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CHARACTERIZATION OF A REGULAR FUNCTION WITH VALUES IN DUAL QUATERNIONS

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Abstract. In this paper, we provide the notions of dual quaternions and their algebraic properties based on matrices. From quaternion analysis, we give the concept of a derivative of functions and and obtain a dual quaternion Cauchy-Riemann system that are equivalent. Also, we research properties of a regular function with values in dual quaternions and relations derivative with a regular function in dual quaternions.

1. Introduction

Let $\mathcal{T}$ be the set of quaternion numbers constructed over a real Euclidean quadratic four dimensional vector space. In 2004 and 2006, Kajiwara, Li and Shon [2, 3] obtained some results for the regeneration in complex, quaternion and Clifford analysis, and for the inhomogeneous Cauchy-Riemann system of quaternions and Clifford analysis in ellipsoid. Naser [12] and Nôno [13] obtained some properties of quaternionic hyperholomorphic functions. In 2011, Koriyama, Mae and Nôno [8, 9] researched for hyperholomorphic functions and holomorphic functions in quaternion analysis. Also, they obtained some results of regularities of octonion functions and holomorphic mappings. In 2012, Gotô and Nôno [1] researched for regular functions with values in a commutative subalgebra $\mathbb{C}(\mathbb{C})$ of matrix algebra $\mathrm{M}(4;\mathbb{R})$. Lim and Shon [10, 11] obtained some properties of hyperholomorphic functions and researched for the hyperholomophic functions and hyperconjugate harmonic functions of octonion variables, and for the dual quaternion functions and its applications. Recently, we [4, 5, 6, 7] obtained some results for the regularity of functions on the ternary quaternion and reduced quaternion field in Clifford analysis, and for the regularity of functions on dual split quaternions in Clifford analysis. Also, we

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investigated the corresponding Cauchy-Riemann systems in special quaternions and properties of each regular functions defined by the corresponding differential operators in special quaternions.

The aim of the paper is to define the representations of dual quaternions, written by a matrix form. Also, we research the conditions of the derivative of functions with values in dual quaternions and the definition of a regular function for Cauchy-Riemann system in dual quaternions.

2. Preliminaries

The field $\mathcal{T}$ of quaternions

$$z = x_0 + e_1x_1 + e_2x_2 + e_3x_3,\ x_j \in \mathbb{R}\ (j = 0, 1, 2, 3),$$

is a four dimensional non-commutative real field such that its four base elements $e_0 = 1, e_1, e_2$ and e_3 satisfying the following :

$$e_1^2 = e_2^2 = e_3^2 = -1,\ e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2.$$

The element $e_0 = 1$ is the identity of $\mathcal{T}$. Identifying the element e_1 with the imaginary unit $\sqrt{-1}$ in the complex field of complex numbers. The dual numbers extended the real numbers by adjoining one new non-zero element ε with the property $\varepsilon^2 = 0$. The collection of dual numbers forms a particular two-dimensional commutative unital associative algebra over the real numbers. Every dual number has the form $z = x + \varepsilon y$ with x and y uniquely determined real numbers. Dual numbers form the coefficients of dual quaternions. If we use matrices, dual numbers can be represented as

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ a + b\varepsilon = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

The sum and product of dual numbers are then calculated with ordinary matrix addition and matrix multiplication; both operations are commutative and associative within the algebra of dual numbers.

3. Dual Quaternions

The algebra

$$\mathbb{DC}(2) = \{Z = z + \varepsilon w \mid z = \sum_{i=0}^{3} e_jx_j, w = \sum_{i=0}^{3} e_jy_j \in \mathcal{T}\} \cong \mathcal{T} \times \mathcal{T},$$

where $x_j,\ y_j \in \mathbb{R}\ (j = 0, 1, 2, 3)$, is a non-commutative subalgebra of $M^2(2;\mathbb{C})$.

We define that the dual quaternionic multiplication of two dual quaternions

$$Z_1 = z_1 + \varepsilon w_1 = \begin{pmatrix} z_1 & w_1 \\ 0 & z_1 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{3} e_j x_j & \sum_{j=0}^{3} e_j y_j \\ 0 & \sum_{j=0}^{3} e_j x_j \end{pmatrix}$$

and

$$Z_2 = z_2 + \varepsilon w_2 = \begin{pmatrix} z_2 & w_2 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{3} e_j \xi_j & \sum_{j=0}^{3} e_j \eta_j \\ 0 & \sum_{j=0}^{3} e_j \xi_j \end{pmatrix}$$

