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ISOGONAL AND ISOTOMIC CONJUGATES OF QUADRATIC RATIONAL Bézier CURVES

Chan Ran Yun [a] and Young Joon Ahn [b,*]

Abstract. In this paper we characterize the isogonal and isotomic conjugates of conic. Every conic can be expressed by a quadratic rational Bézier curve having control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ with weight $w > 0$. We show that the isotomic conjugate of parabola and hyperbola with respect to $\triangle\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ is ellipse, and that the isotomic conjugate of ellipse with the weight $w = \frac{1}{2}$ is identical. We also find all cases of the isogonal conjugate of conic with respect to $\triangle\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. Our characterizations are derived easily due to the expression of conic by the quadratic rational Bézier curve in standard form.

1. Introduction

Quadratic rational Bézier curve is one of the most widely used curves in the field of CAD/CAM and Computer Graphics. Any conic can be expressed by a quadratic rational Bézier curve, and vice versa if the control points of quadratic rational Bézier curve are not collinear. The quadratic rational Bézier curve having control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ is tangent to the lines $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_1\mathbf{b}_2$ at the points $\mathbf{b}_0$ and $\mathbf{b}_2$, respectively.

In triangle geometry there are two well-known conjugates. One is the isogonal conjugate and the other is the isotomic conjugate[2, 9]. For given any point in the triangle, the three lines obtained by reflecting the lines passing through the given point and vertices with respect to the angle bisectors at each vertices meet at one point, which is the isogonal conjugate of the given point with respect to the triangle, as shown in Figure 1. For given any point $\mathbf{x}$ in the triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, and for the point $\mathbf{x}'_i$, $i = 0, 1, 2$ obtained by reflecting the intersection point of lines $\mathbf{x}\mathbf{b}_i$ and $\mathbf{b}_j\mathbf{b}_k$ with respect to the midpoint on the side $\mathbf{b}_j\mathbf{b}_k$, for mutually distinct i, j, k, the

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*Corresponding author.

three lines $\mathbf{b}_0\mathbf{x}'_0$, $\mathbf{b}_1\mathbf{x}'_1$, $\mathbf{b}_2\mathbf{x}'_2$ meet at one point, which is the isotomic conjugate of the given point with respect to the triangle, as shown in Figure 2. The definition of the isogonal and isotomic conjugates shall be more detailedly described in the next section. The conjugates map any point inside the triangle into a point inside the triangle.

We consider the conjugate points of all points on the quadratic rational Bézier curve having control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ with respect to the triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. The motivation of our study is the question: what are the isogonal and isotomic conjugates of the quadratic rational Bézier curve with respect to $\triangle\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, respectively? The researches on the isogonal and isotomic conjugates of conic have been developed[4, 5, 6, 7, 8], but we cannot find the previous results about the isogonal and isotomic conjugates of conic having the control polygon with respect to $\triangle\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. In this paper, using the barycentric coordinates and weight of the quadratic rational Bézier curve we clearly characterize the isogonal and isotomic conjugates for all cases of quadratic rational Bézier curve for all cases.

Our manuscript is organized as follows. In Section 2, the definitions for barycentric coordinates, isogonal and isotomic conjugates, and quadratic rational Bézier curve are introduced. In Section 3, we characterize the isogonal and isotomic conjugates of all quadratic rational Bézier curves which are ellipse, parabola or hyperbola, and summarize our work in Section 4.

2. DEFINITIONS

In a given triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, every point $\mathbf{x}$ can be uniquely expressed by $\mathbf{x} = u\mathbf{b}_0 + v\mathbf{b}_1 + w\mathbf{b}_2$, $u + v + w = 1$. The triple coordinates (u, v, w) are called the *barycentric coordinates* of $\mathbf{x}$, as shown in Figure 3, and the ratio $(ku : kv : kw)$ is called the *homogeneous barycentric coordinates* of $\mathbf{x}$ for some positive k. Note that the ratio of areas of triangles $\triangle\mathbf{x}\mathbf{b}_1\mathbf{b}_2 : \triangle\mathbf{x}\mathbf{b}_2\mathbf{b}_0 : \triangle\mathbf{x}\mathbf{b}_0\mathbf{b}_1$ is equal to $u : v : w$. The centroid and incenter have the homogeneous barycentric coordinates $(1 : 1 : 1)$ and $(a : b : c)$, respectively [9], where a, b, c are lengths of side lines $\overline{\mathbf{b}_1\mathbf{b}_2}, \overline{\mathbf{b}_2\mathbf{b}_0}, \overline{\mathbf{b}_0\mathbf{b}_1}$, in order.

