha > 2$$

for some smooth functions λ_α. Suppose that $\lambda_\alpha \equiv 0$ for all α. Then as before, we can obtain

$$\Theta_\alpha{}^J = 0 \mod \varphi, \quad \alpha > 2, \ \forall J,$$
$$\Phi_\alpha{}^\beta = 0, \quad \alpha > 2 \quad \text{or} \quad \beta > 2,$$

i.e., f is of contact rank 2 and the forms adapted to f satisfy

$$\Phi_\alpha{}^\beta - \varphi_\alpha{}^\beta = 0,$$
$$\Theta_\alpha{}^J - \theta_\alpha{}^J = 0 \mod \varphi.$$

Suppose there exists α such that $\lambda_\alpha \neq 0$. We may assume $\alpha = 3$. After a dilation of ${\Phi_3}^3$, we may assume that at generic points, $\lambda_3 = 1$, i.e.,

$$ {\Phi_3}^3 = \varphi. $$

By differentiating this, we obtain

$$ {\Theta_3}^J \wedge {\Theta_J}^3 = \theta^j \wedge \theta_j \qquad \mathrm{mod}\ \varphi. $$

then by (3.8) and (3.12), we have at most $p - q - 2(n-1)$ linearly independent $(1,0)$ forms on the left-hand side, while on the right-hand side we have $n-1$ linearly independent $(1,0)$ forms. Since we assumed that $p - q < 3(n-1)$, this is a contradiction.

Next we will show that there exists a choice of frames such that

$$ {\Theta_\alpha}^J = {\theta_\alpha}^J. $$

Write

$$ {\Theta_\alpha}^J - {\theta_\alpha}^J = {\eta_\alpha}^J \varphi \tag{3.13} $$

for some ${\eta_\alpha}^J$. Consider the equations obtained by differentiating (3.13):

$$ ({\Psi_\alpha}^\beta - {\psi_\alpha}^\beta) \wedge {\theta_\beta}^J + {\theta_\alpha}^K \wedge ({\Omega_K}^J - {\omega_K}^J) = {\eta_\alpha}^J (\theta^k \wedge \theta_k) \quad \mathrm{mod}\ \varphi, \tag{3.14} $$

where

$$ {\psi_\alpha}^\alpha = \psi, \quad \alpha = 1, \dots, r, \quad {\psi_\alpha}^\beta = 0 \quad \text{otherwise} $$

and

$$ {\omega_K}^J = 0 \quad J > n-1 \text{ or } K > n-1. $$

Let $\alpha > r$. Then left-hand side of (3.14) contains at most one $(1,0)$ form, while the right-hand side contains $(n-1)$ linearly independent $(1,0)$ forms with $n - 1 > 1$ unless ${\eta_\alpha}^J = 0$. Therefore we conclude that

$$ {\eta_\alpha}^J = 0, \quad \alpha > r $$

or equivalently

$$ {\Theta_\alpha}^J = 0, \quad \alpha > r. $$

Finally, define a matrix $({B_\alpha}^J)$ by

$$ {B_\alpha}^J := {\eta_\alpha}^J, $$

where ${\eta_\alpha}^J$ satisfies

$$ {\Theta_\alpha}^J - {\theta_\alpha}^J = {\eta_\alpha}^J \varphi. $$

Consider the change of frame of $S_{p,q}$ discussed after Definition 2.1, given by

$$\widetilde{Z}_\alpha = Z_\alpha, \quad \widetilde{X}_J = X_J + C_J{}^\beta Z_\beta, \quad \widetilde{Y}_\alpha = Y_\alpha + A_\alpha{}^\beta Z_\beta + B_\alpha{}^J X_J$$

such that

$$C_J{}^\alpha := -B_J{}^\alpha$$

and A_α^β satisfies

$$\left(A_\alpha{}^\beta + \overline{A_\beta{}^\alpha}\right) + \sum_J B_\alpha{}^J \overline{B_\beta{}^J} = 0.$$

Since the sum here is hermitian, one can always choose A_α^β with this property. Then $\Phi_\alpha{}^\beta$ remain the same while Θ_α^J change to

$$\Theta_\alpha{}^J - \Phi_\alpha{}^\beta B_\beta{}^J.$$

Therefore the new $\Theta_\alpha{}^J$ satisfies

$$\Theta_\alpha{}^J = \theta_\alpha{}^J.$$

□

4. Second Fundamental Forms and Gauss Equations for CR Embeddings

In this section, we determine second fundamental forms given by $\Omega_J{}^K$. Then we determine $\Psi_\alpha{}^\beta$ and $\Sigma_\alpha{}^J$. By using these forms, we construct a linear subspace of $Gr(q, p+q)$ that contains the image of a given embedding(Lemma 4.1, Lemma 4.2). Their proofs are slight modification of the proof of Proposition 7.1 in [11].