is given by

$$Z_1 Z_2 = \begin{pmatrix} z_1 z_2 & z_1 w_2 + w_1 z_2 \\ 0 & z_1 z_2 \end{pmatrix}$$

$$= \begin{pmatrix} (\sum_{j=0}^{3} e_j x_j) \cdot (\sum_{j=0}^{3} e_j \xi_j) & \begin{matrix} (\sum_{j=0}^{3} e_j x_j) \cdot (\sum_{j=0}^{3} e_j \eta_j) \\ +(\sum_{j=0}^{3} e_j y_j) \cdot (\sum_{j=0}^{3} e_j \xi_j) \end{matrix} \\ 0 & (\sum_{j=0}^{3} e_j x_j) \cdot (\sum_{j=0}^{3} e_j \xi_j) \end{pmatrix}.$$

The dual quaternionic conjugate Z^* of Z is

$$Z^* = \begin{pmatrix} z^* & w^* \\ 0 & z^* \end{pmatrix} = \begin{pmatrix} x_0 - \sum_{j=1}^{3} e_j x_j & y_0 - \sum_{j=1}^{3} e_j y_j \\ 0 & x_0 - \sum_{j=1}^{3} e_j x_j \end{pmatrix}.$$

Then the modulus $|Z|$ and the inverse Z^{-1} of Z in $\mathbb{DC}(2)$ are defined by the following :

$$|Z|^2 = ZZ^* = \begin{pmatrix} zz^* & zw^* + wz^* \\ 0 & zz^* \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{3} x_j^2 & 2\sum_{j=0}^{3} x_j y_j \\ 0 & \sum_{j=0}^{3} x_j^2 \end{pmatrix}$$

and

$$Z^{-1} = \frac{Z^*}{|Z|^2} \ (Z \neq 0).$$

By using the multiplication of $Z \in \mathbb{DC}(2)$, the power of Z is for $n \in \mathbb{N}$,

$$Z^n = (z+\varepsilon w)^n = \begin{pmatrix} z & w \\ 0 & z \end{pmatrix}^n = \begin{pmatrix} z^n & \sum_{k=1}^n z^{n-k} w z^{k-1} \\ 0 & z^n \end{pmatrix}.$$

and the division of two $Z,\ W \in \mathbb{DC}(2)$ can be computed as

$$\frac{Z_1}{Z_2} = \frac{z_1+\varepsilon w_1}{z_2+\varepsilon w_2} = \frac{z_1+\varepsilon w_1}{z_2+\varepsilon w_2}\frac{z_2^*+\varepsilon w_2^*}{z_2^*+\varepsilon w_2^*} = \frac{z_1 z_2^* + \varepsilon(z_1 w_2^* + w_1 z_2^*)}{z_2 z_2^* + \varepsilon(z_2 w_2^* + w_2 z_2^*)}.$$

Since $z_2 z_2^*$ and $z_2 w_2^* + w_2 z_2^*$ are real variables, it can be written by

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{1}{M^2}\{z_1 z_2^* M + \varepsilon(-z_1 z_2^* N + z_1 w_2^* M + w_1 z_2^* M)\} \\ &= \frac{z_1}{z_2} + \varepsilon\Big(\frac{z_1 w_2^*}{z_2 z_2^*} + \frac{w_1}{z_2} - \frac{z_1 w_2^*}{z_2 z_2^*}\Big) = \frac{Z_1}{z_2} = \begin{pmatrix} \frac{z_1}{z_2} & \frac{w_1}{z_2} \\ 0 & \frac{z_1}{z_2} \end{pmatrix}, \end{aligned}$$

where $M := z_2 z_2^*$ and $N := z_2 w_2^* + w_2 z_2^*$.

We use the following differential operators :

$$\begin{aligned} D &:= D_z + \varepsilon D_w = \begin{pmatrix} D_z & D_w \\ 0 & D_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial z_2} & \frac{\partial}{\partial \overline{w_1}} + e_2\frac{\partial}{\partial w_2} \\ 0 & \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial z_2} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^3 e_j \frac{\partial}{\partial x_j} & \sum_{j=0}^3 e_j \frac{\partial}{\partial y_j} \\ 0 & \sum_{j=0}^3 e_j \frac{\partial}{\partial x_j} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} D^* &= D_z^* + \varepsilon D_w^* = \begin{pmatrix} D_z^* & D_w^* \\ 0 & D_z^* \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2} & \frac{\partial}{\partial w_1} - e_2\frac{\partial}{\partial w_2} \\ 0 & \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x_0} - \sum_{j=1}^3 e_j \frac{\partial}{\partial x_j} & \frac{\partial}{\partial y_0} - \sum_{j=1}^3 e_j \frac{\partial}{\partial y_j} \\ 0 & \frac{\partial}{\partial x_0} - \sum_{j=1}^3 e_j \frac{\partial}{\partial x_j} \end{pmatrix}, \end{aligned}$$

where $\frac{\partial}{\partial z_k},\ \frac{\partial}{\partial \overline{z_k}},\ \frac{\partial}{\partial w_k},\ \frac{\partial}{\partial \overline{w_k}}$ $(k=1,2)$ are usual complex differential operations.