2.1. Isogonal conjugate and isotomic conjugate For any point $\mathbf{x}$ inside the triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, the lines $\mathbf{x}\mathbf{b}_i$, $i = 0, 1, 2$, are called by cevians of $\mathbf{x}$ [9]. Let the line ℓ_i, $i = 0, 1, 2$, be the reflection of the cevian $\mathbf{x}\mathbf{b}_i$ with respect to the angle bisector at vertex $\mathbf{b}_i$, respectively. The three lines ℓ_0, ℓ_1 and ℓ_2 are concurrent at

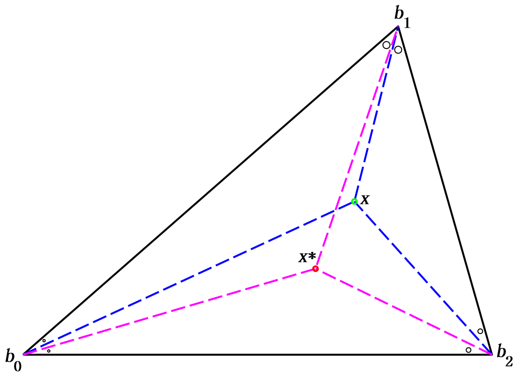

Figure 1. The isogonal conjugate of $\mathbf{x}$ is $\mathbf{x}^*$ with respect to the triangle $\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$.

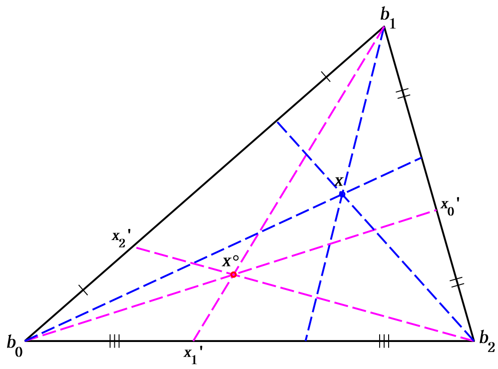

Figure 2. The isotomic conjugate of $\mathbf{x}$ is $\mathbf{x}^\circ$ with respect to the triangle $\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$.

one point, which is called the *isogonal conjugate of* $\mathbf{x}$ with respect to the triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ and it denoted by $\mathbf{x}^*$, as shown in Figure 1. For each $i = 0, 1, 2$, two lines $\mathbf{x}\mathbf{b}_i$ and $\mathbf{x}^*\mathbf{b}_i$ are symmetric with respect to the bisector of $\angle \mathbf{b}_i$. The incenter is the uniquely fixed point under the isogonal conjugate transformation, and the centroid and the orthocenter are the isogonal conjugates of the symmedian point and the circumcenter, respectively, and vice versa. If $\mathbf{x}$ has the homogeneous barycentric coordinates $(u : v : w)$, then its isogonal conjugate $\mathbf{x}^*$ has $(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w})$ [9].