Let f be a CR map of contact rank r with $r \in \{1, 2\}$. Differentiate (3.1) using the structure equations to obtain

$$(\Psi_\alpha{}^\beta - \psi_\alpha{}^\beta) \wedge \theta_\beta{}^J + \theta_\alpha{}^K \wedge (\Omega_K{}^J - \omega_K{}^J) + \varphi_\alpha{}^\beta \wedge (\Sigma_\beta{}^J - \sigma_\beta{}^J) = 0, \tag{4.1}$$

where

$$\sigma_\alpha{}^{(\alpha-1)(n-1)+j} = \sigma^j, \quad \alpha = 1, \dots, r, \; j = 1, \dots, n-1$$

and 0 otherwise.

4.1. Contact rank 1 map Choose $\alpha > 1$ and $J = j$. Then (4.1) takes the form

$$\Psi_\alpha{}^1 \wedge \theta^j = 0, \quad \alpha > 1.$$

By Cartan Lemma we obtain

$$\Psi_\alpha{}^1 = 0 \mod \theta^j$$

for fixed j. Since Ψ is independent of $j = 1, \ldots, n-1$ and we assumed $n-1 > 1$, we obtain

$$\Psi_\alpha{}^1 = 0, \quad \alpha > 1. \tag{4.2}$$

We will show the following lemma.

Lemma 4.1. *There exists $(p-q+2)$-dimensional subspace V_1 and $(q-1)$-dimensional subspace V_2 in $\mathbb{C}^{p+q}$ orthogonal to each other such that $Gr(1, V_1) \oplus V_2$ contains the image $f(S_{n,1})$.*

Proof. Choose an open set $M \subset S_{n,1}$ where f is defined. Let Z, X, Y be constant vector fields of $\mathbb{C}^{p+q}$ forming a $S_{p,q}$-frame at a fixed reference point of $f(M)$ and let

$$\widetilde{Z}_\alpha = \lambda_\alpha^\beta Z_\beta + \eta_\alpha{}^K X_K + \zeta_\alpha{}^\beta Y_\beta, \tag{4.3}$$

$$\widetilde{X}_J = \lambda_J^\beta Z_\beta + \eta_J{}^K X_K + \zeta_J{}^\beta Y_\beta, \tag{4.4}$$

$$\widetilde{Y}_\alpha = \tilde{\lambda}_\alpha^\beta Z_\beta + \tilde{\eta}_\alpha{}^K X_K + \tilde{\zeta}_\alpha{}^\beta Y_\beta \tag{4.5}$$

be an adapted $S_{p,q}$-frame along $f(M)$. Write

$$A = \begin{pmatrix} \lambda_\alpha^\beta & \eta_\alpha{}^K & \zeta_\alpha{}^\beta \\ \lambda_J^\beta & \eta_J{}^K & \zeta_J{}^\beta \\ \tilde{\lambda}_\alpha^\beta & \tilde{\eta}_\alpha{}^K & \tilde{\zeta}_\alpha{}^\beta \end{pmatrix},$$

so that (4.3) - (4.5) take the form

$$\begin{pmatrix} \widetilde{Z} \\ \widetilde{X} \\ \widetilde{Y} \end{pmatrix} = A \begin{pmatrix} Z \\ X \\ Y \end{pmatrix}. \tag{4.6}$$

Since Z, X, Y form an adapted frame at a reference point of M, we may assume that

$$A = I_{p+q} \tag{4.7}$$

at the reference point. Since Z, X, Y are constant vector fields, i.e., $dZ = dX = dY = 0$, differentiating (4.6) and using (2.1) we obtain

$$dA = \begin{pmatrix} \Psi_\alpha^\beta & \Theta_\alpha{}^J & \Phi_\alpha{}^\beta \\ \Sigma_K^\beta & \Omega_K^J & \Theta_K^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha{}^J & \widehat{\Psi}_\alpha^\beta \end{pmatrix} A. \tag{4.8}$$