The Laplacian operator is

$$|D|^2 = DD^* = \begin{pmatrix} D_z D_z^* & D_z D_w^* + D_w D_z^* \\ 0 & D_z D_z^* \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^3 \frac{\partial^2}{\partial x_j^2} & 2\sum_{j=0}^3 \frac{\partial^2}{\partial x_j \partial y_j} \\ 0 & \sum_{j=0}^3 \frac{\partial^2}{\partial x_j^2} \end{pmatrix}.$$

Let S be a bounded open subset in $\mathcal{T} \times \mathcal{T}$. A function $F(Z)$ is defined by the following form in S with values in M(2;$\mathbb{C}$) :

$$\begin{aligned} F(Z) &= F(z+\varepsilon w) = f(z,w) + \varepsilon g(z,w) \\ &= \begin{pmatrix} f(z,w) & g(z,w) \\ 0 & f(z,w) \end{pmatrix} = \begin{pmatrix} f_1 + f_2 e_2 & g_1 + g_2 e_2 \\ 0 & f_1 + f_2 e_2 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^{3} e_j u_j & \sum_{j=0}^{3} e_j v_j \\ 0 & \sum_{j=0}^{3} e_j u_j \end{pmatrix}, \end{aligned}$$

where $u_j = u_j(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3)$ and $v_j = v_j(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3)$ are real valued functions.

Remark 3.1. Using differential operators, we have the following equations:

$$DF = \begin{pmatrix} D_z f & D_z g + D_w f \\ 0 & D_z f \end{pmatrix}, \ D^*F = \begin{pmatrix} D_z^* f & D_z^* g + D_w^* f \\ 0 & D_z^* f \end{pmatrix},$$

$$FD = \begin{pmatrix} fD_z^* & fD_w + gD_z \\ 0 & fD_z \end{pmatrix}, \ FD^* = \begin{pmatrix} fD_z^* & fD_w^* + gD_z^* \\ 0 & fD_z^* \end{pmatrix},$$

where

$$D_z f = \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial \overline{f_1}}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial \overline{z_1}}\Big)e_2, \ D_z^* f = \Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial \overline{f_1}}{\partial \overline{z_2}}\Big)e_2,$$

$$fD_z = \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_1}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial z_1}\Big)e_2, \ fD_z^* = \Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\Big)e_2.$$

Definition 3.2. Let S be a bounded open subset in $\mathcal{T} \times \mathcal{T}$. A function $F = f + \varepsilon g$ is said to be M-*regular* in S if f and g of F are continuously differential quaternion valued functions in S such that $D^*F = 0$.

Remark 3.3. The equation $D^*F = 0$ is equivalent to

$$D_z^* f = 0, \ D_z^* g + D_w^* f = 0.$$

Also, it is equivalent to

$$
\begin{cases}
\dfrac{\partial f_1}{\partial z_1} = -\dfrac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \dfrac{\partial f_2}{\partial z_1} = \dfrac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\[2ex]
\dfrac{\partial f_1}{\partial w_1} + \dfrac{\partial g_1}{\partial z_1} = -\dfrac{\partial \overline{f_2}}{\partial \overline{w_2}} - \dfrac{\partial \overline{g_2}}{\partial \overline{z_2}}, \\[2ex]
\dfrac{\partial f_2}{\partial w_1} + \dfrac{\partial g_2}{\partial z_1} = \dfrac{\partial \overline{f_1}}{\partial \overline{w_2}} + \dfrac{\partial \overline{g_1}}{\partial \overline{z_2}}.
\end{cases} \tag{3.1}
$$

The above system is called a dual quaternion Cauchy-Riemann system in dual quaternions.

Let Ω be an open subset of $\mathbb{DC}(2)$, for $Z_0 = z_0 + \varepsilon w_0 \in \Omega$,

$$F : \Omega \ \rightarrow \ \mathbb{DC}(2)$$

is called a dual-quaternion function in $\mathbb{DC}(2)$.