Let the point $\mathbf{x}_i$, $i = 0, 1, 2$, be the intersection point of the cevian $\mathbf{x}\mathbf{b}_i$ and the opposite side $\mathbf{b}_j\mathbf{b}_k$, where $i, j, k \in \{0, 1, 2\}$ are mutually distinct. Let $\mathbf{x}'_i$, $i = 0, 1, 2$, be the reflection point of $\mathbf{x}_i$ with respect to the midpoint of the line segment $\mathbf{b}_j\mathbf{b}_k$. The three lines $\mathbf{x}'_0\mathbf{b}_0$, $\mathbf{x}'_1\mathbf{b}_1$, and $\mathbf{x}'_2\mathbf{b}_2$ are concurrent at one point, which is called

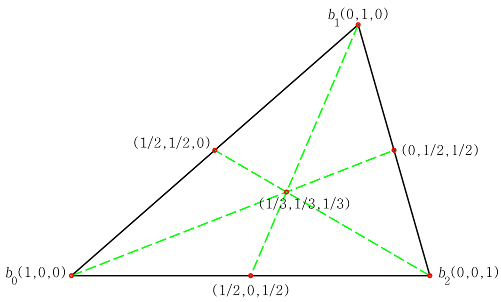

Figure 3. Barycentric coordinates with respect to $\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$.

the *isotomic conjugate of* $\mathbf{x}$ with respect to the triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ and we denote it by $\mathbf{x}^\circ$, as shown in Figure 2. The centroid is the uniquely fixed point under the isotomic conjugate transformation. If $\mathbf{x}$ has the homogeneous barycentric coordinates $(u : v : w)$, then its isotomic conjugate $\mathbf{x}^*$ has $(\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$.

2.2. Quadratic rational Bézier curve The quadratic rational Bézier curve $\mathbf{r}(t)$ having the control points $\mathbf{b}_i$, $i = 0, 1, 2$ and weights $w_i > 0$ is defined by

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{2} B_i(t) w_i \mathbf{b}_i}{\sum_{i=0}^{2} B_i(t) w_i} \qquad t \in [0, 1]$$

where $B_i(t) = \frac{2!}{i!(2-i)!} t^i (1-t)^{2-i}$ is the Bernstein polynomial of degree two [1, 3]. By changing the variable, it can be rewritten without change of the shape in the standard form

$$\mathbf{r}(t) = \frac{B_0(t)\mathbf{b}_0 + wB_1(t)\mathbf{b}_1 + B_2(t)\mathbf{b}_2}{B_0(t) + wB_1(t) + B_2(t)}$$

where $w = \frac{w_1}{\sqrt{w_0 w_2}}$. For each $t \in [0, 1]$, the point $\mathbf{r}(t)$ has the barycentric coordinates

$$\left(\frac{B_0(t)}{B_0(t) + wB_1(t) + B_2(t)}, \frac{wB_1(t)}{B_0(t) + wB_1(t) + B_2(t)}, \frac{B_2(t)}{B_0(t) + wB_1(t) + B_2(t)} \right).$$

For nonlinear polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ the quadratic rational Bézier curve $\mathbf{r}(t)$ is a conic, which is ellipse iff $w < 1$, parabola iff $w = 1$, and hyperbola iff $w > 1$, as shown in Figure 4.

3. Isotomic and Isogonal Conjugate of Quadratic Rational Bézier Curves.

In this section we characterize the isotomic and isogonal conjugates of conic with respect to $\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ using the expression of conic by the quadratic rational Bézier

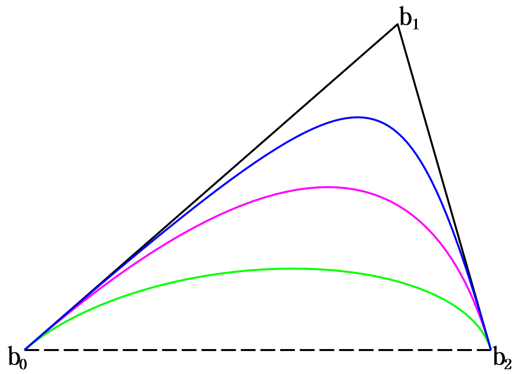

Figure 4. Quadratic rational Bézier curve having control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ with weight $w > 0$ is ellipse(green) if $w < 1$, parabola(magenta) if $w = 1$, and hyperbola(blue) if $w > 1$.

curve having the polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. The isotomic and isogonal conjugates relative to the control polygon have a particularly nice form.