Next, it follows from Lemma 3.2 and (4.2) that

$$d\widetilde{Z}_\alpha = \sum_{\beta > 1} \Psi_\alpha{}^\beta \widetilde{Z}_\beta, \quad \alpha > 1,$$

in particular, the span of $\widetilde{Z}_\alpha$, $\alpha > 1$, is independent of the point in M. Hence together with (4.3) and (4.7), we conclude

$$\eta_\alpha{}^K = \zeta_\alpha{}^\beta = 0, \quad \alpha > 1. \tag{4.9}$$

Furthermore, (4.8) for $\alpha = 1$ together with Lemma 3.2 and (4.2) (and with the symmetry relations analogous to (2.2)) we obtain

$$\begin{pmatrix} d\zeta_1{}^\beta \\ d\zeta_J{}^\beta \\ d\tilde{\zeta}_1{}^\beta \end{pmatrix} = \begin{pmatrix} \Psi_1{}^\gamma & \theta_1{}^L & \varphi \\ \Sigma_J^\gamma & \Omega_J^L & \theta_J^1 \\ \Xi_1^\gamma & \Sigma_1^L & \widehat{\Psi}_1{}^1 \end{pmatrix} \begin{pmatrix} \zeta_\gamma{}^\beta \\ \zeta_L{}^\beta \\ \tilde{\zeta}_1{}^\beta \end{pmatrix}. \tag{4.10}$$

Now with (4.9) taken into account, (4.10) becomes

$$\begin{pmatrix} d\zeta_1{}^\beta \\ d\zeta_J{}^\beta \\ d\tilde{\zeta}_1{}^\beta \end{pmatrix} = \begin{pmatrix} \Psi_1{}^1 & \theta_1{}^L & \varphi_1{}^\beta \\ \Sigma_J^1 & \Omega_J^L & \theta_J^1 \\ \Xi_1^1 & \Sigma_1^L & \widehat{\Psi}_1{}^1 \end{pmatrix} \begin{pmatrix} \zeta_1{}^\beta \\ \zeta_L{}^\beta \\ \tilde{\zeta}_1{}^\beta \end{pmatrix}.$$

Thus each of the vector valued functions $\zeta^\beta := (\zeta_1{}^\beta, \zeta_J{}^\beta, \tilde{\zeta}_1{}^\beta)$ for a fixed β satisfies a complete system of linear first order differential equations. Then by the initial condition (4.7) and the uniqueness of solutions, we conclude, in particular, that

$$\zeta^\beta = 0, \quad \beta > 1$$

Hence (4.3) implies

$$\widetilde{Z}_1 = \lambda_1{}^\beta Z_\beta + \eta_1{}^K X_K + \zeta_1{}^1 Y_1.$$

Now setting

$$\widehat{Z}_1 := \widetilde{Z}_1 - \sum_{\beta>1} \lambda_1^\beta Z_\beta, \tag{4.11}$$

we still have

$$\operatorname{span}\{\widehat{Z}_1, \widetilde{Z}_2, \dots, \widetilde{Z}_q\} = \operatorname{span}\{\widetilde{Z}_\alpha\},$$

whereas (4.11) becomes

$$\widehat{Z}_1 = \lambda_1{}^1 Z_1 + \eta_1{}^K X_K + \zeta_1{}^1 Y_1,$$

implying

$$\operatorname{span}\{\widehat{Z}_1\} \subset \operatorname{span}\{Z_1, X_1, \dots, X_{p-q}, Y_1\}.$$

Then we conclude that

$$\begin{aligned} f(M) &= \operatorname{span}\{\widetilde{Z}_\alpha\} = \operatorname{span}\{\widehat{Z}_1\} \oplus \operatorname{span}\{\widetilde{Z}_2, \dots, \widetilde{Z}_q\} \\ &= \operatorname{span}\{\widehat{Z}_1\} \oplus \operatorname{span}\{Z_2, \dots, Z_q\} \subset Gr(1, V_1) \oplus V_2, \end{aligned}$$

where

$$V_1 = \text{span}\,\{Z_1, X_1, \dots, X_{p-q}, Y_1\}, \quad V_2 = \text{span}\,\{Z_2, \dots, Z_q\}.$$

□

4.2. Contact rank 2 map Choose $\alpha > 2$ and $J = j$ or $J = n-1+j$. Then (4.1) takes the form

$$\Psi_\alpha{}^1 \wedge \theta^j = \Psi_\alpha{}^2 \wedge \theta^j = 0, \quad \alpha > 2.$$