Definition 3.4. A function F is said to be *continuous* at $Z_0 = z_0 + \varepsilon w_0$ if

$$\lim_{Z \to Z_0} F(Z) = F(Z_0),$$

where the limit has

$$\lim_{Z \to Z_0} F(Z) = \lim_{z \to z_0,\ w \to w_0} F(Z) = F(Z_0).$$

Definition 3.5. The dual quaternion function F is said to be *differentiable* in dual quaternions if the limit

$$\frac{dF}{dZ} := \lim_{z \to z_0,\ w \to w_0} \frac{F(Z) - F(Z_0)}{Z - Z_0}$$

exists and the limit is called the derivative of F in dual quaternions.

Remark 3.6. From the definition of derivative of f and properties of differential operations of quaternion valued functions, we have

$$
\begin{aligned}
\frac{\partial f}{\partial z} \ &:= \ \lim_{z \to z_0} \frac{f(z,w) - f(z_0,w_0)}{z - z_0} \\
&= \ \sum_{r=0}^{3} e_r \lim_{x_r \to x_r^0} \frac{u_r(x_0,x_1,x_2,x_3) - u_r(x_0^0,x_1^0,x_2^0,x_3^0)}{x_r - x_r^0} = \sum_{r=0}^{3} e_r \frac{\partial u_r}{\partial x_r},
\end{aligned} \tag{3.2}
$$

where $(z_0, w_0) = (x_0^0, x_1^0, x_2^0, x_3^0)$ is a constant in a domain of f (see [2, 11]). Since the equation (3.2) is equivalent to $D_z f$, we can express $\frac{\partial f}{\partial z} = D_z f$. Hence, by the representations of DF and properties of limit, calculating the division for $\frac{F(Z)-F(Z_0)}{Z-Z_0}$,

$$\begin{aligned}\frac{dF}{dZ} &= \frac{\partial f}{\partial z} + \varepsilon\frac{\partial g}{\partial z} + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0) \\ &= \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ 0 & \frac{\partial f}{\partial z} \end{pmatrix} + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0}\Big(\frac{w-w_0}{z-z_0}\Big)^2 \\ &= \begin{pmatrix} D_z f & D_z g \\ 0 & D_z f \end{pmatrix} + \begin{pmatrix} 0 & \frac{\partial f}{\partial w} \\ 0 & 0 \end{pmatrix} = DF.\end{aligned}$$

Therefore, we can represent $\frac{\partial F}{\partial Z} = DF$.

Theorem 3.7. *Let $F = f + \varepsilon g$ be a dual quaternion function in $\Omega \subset \mathbb{DC}(2)$. If F satisfies the equation $Df = 0$, then the derivative of F satisfies the following equation:*

$$\frac{dF}{dZ} := \lim_{Z\to Z_0} \frac{F(Z)-F(Z_0)}{Z-Z_0} = D_z F.$$

Proof. By the division of dual quaternions, we have

$$\begin{aligned}\frac{F(Z)-F(Z_0)}{Z-Z_0} &= \frac{f(z,w)-f(z_0,w_0)}{z-z_0} + \varepsilon\frac{g(z,w)-g(z_0,w_0)}{z-z_0} \\ &\quad +\varepsilon\frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0).\end{aligned}$$

Then, the limit

$$\begin{aligned}&\lim_{Z\to Z_0} \frac{F(Z)-F(Z_0)}{Z-Z_0} \\ =& \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0} + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{g(z,w)-g(z_0,w_0)}{z-z_0} \\ &+\varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0) \\ =& \frac{\partial f}{\partial z} + \varepsilon\frac{\partial g}{\partial z} - \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0) \\ =& D_z F + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0}\,\frac{w-w_0}{z-z_0} \\ =& D_z F + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{w-w_0}\Big(\frac{w-w_0}{z-z_0}\Big)^2\end{aligned}$$

exists if and only if $\frac{w-w_0}{z-z_0}$ has two cases to deal with
Case 1)

$$\lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0) = \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0}\frac{w-w_0}{z-z_0}.$$

If

$$\lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0} = 0,$$

then the limit exists and the derivative can be written by

$$\frac{df(Z_0)}{dZ} = \varepsilon\frac{\partial g(z_0,w_0)}{\partial z}.$$

Case 2)

$$\lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{(z-z_0)^2}(w-w_0) = \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{w-w_0}\Big(\frac{w-w_0}{z-z_0}\Big)^2.$$