Theorem 3.1. *With respect to $\triangle\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ the isotomic and isogonal conjugates of the quadratic rational Bézier curve having the control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ are also quadratic rational Bézier curves with control polygon $\mathbf{b}_2\mathbf{b}_1\mathbf{b}_0$.*

Proof. The quadratic rational Bézier curve $\mathbf{r}(t)$ having the control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ with weight w is

$$\mathbf{r}(t) = \tau_0\mathbf{b}_0 + \tau_1\mathbf{b}_1 + \tau_2\mathbf{b}_2$$

where

$$\tau_0 = \frac{B_0(t)}{B_0(t) + wB_1(t) + B_2(t)}$$
$$\tau_1 = \frac{wB_1(t)}{B_0(t) + wB_1(t) + B_2(t)}$$
$$\tau_2 = \frac{B_2(t)}{B_0(t) + wB_1(t) + B_2(t)}$$

which are the barycentric coordinates of $\mathbf{r}(t)$. Let $\mathbf{r}^\circ(t)$ and $\mathbf{r}^*(t)$ be the isotomic and isogonal conjugates of the quadratic rational Bézier curve $\mathbf{r}(t)$. We have

$$\mathbf{r}^\circ(t) = \tau_0^\circ\mathbf{b}_0 + \tau_1^\circ\mathbf{b}_1 + \tau_2^\circ\mathbf{b}_2$$

where

$$\tau_0^\circ = \frac{\frac{1}{\tau_0}}{\frac{1}{\tau_0} + \frac{1}{\tau_1} + \frac{1}{\tau_2}} = \frac{B_2(t)}{B_2(t) + \frac{1}{4w}B_1(t) + B_0(t)}$$

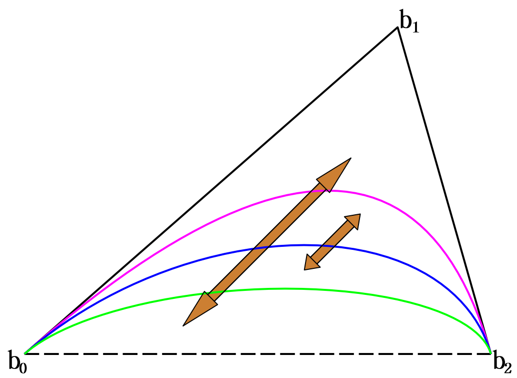

Figure 5. Isotomic conjugate transformation maps parabola (magenta) into an ellipse(green, $w = \frac{1}{4}$). The ellipse with weight $w = \frac{1}{2}$(blue) is fixed uniquely under this transformation.

$$\tau_1^\circ = \frac{\frac{1}{\tau_1}}{\frac{1}{\tau_0}+\frac{1}{\tau_1}+\frac{1}{\tau_2}} = \frac{\frac{1}{4w}B_1(t)}{B_2(t)+\frac{1}{4w}B_1(t)+B_0(t)}$$
$$\tau_2^\circ = \frac{\frac{1}{\tau_2}}{\frac{1}{\tau_0}+\frac{1}{\tau_1}+\frac{1}{\tau_2}} = \frac{B_0(t)}{B_2(t)+\frac{1}{4w}B_1(t)+B_0(t)}.$$

Thus $\mathbf{r}^\circ(t)$ is the quadratic rational Bézier curve having control points $\mathbf{b}_2, \mathbf{b}_1, \mathbf{b}_0$ with weight $\frac{1}{4w}$.

Also we have

$$\mathbf{r}^*(t) = \tau_0^*\mathbf{b}_0 + \tau_1^*\mathbf{b}_1 + \tau_2^*\mathbf{b}_2$$

where

$$\tau_0^* = \frac{\frac{a^2}{\tau_0}}{\frac{a^2}{\tau_0}+\frac{b^2}{\tau_1}+\frac{c^2}{\tau_2}} = \frac{a^2B_2(t)}{a^2B_2(t)+\frac{b^2}{4w}B_1(t)+c^2B_0(t)}$$
$$\tau_1^* = \frac{\frac{b^2}{\tau_1}}{\frac{a^2}{\tau_0}+\frac{b^2}{\tau_1}+\frac{c^2}{\tau_2}} = \frac{\frac{b^2}{4w}B_1(t)}{a^2B_2(t)+\frac{b^2}{4w}B_1(t)+c^2B_0(t)}$$
$$\tau_2^* = \frac{\frac{c^2}{\tau_2}}{\frac{a^2}{\tau_0}+\frac{b^2}{\tau_1}+\frac{c^2}{\tau_2}} = \frac{c^2B_0(t)}{a^2B_2(t)+\frac{b^2}{4w}B_1(t)+c^2B_0(t)}.$$