Since Ψ is independent of $j = 1, \dots, n-1$ and we assumed $n-1 > 1$, by Cartan Lemma we obtain

$$\Psi_\alpha{}^1 = \Psi_\alpha{}^2 = 0, \quad \alpha > 2. \tag{4.12}$$

Use (4.1) for either $\alpha = 1$ and $J = n-1+j$ or $\alpha = 2$ and $J = j$ or $\alpha = 1, 2$ and $J > 2(n-1)$ to obtain

$$\Psi_1{}^2 \wedge \theta^j + \theta^k \wedge \Omega_k{}^{n-1+j} + \varphi \wedge \Sigma_1{}^{n-1+j} = 0, \quad j \le n-1, \tag{4.13}$$

$$\Psi_2{}^1 \wedge \theta^j + \theta^k \wedge \Omega_{n-1+k}{}^j + \varphi \wedge \Sigma_2{}^j = 0, \quad j \le n-1, \tag{4.14}$$

$$\theta^k \wedge \Omega_k{}^J + \varphi \wedge \Sigma_1{}^J = \theta^k \wedge \Omega_{n-1+k}{}^J + \varphi \wedge \Sigma_2{}^J = 0, \quad J > 2(n-1). \tag{4.15}$$

By Cartan's Lemma, we obtain

$$\Omega_k{}^{n-1+j} = \Sigma_1{}^{n-1+j} = \Omega_{n-1+k}{}^j = \Sigma_2{}^j = 0 \mod \{\theta, \varphi\}, \quad j, k \le n-1, \tag{4.16}$$

$$\Omega_k{}^J = \Sigma_1{}^J = \Omega_{n-1+k}{}^J = \Sigma_2{}^J = 0 \mod \{\theta, \varphi\}, \quad k \le n-1,\ J > n-1, \tag{4.17}$$

where θ is an ideal generated by $\theta^1, \dots, \theta^{n-1}$. Since

$$\Omega_k^{n-1+j} = -\overline{\Omega_{n-1+j}{}^k},$$

by using (4.16), we conclude that

$$\Omega_k{}^{n-1+j} = 0 \mod \varphi. \tag{4.18}$$

Moreover, since Ψ is independent of j, substituting (4.18) into (4.13) and (4.14), we obtain

$$\Psi_1{}^2 = \Psi_2{}^1 = 0 \mod \varphi. \tag{4.19}$$

Next we will determine second fundamental forms of f as in [16]. We will show that it has a trivial solution only. For details, we refer [16].

Use (4.1) for $\alpha = 1$ and $J = j \le (n-1)$ to obtain

$$\left[\delta_k{}^j \left(\Psi_1{}^1 - \psi\right) - \left(\Omega_k{}^j - \omega_k{}^j\right)\right] \wedge \theta^k + \varphi \wedge \left(\Sigma_1{}^j - \sigma^j\right) = 0.$$

Then by Cartan Lemma, we obtain

$$\delta_k{}^j \left(\Psi_1{}^1 - \psi\right) - \left(\Omega_k{}^j - \omega_k{}^j\right) = 0 \qquad \mod \theta, \varphi.$$

By symmetry relation for Ω, we obtain

$$\begin{aligned} \Omega_k{}^j &= \omega_k{}^j \qquad \mod \varphi, \quad j \neq k \\ \Psi_1{}^1 - \Omega_j{}^j &= \psi - \omega_j{}^j \qquad \mod \varphi. \end{aligned}$$

Furthermore, differentiation of

$$\Phi_1{}^1 - \varphi = 0$$

by using the structure equations yields

$$\left(\Psi_1{}^1 - \psi - \hat{\Psi}_1{}^1 + \hat{\psi}\right) \wedge \varphi = 0,$$

or equivalently

$$\left(\Psi_1{}^1 - \psi + \overline{\Psi_1{}^1} - \overline{\psi}\right) \wedge \varphi = 0.$$

Therefore we obtain

$$\Psi_1{}^1 - \psi = \hat{\Psi}_1{}^1 - \hat{\psi} + g\varphi$$

for some pure imaginary function g.

Similar computation for (4.1) with $\alpha = 2$ and $J = n - 1 + j$ together with the relation

$$\Phi_2{}^2 - \varphi = 0$$

yields

$$\begin{aligned} \Omega_{n-1+k}{}^{n-1+j} &= \omega_k{}^j \qquad \mod \varphi, \quad j \neq k, \\ \Psi_2{}^2 - \Omega_{n-1+j}{}^{n-1+j} &= \psi - \omega_j{}^j \qquad \mod \varphi. \end{aligned}$$

and

$$\Psi_2{}^2 - \psi = \hat{\Psi}_2{}^2 - \hat{\psi} + h\varphi$$

for a pure imaginary function h.