If

$$\lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{w-w_0} = 0,$$

then the limit exists and the derivative can be written by

$$\frac{df(Z_0)}{dZ} = D_zF.$$

Therefore, the equation $\frac{dF}{dZ} = D_zF$ is obtained. □

Theorem 3.8. *Let $F = f+\varepsilon g$ be a dual quaternion function in $\Omega \subset \mathbb{DC}(2)$. If F is a M-regular function in dual quaternions, that is, F satisfies the equation $D^*F = 0$, then the derivative of F satisfies the following equation:*

$$\frac{dF}{dZ} = DF = \frac{\partial F}{\partial x_0}.$$

Proof. From the proof of Theorem 3.7, we have

$$\begin{aligned}\frac{dF(Z_0)}{dZ} &= D_zF + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{z-z_0}\frac{w-w_0}{z-z_0} \\ &= D_zF + \varepsilon \lim_{z\to z_0,\ w\to w_0} \frac{f(z,w)-f(z_0,w_0)}{w-w_0}\Big(\frac{w-w_0}{z-z_0}\Big)^2.\end{aligned}$$

Since F satisfies a dual quaternion Cauchy-Riemann system (3.1), we have

$$
\begin{aligned}
D_zF &= D_zf+\varepsilon D_zg=\Big(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial \overline{z_1}}\Big)+\Big(\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial \overline{z_1}}\Big)e_2\\
&+\varepsilon\Big(\frac{\partial g_1}{\partial z_1}+\frac{\partial g_1}{\partial \overline{z_1}}\Big)+\varepsilon\Big(\frac{\partial g_2}{\partial z_1}+\frac{\partial g_2}{\partial \overline{z_1}}\Big)e_2.
\end{aligned}
$$

Therefore, since $\frac{\partial}{\partial z_1}+\frac{\partial}{\partial \overline{z_1}}=\frac{\partial}{\partial x_0}$, we have

$$\frac{dF(Z_0)}{dZ}=\frac{\partial F(Z_0)}{\partial x_0}.$$

□

References

1. S. Gotô & K. Nôno: Regular Functions with Values in a Commutative Subalgebra $\mathbb{C}(\mathbb{C})$ of Matrix Algebra $M(4;\mathbb{R})$. *Bull. Fukuoka Univ. Ed.* **61** part III (2012), 9-15.
2. J. Kajiwara, X.D. Li & K.H. Shon: *Regeneration in complex, quaternion and Clifford analysis.* Proc. the 9th International Conf. on Finite or Infinite Dimensional Complex Analysis and Applications, Adv. Complex Anal. Appl., Kluwer Acad. Pub., Hanoi, **2** (2004), no. 9, 287-298.
3. ______: *Function spaces in complex and Clifford analysis, Inhomogeneous Cauchy Riemann system of quaternion and Clifford analysis in ellipsoid.* Proc. the 14th International Conf. on Finite or Infinite Dimensional Complex Analysis and Applications, National Univ. Pub., Hanoi, Vietnam, Hue Univ., **14** (2006), 127-155.
4. J.E. Kim, S.J. Lim & K.H. Shon: Taylor series of functions with values in dual quaternion. *J. Korean Soc. Math. Educ., Ser. B, Pure Appl. Math.* **20** (2013), no. 4, 251-258.
5. ______: Regular functions with values in ternary number system on the complex Clifford analysis. *Abstr. Appl. Anal.* **2013** Article ID. 136120 (2013), 7 pages.
6. ______: Regularity of functions on the reduced quaternion field in Clifford analysis. *Abstr. Appl. Anal.* **2014** Article ID. 654798 (2014), 8 pages.
7. ______: The Regularity of functions on Dual split quaternions in Clifford analysis. *Abstr. Appl. Anal.* **2014** Article ID. 369430 (2014), 8 pages.
8. H. Koriyama, H. Mae & K. Nôno: Hyperholomorphic fucntions and holomorphic functions in quaternionic analysis. *Bull. Fukuoka Univ. Ed.* **60** part III (2011), 1-9.
9. H. Koriyama & K. Nôno: On regularities of octonionic functions and holomorphic mappings. *Bull. Fukuoka Univ. Ed.* **60** part III (2011), 11-28.
10. S.J. Lim & K.H. Shon: Properties of hyperholomorphic functions in Clifford analysis. *East Asian Math. J.* **28** (2012), no. 5, 553-559.
11. ______: Dual quaternion functions and its applications. *J. Applied Math.* Article ID 583813 (2013), 6 pages.

12. M. Naser: Hyperholomorphic functions. *Silberian Math. J.* **12** (1971), 959-968.
13. K. Nôno: Hyperholomorphic functions of a quaternion variable. *Bull. Fukuoka Univ. Ed.* **32** (1983), 21-37.

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