Thus this quadratic rational Bézier curve has control points $\mathbf{b}_2, \mathbf{b}_1, \mathbf{b}_0$ with weight $a^2, \frac{b^2}{4w}, c^2$, in order, which can be expressed in standard form with the weight $\frac{b^2}{4acw}$.

□

		$\mathbf{r}^\circ(t)$		
		ellipse	parabola	hyperbola
$\mathbf{r}(t)$	ellipse	$\frac{1}{4} < w < 1$	$w = \frac{1}{4}$	$w < \frac{1}{4}$
	parabola	$w = 1$	X	X
	hyperbola	$w > 1$	X	X

Table 1. Isotomic conjugate $\mathbf{r}^\circ(t)$ of quadratic rational Bézier curve $\mathbf{r}(t)$.

Remark 3.2. The isotomic conjugate of the quadratic rational Bézier curve can be written in standard form

$$\mathbf{r}^\circ(t) = \frac{B_0(t)\mathbf{b}_2 + \frac{1}{4w}B_2(t)\mathbf{b}_1 + B_2(t)\mathbf{b}_0}{B_0(t) + \frac{1}{4w}B_1(t) + B_2(t)}.$$

Thus the isotomic conjugate transformation maps the weight w into $\frac{1}{4w}$, control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ into $\mathbf{b}_2\mathbf{b}_1\mathbf{b}_0$. The quadratic rational Bézier curve with weight $w = \frac{1}{2}$ is fixed uniquely under the transformation. This transformation maps hyperbola and parabola into only ellipse. Also it maps ellipse with weight w less than, equal to, and greater than $\frac{1}{4}$ into hyperbola, parabola, and ellipse, respectively, as shown in Figure 5 and Table 1.

Remark 3.3. The isogonal conjugate transformation maps the weight w into $\frac{b^2}{4acw}$, control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ into $\mathbf{b}_2\mathbf{b}_1\mathbf{b}_0$. The quadratic rational Bézier curve with weight $w = \frac{b}{2\sqrt{ac}}$ is uniquely fixed under the transformation. Thus there are three cases: $b^2 - 4ac$ is negative, zero or positive. It looks like the discriminant of quadratic equation.

If $b^2 - 4ac < 0$, the isogonal conjugate transformation maps hyperbola and parabola into ellipse. The ellipse with weight $w < \frac{b^2}{4ac}$, $w = \frac{b^2}{4ac}$ and $\frac{b^2}{4ac} < w < 1$ are mapped into hyperbola, parabola and ellipse, respectively, as shown in Figure 6 and Table 2.

If $b^2 - 4ac = 0$, the isogonal conjugate transformation maps ellipse into hyperbola and vice versa, and parabola is fixed uniquely, as shown in Figure 7 and Table 3.

If $b^2 - 4ac > 0$, the isogonal conjugate transformation maps ellipse and parabola into hyperbola. The hyperbola with weight $w > \frac{b^2}{4ac}$, $w = \frac{b^2}{4ac}$ and $1 < w < \frac{b^2}{4ac}$ are mapped into ellipse, parabola and hyperbola, respectively, as shown in Figure 8 and Table 4.

Our results in this section are the improvement and reconstruction of thesis[10] of the first author in this paper.