Take a real vector change of $S_{p,q}$ defined by

$$\widetilde{Y}_1 = Y_1 + \frac{g}{2} Z_1 + \mu Z_2,$$

$$\widetilde{Y}_2 = Y_2 + \frac{h}{2} Z_2 + \bar{\mu} Z_1$$

for a smooth function μ satisfying

$$\Psi_1{}^2 = \mu\varphi$$

in (4.19) and fixing the rest. Then after the frame change, we obtain

$$\Psi_1{}^1 - \psi = \hat{\Psi}_1{}^1 - \hat{\psi}, \tag{4.20}$$

$$\Psi_2{}^2 - \psi = \hat{\Psi}_2{}^2 - \hat{\psi}, \tag{4.21}$$

$$\Psi_1{}^2 = 0. \tag{4.22}$$

By differentiating (4.20),(4.21),(4.22) and substituting (4.12) and (4.19), we obtain

$$\theta^k \wedge (\Sigma_k{}^1 - \sigma_k) = (\Sigma_1{}^k - \sigma^k) \wedge \theta_k \qquad \text{mod } \varphi,$$

$$\theta^k \wedge (\Sigma_{n-1+k}^{\ 2} - \sigma_k) = (\Sigma_2{}^{n-1+k} - \sigma^k) \wedge \theta_k \qquad \text{mod } \varphi,$$

$$\theta^j \wedge \Sigma_j{}^2 = 0 \qquad \text{mod } \varphi.$$

Then by Cartan Lemma, we obtain

$$\Sigma_1{}^j - \sigma^{\ j} = \Sigma_2{}^{n-1+j} - \sigma^{\ j} = 0 \qquad \text{mod } \theta, \bar{\theta}, \varphi,$$

$$\Sigma_j{}^2 = 0 \qquad \text{mod } \theta, \varphi.$$

By (4.16) and symmetry relation for Σ, we obtain

$$\Sigma_2{}^j = 0 \qquad \text{mod } \varphi. \tag{4.23}$$

Now let

$$\Sigma_1{}^j - \sigma^{\ j} = g_k{}^j \theta^k \qquad \text{mod } \bar{\theta}, \varphi,$$

$$\Sigma_2{}^{n-1+j} - \sigma^{\ j} = h_k{}^j \theta^k \qquad \text{mod } \bar{\theta}, \varphi,$$

Then (4.1) implies

$$\delta_k{}^j \left(\Psi_1{}^1 - \psi\right) - \left(\Omega_k{}^j - \omega_k{}^j\right) = g_k{}^j \varphi, \tag{4.24}$$

$$\delta_k{}^j \left(\Psi_2{}^2 - \psi\right) - \left(\Omega_{n-1+k}^{\ n-1+j} - \omega_k{}^j\right) = h_k{}^j \varphi. \tag{4.25}$$

Write

$$\Omega_j{}^J = h_j{}^J{}_\ell\, \theta^\ell \quad \text{mod } \varphi, \quad K > 2(n-1).$$

Differentiate (4.24) and substitute (4.18) to obtain

$$\theta^\ell \wedge (\Sigma_\ell{}^1 - \sigma_\ell) + \sum_{K>2(n-1)} \Omega_k{}^K \wedge \Omega_K^{\ j} = g_k{}^j \left(\theta^\ell \wedge \theta_\ell\right) \quad \text{mod } \varphi,$$

which implies

$$\sum_{K>2(n-1)} h_k{}^K{}_\ell h^j{}_K{}^m = g_\ell{}^m \delta_k{}^j + g_\ell{}^j \delta_k{}^m + g_k{}^m \delta_\ell{}^j + g_k{}^j \delta_\ell{}^m. \tag{4.26}$$

If $p-q < 3(n-1)$, then (4.26) has trivial solution only.(See [3].) Therefore we obtain

$$h_k{}^J{}_\ell = 0$$

or equivalently

$$\Omega_k{}^J = 0 \qquad \mod \varphi, \quad J > 2(n-1). \tag{4.27}$$

Similar computation for $\Omega_{n-1+k}^{\ J}$, $J > 2(n-1)$ using (4.25) yields

$$\Omega_{n-1+k}^{\ J} = 0 \qquad \mod \varphi, \quad J > 2(n-1).$$

By (4.18) and (4.27), we can write

$$\Omega_k{}^J = \eta_k{}^J \varphi, \quad J > n-1.$$

By differentiating this, we obtain

$$\begin{cases} \Sigma_k{}^2 \wedge \theta^j + \theta_k \wedge \Sigma_1{}^J = \eta_k{}^J \theta^\ell \wedge \theta_\ell \quad \mod \varphi, \quad J = n-1+j, \\ \theta_k \wedge \Sigma_1{}^J = \eta_k{}^J \theta^\ell \wedge \theta_\ell \quad \mod \varphi, \quad J > 2(n-1). \end{cases} \tag{4.28}$$

By (4.17) and (4.23) we can show that the left-hand side of (4.28) contains at most one $(0,1)$ form, while the right-hand side contains $(n-1)$ linearly independent $(0,1)$ forms unless $\eta_k{}^J = 0$. Hence we conclude that

$$\eta_k{}^J = 0$$

or equivalently

$$\Omega_k{}^J = 0, \quad J > n-1 \tag{4.29}$$

and therefore by substituting (4.17) and (4.23) into (4.28), we obtain

$$\Sigma_1{}^J = 0 \qquad \mod \varphi, \quad J > n-1. \tag{4.30}$$

Similar computation for $\Omega_{n-1+k}^{\ J}$, $J > 2(n-1)$ implies

$$\begin{aligned} \Omega_{n-1+k}^{\ J} &= 0 \quad J > 2(n-1), \\ \Sigma_2{}^J &= 0 \qquad \mod \varphi, \quad J > 2(n-1). \end{aligned}$$

Furthermore, by substituting (4.29) to(4.15) with $J = j$, we obtain

$$\Sigma_2{}^j = 0 \qquad \mod \varphi, \quad j \le n-1.$$

Finally we will determine Ψ and Σ. By (4.19), we can write

$$\Psi_2{}^1 = \mu\varphi.$$

By differentiating this and substituting (4.12) and (4.19), we obtain

$$\theta^k \wedge \Sigma_{n-1+k}^{\ 1} = \mu \theta^\ell \wedge \theta_\ell \qquad \mod \varphi.$$

By (4.30), this implies

$$\mu = 0$$

or equivalently

$$\Psi_2{}^1 = 0.$$

Let

$$\Sigma_1{}^J = \mu^J \varphi, \quad J > (n-1).$$

By differentiation, we obtain

$$\begin{aligned} \Xi_1{}^2 \wedge \theta^j &= \mu^{n-1+j}\ \theta^\ell \wedge \theta_\ell \qquad \mod \varphi, \quad j \leq n-1 \\ 0 &= \mu^{n-1+j}\ \theta^\ell \wedge \theta_\ell \qquad \mod \varphi, \quad J > 2(n-1), \end{aligned}$$

which yield

$$\mu^J = 0$$

or equivalently

$$\Sigma_1{}^J = 0.$$

Since $\Xi_1{}^2$ is independent of j, we obtain

$$\Xi_1{}^2 = 0.$$

Similar computation for $\Sigma_2{}^J$ yields

$$\Sigma_2{}^j = \Sigma_2{}^J = 0, \quad j < n,\ J > 2(n-1).$$

Summing up we obtain the following:
For any contact rank 2 local CR embedding f from $S_{n,1}$ into $S_{p,q}$, there is a choice of frames such that

$$\Psi_1{}^2 = \Psi_2{}^1 = \Psi_\alpha{}^1 = \Psi_\alpha{}^2 = 0, \quad \alpha > 2, \tag{4.31}$$

$$\Omega_k{}^J = \Sigma_1{}^J = 0, \quad k < n,\ J > n-1, \tag{4.32}$$

$$\Omega_{n-1+k}^{\ J} = \Sigma_2{}^j = \Sigma_2{}^J = 0, \quad j,k < n,\ J > 2(n-1), \tag{4.33}$$

$$\Xi_1{}^2 = 0. \tag{4.34}$$

We will show the following lemma.