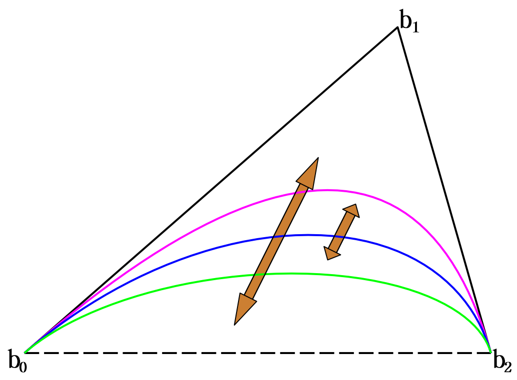

Figure 6. When $b^2 - 4ac < 0$, the isogonal conjugate transformation maps parabola(magenta) into an ellipse(green, $w = \frac{b^2}{4ac}$), and the ellipse with weight $w = \frac{b}{2\sqrt{ac}}$(blue) is uniquely fixed under this transformation.

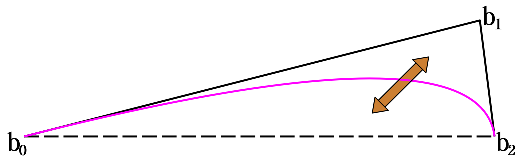

Figure 7. For $b^2 - 4ac = 0$, the parabola(magenta) is uniquely fixed under the isogonal conjugate transformation.

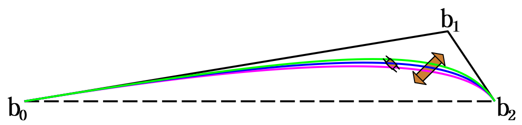

Figure 8. When $b^2 - 4ac > 0$, the isogonal conjugate transformation maps parabola(magenta) into a hyperbola(green, $w = \frac{b^2}{4ac}$), and the hyperbola with weight $w = \frac{b}{2\sqrt{ac}}$(blue) is uniquely fixed under this transformation.

4. CONCLUSION

In this paper we characterized all cases of the isogonal and isotomic conjugates of quadratic rational Bézier curve having control polygon $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ with respect to

		$\mathbf{r}^*(t)$		
		ellipse	parabola	hyperbola
$\mathbf{r}(t)$	ellipse	$\frac{b^2}{4ac} < w < 1$	$w = \frac{b^2}{4ac}$	$w < \frac{b^2}{4ac}$
	parabola	$w = 1$	X	X
	hyperbola	$w > 1$	X	X

Table 2. Isogonal conjugate for $b^2 - 4ac < 0$.

		$\mathbf{r}^*(t)$		
		ellipse	parabola	hyperbola
$\mathbf{r}(t)$	ellipse	X	X	$w < 1$
	parabola	X	$w = 1$	X
	hyperbola	$w > 1$	X	X

Table 3. Isogonal conjugate for $b^2 - 4ac = 0$.

		$\mathbf{r}^*(t)$		
		ellipse	parabola	hyperbola
$\mathbf{r}(t)$	ellipse	X	X	$w < 1$
	parabola	X	X	$w = 1$
	hyperbola	$w > \frac{b^2}{4ac}$	$w = \frac{b^2}{4ac}$	$1 < w < \frac{b^2}{4ac}$

Table 4. Isogonal conjugate for $b^2 - 4ac > 0$.

$\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. We showed that the isotomic conjugate transformation maps parabola, hyperbola and ellipse with the weight $w > 1/4$ into ellipse, maps ellipse with $w = \frac{1}{4}$ into parabola, and maps ellipse with $w < \frac{1}{4}$ into hyperbola. The isogonal conjugate transformation of conic is more complicated. We also identified all cases of the isogonal conjugates of quadratic rational Bézier curves which are ellipse, parabola or hyperbola. We could derive our characterizations very easily due to the expression of conic by the quadratic rational Bézier curve in standard form.

There are many natural and interesting problems along the modern geometry with the isogonal and isotomic conjugate transformations. In future work, we plan to exploit these problems concerned with the rational Bézier curves and surfaces of higher degree than two.

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[a]Department of Mathematics Education, Chosun University, Gwangju 501-759, Republic of Korea
Email address: serenachanran@gmail.com

[b]Department of Mathematics Education, Chosun University, Gwangju 501-759, Republic of Korea
Email address: ahn@chosun.ac.kr