Lemma 4.2. *There exist $(n+1)$-dimensional subspaces V_1, V_2 and $(q-2)$-dimensional subspace V_3 in $\mathbb{C}^{p+q}$ orthogonal to each other such that $Gr(1, V_1) \oplus Gr(1, V_2) \oplus V_3$ contains the image $f(S_{n,1})$.*

Proof. We use the same method in Lemma 4.1. Let $M \subset S_{n,1}$, Z, X, Y and

$$\begin{aligned}\widetilde{Z}_\alpha &= \lambda_\alpha{}^\beta Z_\beta + \eta_\alpha{}^K X_K + \zeta_\alpha{}^\beta Y_\beta, \\ \widetilde{X}_J &= \lambda_J{}^\beta Z_\beta + \eta_J{}^K X_K + \zeta_J{}^\beta Y_\beta, \\ \widetilde{Y}_\alpha &= \tilde{\lambda}_\alpha{}^\beta Z_\beta + \tilde{\eta}_\alpha{}^K X_K + \tilde{\zeta}_\alpha{}^\beta Y_\beta\end{aligned} \tag{4.35}$$

be as in Lemma 4.1.

It follows from Lemma 3.2 and (4.31) that

$$d\widetilde{Z}_\alpha = \sum_{\beta>2} \Psi_\alpha{}^\beta \widetilde{Z}_\beta, \quad \alpha > 2, \tag{4.36}$$

in particular, the span of $\widetilde{Z}_\alpha$, $\alpha > 2$, is independent of the point in M. Hence as in Lemma 4.1, we conclude

$$\lambda_\alpha{}^1 = \lambda_\alpha{}^2 = \eta_\alpha{}^K = \zeta_\alpha{}^\beta = 0, \quad \alpha > 2. \tag{4.37}$$

Furthermore, (4.8) implies

$$\begin{pmatrix} d\eta_\alpha{}^K \\ d\eta_J{}^K \\ d\tilde{\eta}_\alpha{}^K \end{pmatrix} = \begin{pmatrix} \Psi_\alpha{}^\beta & \Theta_\alpha{}^L & \Phi_\alpha{}^\beta \\ \Sigma_J{}^\beta & \Omega_J{}^L & \Theta_J{}^\beta \\ \Xi_\alpha{}^\beta & \Sigma_\alpha{}^L & \widehat{\Psi}_\alpha{}^\beta \end{pmatrix} \begin{pmatrix} \eta_\beta{}^K \\ \eta_L{}^K \\ \tilde{\eta}_\beta{}^K \end{pmatrix}.$$

In particular, restricting to $\alpha = 1$ and $J = j \le n$ with (4.31)-(4.34) and (4.37) taken into account, we obtain

$$\begin{pmatrix} d\eta_1{}^K \\ d\eta_j{}^K \\ d\tilde{\eta}_1{}^K \end{pmatrix} = \begin{pmatrix} \Psi_1{}^1 & \theta^\ell & \varphi \\ \Sigma_j{}^1 & \Omega_j{}^\ell & \theta_j \\ \Xi_1{}^1 & \Sigma_1{}^\ell & \widehat{\Psi}_1{}^1 \end{pmatrix} \begin{pmatrix} \eta_1{}^K \\ \eta_\ell{}^K \\ \tilde{\eta}_1{}^K \end{pmatrix}.$$

Repeating the above argument for λ and ζ instead of η, we obtain

$$\begin{pmatrix} d\lambda_1{}^2 \\ d\lambda_j{}^2 \\ d\tilde{\lambda}_1{}^2 \end{pmatrix} = \begin{pmatrix} \Psi_1{}^1 & \theta^\ell & \varphi \\ \Sigma_j{}^1 & \Omega_j{}^\ell & \theta_j \\ \Xi_1{}^1 & \Sigma_1{}^\ell & \widehat{\Psi}_1{}^1 \end{pmatrix} \begin{pmatrix} \lambda_1{}^2 \\ \lambda_\ell{}^2 \\ \tilde{\lambda}_1{}^2 \end{pmatrix}.$$

and

$$\begin{pmatrix} d\zeta_1{}^\beta \\ d\zeta_j{}^\beta \\ d\tilde{\zeta}_1{}^\beta \end{pmatrix} = \begin{pmatrix} \Psi_1{}^1 & \theta^\ell & \varphi \\ \Sigma_j{}^1 & \Omega_j{}^\ell & \theta_j \\ \Xi_1{}^1 & \Sigma_1{}^\ell & \widehat{\Psi}_1{}^1 \end{pmatrix} \begin{pmatrix} \zeta_1{}^\beta \\ \zeta_\ell{}^\beta \\ \tilde{\zeta}_1{}^\beta \end{pmatrix}.$$

Thus each of the vector valued functions $\lambda^2 = (\lambda_1{}^2, \lambda_j{}^2, \tilde{\lambda}_1{}^2)$, $\eta^K := (\eta_1{}^K, \eta_j{}^K, \tilde{\eta}_1{}^K)$ for a fixed K and $\zeta^\beta := (\zeta_1{}^\beta, \zeta_j{}^\beta, \tilde{\zeta}_1{}^\beta)$ for a fixed β satisfies a complete system of linear first order differential equations. Then as in Lemma 4.1 we conclude, in particular, that

$$\lambda_1{}^2 = 0$$

and

$$\eta^K = \zeta^\beta = 0, \quad K > n,\, \beta > 1.$$

Hence (4.35) implies

$$\widetilde{Z}_1 = \sum_{\beta\neq 2} {\lambda_1}^\beta Z_\beta + {\eta_1}^k X_k + {\zeta_1}^1 Y_1. \tag{4.38}$$

Similar computation for $\widetilde{Z}_2$ implies

$$\widetilde{Z}_2 = \sum_{\beta\neq 1} {\lambda_2}^\beta Z_\beta + {\eta_2}^{n-1+k} X_{n-1+k} + {\zeta_2}^2 Y_2. \tag{4.39}$$

Now setting

$$\widehat{Z}_\alpha := \widetilde{Z}_\alpha - \sum_{\beta>2} {\lambda_\alpha}^\beta Z_\beta, \quad \alpha = 1, 2,$$

we still have

$$\mathrm{span}\,\{\widehat{Z}_1, \widehat{Z}_2, \widetilde{Z}_3, \dots, \widetilde{Z}_q\} = \mathrm{span}\,\{\widetilde{Z}_\alpha\},$$

whereas (4.38), (4.39) become

$$\widehat{Z}_1 = {\lambda_1}^1 Z_1 + {\eta_1}^k X_k + {\zeta_1}^1 Y_1,$$

$$\widehat{Z}_2 = {\lambda_2}^2 Z_2 + {\eta_2}^{n-1+k} X_{n-1+k} + {\zeta_2}^2 Y_2,$$

implying

$$\mathrm{span}\,\{\widehat{Z}_1\} \subset \mathrm{span}\,\{Z_1, X_1, \dots, X_{n-1}, Y_1\},$$

$$\mathrm{span}\,\{\widehat{Z}_2\} \subset \mathrm{span}\,\{Z_2, X_n, \dots, X_{2n-2}, Y_2\}.$$

Then together with (4.36) we conclude that

$$f(M) = \mathrm{span}\,\{\widetilde{Z}_\alpha\} = \mathrm{span}\,\{\widehat{Z}_1\} \oplus \{\widehat{Z}_2\} \oplus \mathrm{span}\,\{\widetilde{Z}_3, \dots, \widetilde{Z}_q\}$$

$$= \mathrm{span}\,\{\widehat{Z}_1\} \oplus \{\widehat{Z}_2\} \oplus \mathrm{span}\,\{Z_3, \dots, Z_q\} \subset Gr(1, V_1) \oplus Gr(1, V_2) \oplus V_3$$

where

$$V_1 = \mathrm{span}\,\{Z_1, X_1, \dots, X_{n-1}, Y_1\}, \quad V_2 = \mathrm{span}\,\{Z_2, X_n, \dots, X_{2n-2}, Y_2\},$$
$$V_3 = \mathrm{span}\,\{Z_3, \dots, Z_q\}.$$

□

5. Proof of Theorem 1.2

Suppose f is of contact rank 1. Then by Lemma 4.1, there exist $(p-q+2)$-dimensional subspace V_1 and $(q-1)$-dimensional subspace V_2 such that the image of f is contained in $Gr(1, V_1) \oplus V_2$. The V_2-component of f is a constant map. Therefore it is enough to show that $Gr(1, V_1)$-component of f is either a linear map or Whitney map. But $Gr(1, V_1) = \mathbb{P}^{p-q+1}$. Therefore by the result of [9] under the condition $n > 3$ and $(p-q) < 3n-4$, we conclude that $Gr(1, V_1)$-component of f is either a flat embedding or D'Angelo map.

Suppose f is of contact rank 2, then by Lemma 4.2, there exist $(n+1)$-dimensional subspaces V_1, V_2 and $(q-2)$-dimensional subspace V_3 such that the image of f is contained in $Gr(1, V_1) \oplus Gr(1, V_2) \oplus V_3$. As before, it is enough to show that $Gr(1, V_1)$ and $Gr(1, V_2)$-components of f are linear. Since V_1 and V_2 are of dimension $(n+1)$, each component of f is a CR automorphism of $S_{n,1}$. Therefore, it is projective linear, which completes the proof.

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